Tree self-similarity based on Horton ordering and Tokunaga indexing

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Introduction.

There exist two important types of tree self-similarity related to the Horton-Strahler ordering and Tokunaga indexing schemes for tree branches.

The Horton-Strahler indexing assigns orders to the tree branches according to their relative importance in the hierarchy.

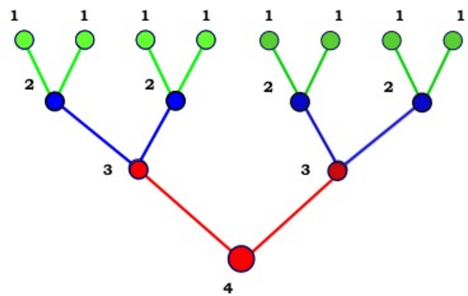
- Introduced in hydrology in the 1950s to describe the dendritic structure of river networks.

- Applications: ranking river tributaries, analysis of brain structure, designing optimal computer codes, etc.

Horton-Strahler ordering.

Consider a **rooted tree** mod series reduction (removing degree two vertices).

- *Horton-Strahler orders* measure “importance” of tree branches within the hierarchy
- In a perfect binary tree (all leaves having the same depth) the orders are proportional to depth



- How to assign orders in a non-perfect tree?

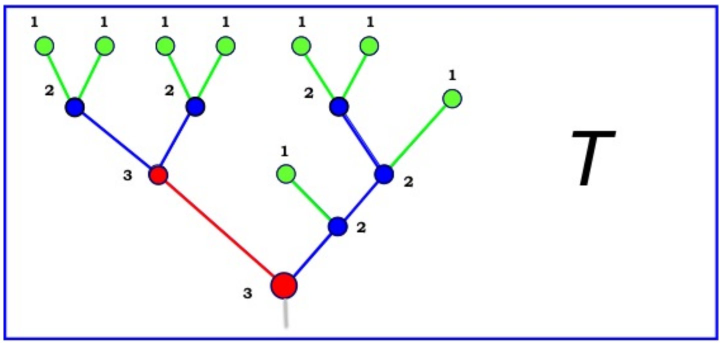
Horton-Strahler order (via pruning).

- Pruning $\mathcal{R}(T)$ of a finite tree T cuts the leaves, followed by *series reduction*.

- A chain of the same order vertices with edges connecting to parent vertices is called branch.

- Branches cut at k-th pruning, $\mathcal{R}^{k-1}(T) \setminus \mathcal{R}^k(T)$, have order k, $k \geq 1$.

- N_k denotes the number of branches of order k in a finite tree T



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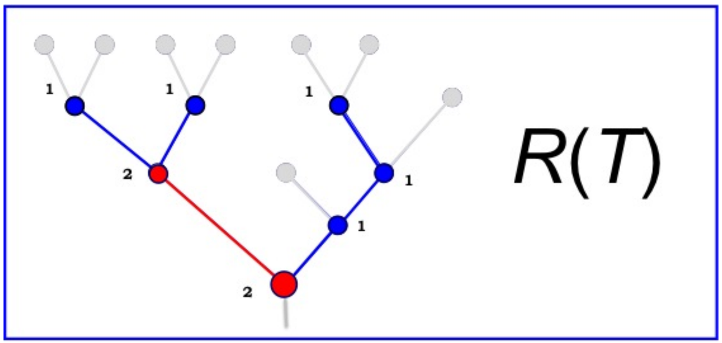

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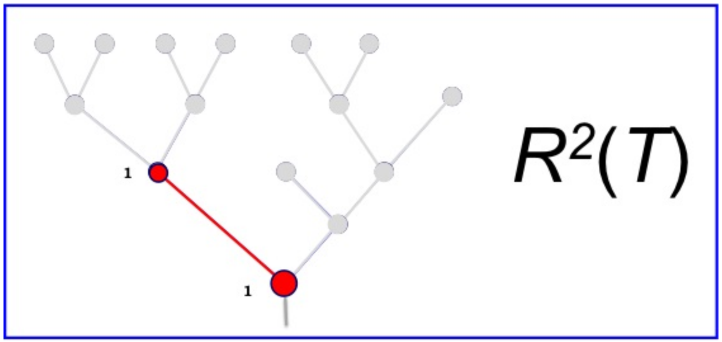

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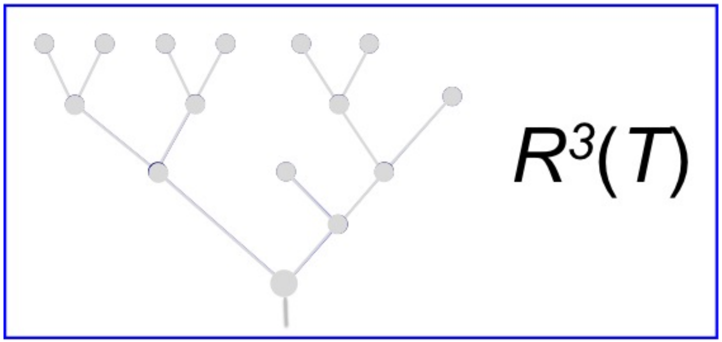

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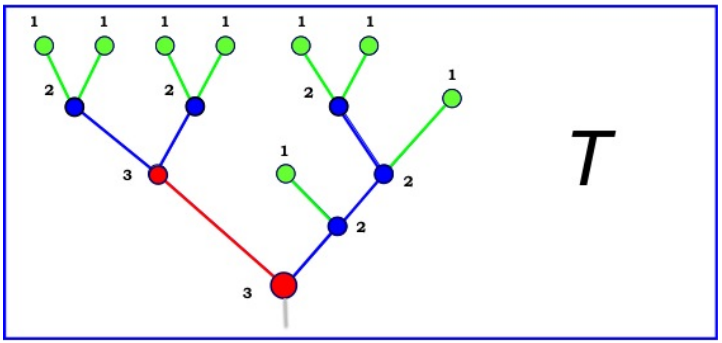
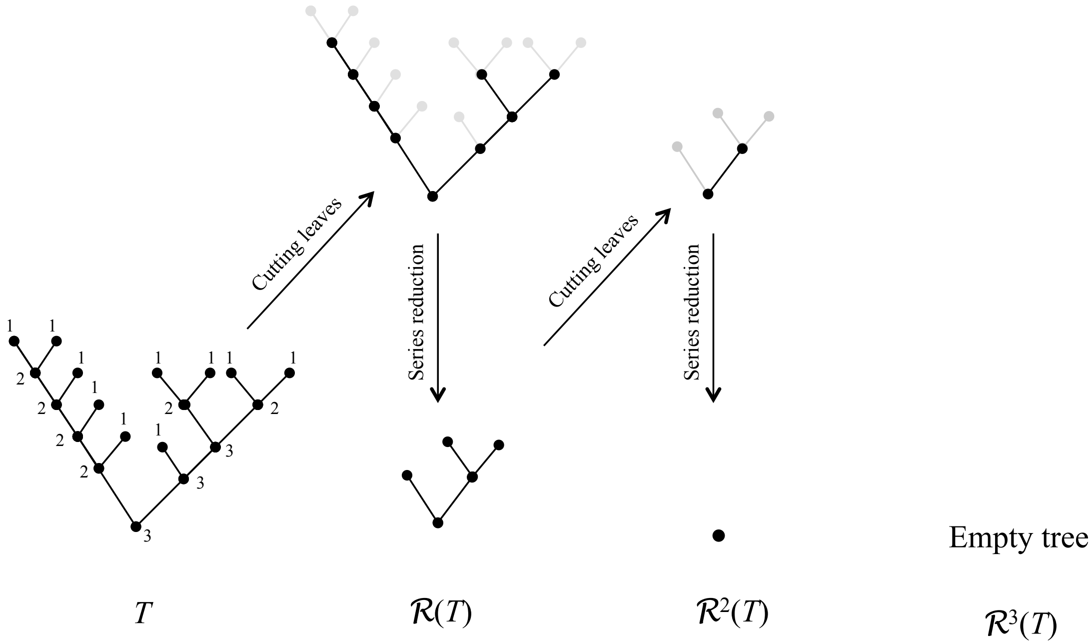
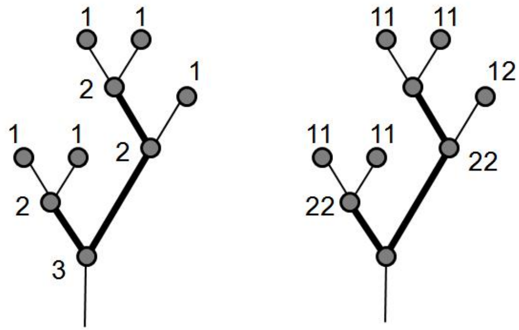


Pruning of a tree mod series reduction



The order of the tree is $k(T) = 3$ with $N_1 = 10$, $N_2 = 3$, $N_3 = 1$, and $N_{1,2} = 3$, $N_{1,3} = 1$, $N_{2,3} = 1$.

Horton-Strahler ordering.



(a) Horton-Strahler orders (b)Tokunaga indices

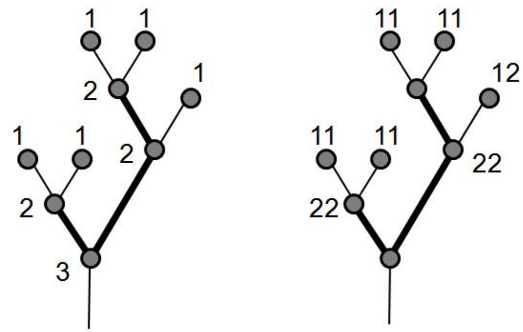
The **Horton-Strahler ordering** of the vertices of a finite rooted labeled binary tree is performed in a hierarchical fashion, from leaves to the root:

(i) each leaf has order $r(\text{leaf}) = 1$;

(ii) when both children, c_1, c_2, of a parent vertex p have the same order r, the vertex p is assigned order $r(p) = r + 1$;

(iii) when two children of vertex p have different orders, the vertex p is assigned the higher order of the two.

Horton-Strahler ordering and Tokunaga indexing.



(a) Horton-Strahler orders (b)Tokunaga indices

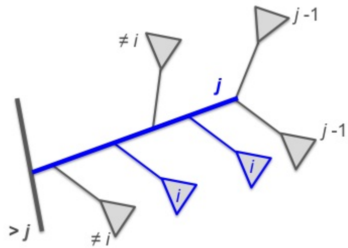
Example: (a) Horton-Strahler ordering

(b) Tokunaga indexing.

Two order-2 branches are depicted by heavy lines in both panels. The Horton-Strahler orders refer, interchangeably, to the tree nodes or to their parent links. The Tokunaga indices refer to entire branches, and not to individual vertices.

Tokunaga indexing: finite binary tree.

- Let $\tau_{ij}^{(k)}$, $1 \le k \le N_j$, $1 \le i < j \le K$, denote the number of branches of order i that merge into the the k-th branch of order j



- Let N_{ij} be the total number of instances when an order-i branch merges an order-j branch
$$N_{ij} = \sum_k \tau_{ij}^{(k)}, i < j$$

- The Tokunaga index T_{ij} is the average number of order-i branches that join an order-j branch:
$$T_{ij} = \frac{N_{ij}}{N_j}$$

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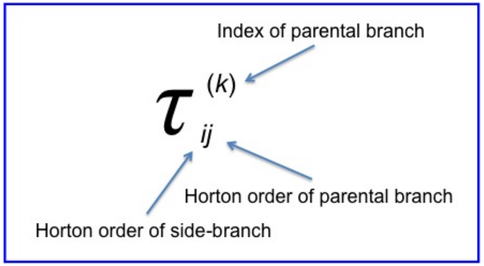

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Tokunaga indexing.

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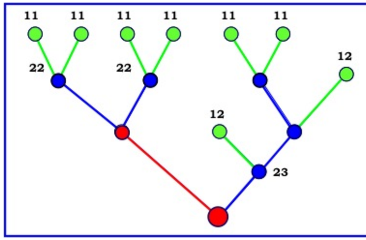

$i \backslash j$	1	2	3	N_i
1	6	2	0	8
2		2	1	2

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Tree self-similarity

Consider probability measures $\{\mu_K\}_{K\geq 1}$, each defined on the set $\mathcal{T}_K$ of finite binary trees of Horton-Strahler order K.

Define the average Horton numbers:

$$\mathcal{N}_k[K] = \mathsf{E}_K[N_k], \quad 1 \leq k \leq K, \quad K \geq 1.$$

Define the respective expectation

$$\mathcal{N}_{ij}[K] = \mathsf{E}_K[N_{ij}].$$

The Tokunaga coefficients $T_{ij}[K]$ for subspace $\mathcal{T}_K$ are defined as

$$T_{ij}[K] = \frac{\mathcal{N}_{ij}[K]}{\mathcal{N}_j[K]}, \quad 1 \leq i < j \leq K.$$

Tree self-similarity

Definition. A set of measures $\{\mu_K\}$ on $\{\mathcal{T}_K\}$ is called coordinated if $T_{ij} := T_{ij}[K]$ for all $K \geq 2$ and $1 \leq i < j \leq K$.

There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_{1,2} & T_{1,3} & \dots & T_{1,K} \\ 0 & 0 & T_{2,3} & \dots & T_{2,K} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_{K-1,K} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

coincides with the restriction of $\mathbb{T}_M$, $M > K$, to the first K coordinates.

Definition. Coordinated probability measures $\{\mu_K\}$ are (mean) self-similar if $T_{ij} = T_{j-i}$ for some sequence $T_k \geq 0$ known as Tokunaga coefficients and any $K \geq 2$.

Tree self-similarity

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There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_1 & T_2 & \dots & T_{K-1} \\ 0 & 0 & T_1 & \dots & T_{K-2} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here **pruning** is equivalent to deleting the first row and first column.

Tokunaga self-similarity

Definition. A random self-similar tree is Tokunaga self-similar if

$$T_{k+1}/T_k = c \quad \Leftrightarrow \quad T_k = a\,c^{k-1} \quad a, c > 0,\ 1 \le k \le K-1.$$

There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & a & ac & ac^2 & \dots & ac^{K-2} \\ 0 & 0 & a & ac & \dots & ac^{K-3} \\ 0 & 0 & 0 & a & \dots & ac^{K-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & a \end{bmatrix}.$$

McConnell and Gupta 2008 showed that Tokunaga self-similarity implies strong Horton law:

$$\lim_{K\to\infty} \frac{\mathcal{N}_k[K]}{\mathcal{N}_1[K]} = R^{1-k} < \infty \quad \text{for any } k \ge 1.$$

Tree self-similarity

Theorem (YK and Zaliapin, Fractals 2016).
Consider a sequence of coordinated self-similar probability measures $\{\mu_K\}$ such that

$$\limsup_{j\to\infty} T_j^{1/j} < \infty.$$

Then $\{\mu_K\}$ satisfy the strong Horton law for some $R > 0$: for each integer $j > 0$,

$$\lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]} = R^{1-j}.$$

Moreover, $1/R = w_0$ is the only real root of the function

$$\hat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$$

in the interval $\left(0, \frac{1}{2}\right]$.

Conversely, if $\limsup_{j\to\infty} T_j^{1/j} = \infty$, then the limit $\lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]}$ does not exist at least for some j.

Horton self-similarity: more generally.

A sequence of probability measures $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ over binary trees has well-defined asymptotic Horton-Strahler orders if for each $k \in \mathbb{N}$, the following limit law is satisfied:

$$\frac{N_k^{(\mathcal{P}_N)}}{N} \longrightarrow \mathcal{N}_k \quad \text{in probability as} \quad N \to \infty,$$

where $\mathcal{N}_k$ is called the asymptotic ratio of the branches of order k.

Horton self-similarity: sequence $\mathcal{N}_k$ decreases in a regular geometric fashion with $k \to \infty$.

Informally,

$$\mathcal{N}_k \asymp N_0 \cdot R^{-k}$$

Horton self-similarity: more generally.

$$\mathcal{N}_k \asymp N_0 \cdot R^{-k}$$

A sequence $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ with well-defined asymptotic Horton-Strahler orders obeys a Horton self-similarity law if and only if at least one of the following limits exists and is finite and positive:

$$\text{(a) root law}: \quad \lim_{k\to\infty} \left(\mathcal{N}_k\right)^{-\frac{1}{k}} = R > 0,$$

$$\text{(b) ratio law}: \quad \lim_{k\to\infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = R > 0,$$

$$\text{(c) geometric law}: \quad \lim_{k\to\infty} \mathcal{N}_k \cdot R^k = N_0 > 0.$$

The constant R is called the Horton exponent.

Galton-Watson tree.

• Critical binary Galton-Watson tree exhibits both Horton and Tokunaga self-similarities (Burd, Waymire, and Winn, 2000). This model has $R = 4$ and $(a, c) = (1, 2)$.

Theorem (Shreve, 1969; Burd et al., 2000). A critical binary Galton-Watson tree is Tokunaga self-similar with

$$(a, c) = (1, 2),$$

that is

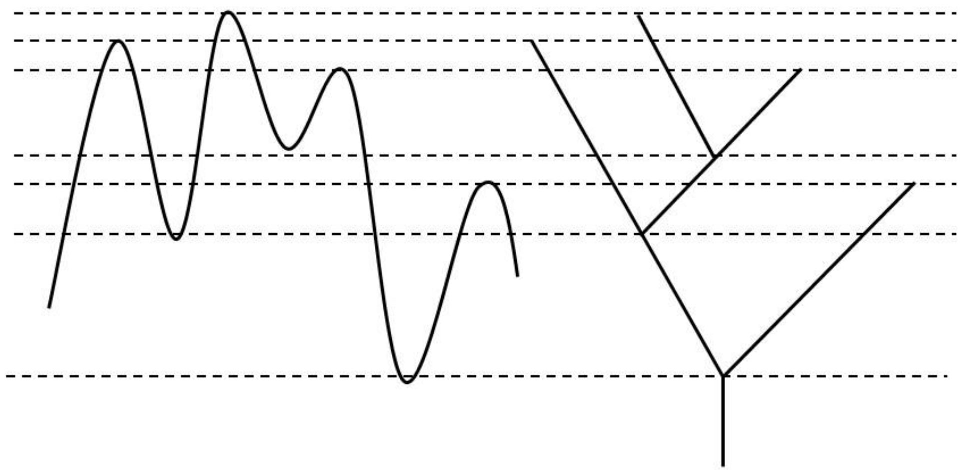
$$T_k = 2^{k-1} \quad \text{and} \quad R = 4.$$

Theorem (Burd et al., 2000).

1. Let $P_{\mathsf{GW}}(p_k)$ denote a Galton-Watson distribution on the space of finite trees with branching probabilities $\{p_k\}$. Then $P_{\mathsf{GW}}(p_k)$ is tree self-similar if and only if $\{p_k\}$ is the critical binary distribution $p_0 = p_2 = 1/2$.

2. Any critical Galton-Watson tree T, $\sum k\, p_k = 1$, converges to the binary critical tree under the operation of pruning, $\mathcal{R}^n(T)$, $n \to \infty$.

Pruning of a level-set tree of a function.

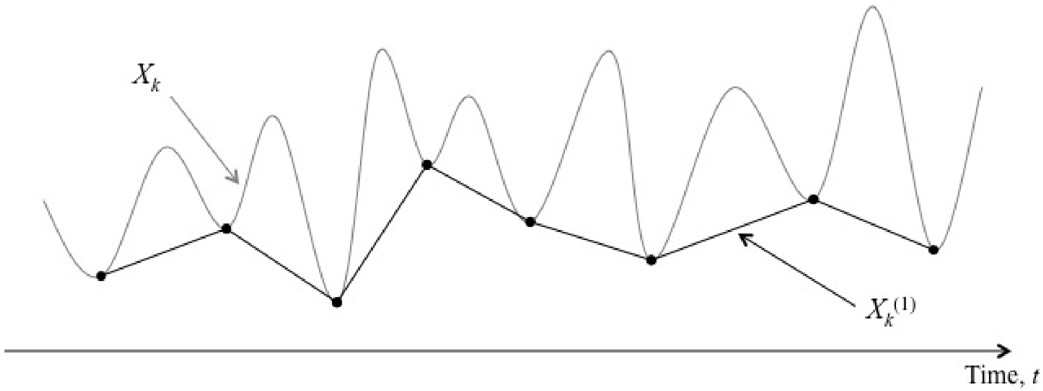


(a) Function X_t (b) Tree LEVEL(X)

Function X_t (panel a) with a finite number of local extrema and its level-set tree level(X) (panel b).

Pruning of time series

Proposition (Zaliapin and YK, CSF 2012). The transition from a time series X_k to the time series $X_k^{(1)}$ of its local minima corresponds to the pruning of the level-set tree level(X).



Horton and Tokunaga self-similarity: Markov chains

Let X_k be a symmetric homogeneous Markov chain and $T = \text{shape}\,(\text{level}(X))$ be the combinatorial level set tree of X_k.

Theorem (Zaliapin and YK, CSF 2012).

1. Tree T is Tokunaga self-similar with parameters $(a, c) = (1, 2)$:

$$\mathsf{E}\left[\tau_{i,i+k}^{(j)}\right] =: T_k = 2^{k-1},$$

   and geometric-Horton self-similar, asymptotically in N, with $R = 4$.

2. Accordingly, a combinatorial level-set tree for regular Brownian motion is Tokunaga and Horton self-similar, with $(a, c) = (1, 2)$, and $R = 4$.

Level-set tree of a homogeneous Markov chai.

Zaliapin and YK, CSF 2012:

• Proved Horton and Tokunaga self-similarity for the level-set tree representation of a homogeneous discrete Markov chain and infinite-tree representation of a regular Brownian motion in continuous time.

• Infinite trees **built from the leafs** representing the complete history of time series. Strong limit laws.

• **Conjecture.** The tree of a fractional Brownian motion B_t^H, $t \in [0,1]$ with the Hurst index $0 < H < 1$ is Tokunaga self-similar with

$$T_{i,i+k} = T_k = c^{k-1}, \quad i,k \geq 1$$

with $c = 2H + 1$.

Root-Horton law for the Kingman's coalescent

In **YK & Zaliapin (AIHP 2016)** we

• Established the root-Horton law for the Kingman's coalescent.

• Showed that the tree for Kingman's coalescent is combinatorially equivalent to the level-set tree of iid time series (the two measures are **one pruning apart**).

• Numerical experiments that suggest stronger Horton laws: ratio, geometric.

Root-Horton law for the Kingman's coalescent.

In **YK & Zaliapin (AIHP 2016)**, we prove the limit law (in probability) for the asymptotics of the number N_k of branches of Horton-Strahler order k in Kingman's N-coalescent process with constant collision kernel:

$$\mathcal{N}_k = \lim_{N\to\infty} N_k/N$$

We show that

$$\mathcal{N}_k = \frac{1}{2}\int_0^\infty g_k^2(x)\,dx,$$

where the sequence $g_k(x)$ solves:

$$g'_{k+1}(x) - \frac{g_k^2(x)}{2} + g_k(x)g_{k+1}(x) = 0, \qquad x \geq 0$$

with $g_1(x) = 2/(x+2)$, $g_k(0) = 0$ for $k \geq 2$.

Root-Horton law for the Kingman's coalescent.

Theorem (YK & Zaliapin, AIHP 2016). The asymptotic Horton ratios $\mathcal{N}_k$ exist and finite and satisfy the convergence $\lim\limits_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$ with $2 \le R \le 4$.

Conjecture. The tree associated with Kingman's coalescent process is Horton self-similar with

$$\lim_{k\to\infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = \lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R \quad \text{and} \quad \lim_{k\to\infty} (\mathcal{N}_k R^k) = const.,$$

where $R = 3.043827\ldots$ and Tokunaga self-similar, asymptotically in k:

$$\lim_{i\to\infty} T_{i,i+k} =: T_k \quad \text{and} \quad \lim_{k\to\infty} \frac{T_k}{c^{k-1}} = a$$

for some positive a and c.

Root-Horton law for the Kingman's coalescent.

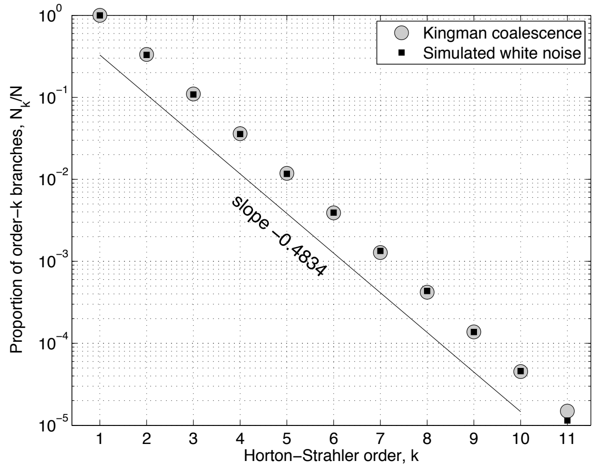
• **Theorem (YK & Zaliapin, AIHP 2016).** The asymptotic Horton ratios $\mathcal{N}_k$ exist and finite and satisfy the convergence $\lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$ with $2 \leq R \leq 4$.

• Consider now a time series X with N local maxima separated by $N-1$ internal local minima such that the latter form a discrete white noise; we call X an *extended discrete white noise*.

Theorem (YK & Zaliapin, AIHP 2016). The combinatorial level set tree of the extended discrete white noise has the same distribution on $\mathcal{T}_N$ as the combinatorial tree generated by Kingman's N-coalescent.

• **Corollary (YK & Zaliapin, AIHP 2016).** The combinatorial level set tree of iid time series is root-Horton self similar with the same Horton exponent R as for Kingman's coalescent.

Root-Horton law for the Kingman's coalescent.



Filled circles: The asymptotic ratio $\mathcal{N}_k$ of the number N_k of branches of order k to N in Kingman's coalescent, as $N \to \infty$. Black squares: The empirical ratio N_k/N_1 in a level-set tree for a single trajectory of a iid time series of length $N = 2^{18}$.

Hierarchical Branching Processes.

Consider a branching process where we begin with a root of hierarchical order K with probability p_K.

• Each branch of order j branches out an offspring of order $i < j$ with rate $\lambda_j T_{j-i}$.

• The branch of order j terminates with rate λ_j, at which moment,

(i) the branch of order $j \geq 2$ splits into two branches, each of order $j - 1$

(ii) the branch of order $j = 1$ terminates without leaving offsprings.

The branching history of the process creates a random planar binary tree, with $T_{i,j} = T_{j-i}$.

Hierarchical Branching Processes.

For $\{p_K\}_{K=1,2,\ldots}$, we generate a random forest. Let $x(s)$ be the vector with coordinates representing the average number of branches of respective orders at time s in a tree.

Initial distribution is $x(0) = \pi := \sum\limits_{K=1}^{\infty} p_K e_K$, and

$$x(s) = e^{\mathbb{G}\Lambda s}\pi,$$

where Λ is a diagonal operator with the diagonal entries $\lambda_1, \lambda_2, \ldots$ and

$$\mathbb{G} := \begin{bmatrix} -1 & T_1+2 & T_2 & T_3 & \ldots \\ 0 & -1 & T_1+2 & T_2 & \ldots \\ 0 & 0 & -1 & T_1+2 & \ddots \\ 0 & 0 & 0 & -1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hierarchical Branching Processes.

Consider the width function at time $s \geq 0$

$$C(s) = \langle \mathbf{1}, x(s) \rangle = \langle \mathbf{1}, e^{\mathbb{G}\Lambda s} \pi \rangle.$$

For μ to be self-similar under pruning need

- $\{p_K\}$ to be geometric

- the sequence λ_j to be geometric: $\lambda_j = \gamma \cdot \lambda^j$ for some $\gamma, \lambda > 0$,

Assuming the above, the critical probability p_c is defined as

$$\lim_{s \to \infty} C(s) = \infty \quad \text{if and only if } p < p_c$$

Hierarchical Branching Processes.

Recall:

$$\widehat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$$

and $w_0 = 1/R$ is the only real root within the radius of convergence.

Suppose $\{p_K\}$ is geometric with parameter p and $\lambda_j = \gamma \cdot \lambda^j$, then

$$x(s) = e^{\mathbb{G}\Lambda s}\pi = \pi + \sum_{m=1}^{\infty} s^m \left[\prod_{i=1}^{m} \widehat{t}\big(\lambda^i(1-p)\big)\right] \Lambda^m \pi.$$

The convergence requirement here is that $\lambda \le 1$.

Hierarchical Branching Processes.

Recall: $\hat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$ and $w_0 = 1/R$ is the only real root within the radius of convergence.

For $\{p_K\}$ geometric and $\lambda_j = \gamma \cdot \lambda^j$ for some $\gamma, \lambda > 0$,

$$x(s) = e^{\mathbb{G}\Lambda s}\pi = \pi + \sum_{m=1}^{\infty} s^m \left[\prod_{i=1}^{m} \hat{t}\big(\lambda^i(1-p)\big) \right] \Lambda^m \pi.$$

Thus $p_c = 1 - \frac{1}{\lambda R}$ and $C(s) = \langle \mathbf{1}, x(s) \rangle = 1$ at criticality.

Forest invariance property at criticality: $e^{\mathbb{G}\Lambda s}\pi = \pi$.

Hierarchical Branching Processes.

$$p_c = 1 - \frac{1}{\lambda R}$$

Observe that for a hierarchical branching process with $\lambda_j = \gamma 2^{-j}$ and $T_{i,j} = 2^{j-i-1}$, the critical probability is

$$p_c = \frac{1}{2}.$$

Therefore, $R = \frac{1}{\lambda(1-p_c)} = 4$.

Recall that the critical binary Galton-Watson tree exhibits both Horton and Tokunaga self-similarities (Burd, Waymire, and Winn, 2000) with parameters $R = 4$, $(a, c) = (1, 2)$ and

$$T_k = a \cdot c^{k-1} = 2^{k-1}.$$

Hierarchical Branching Processes.

Theorem (YK and Zaliapin, 2016). Consider a hierarchical branching process with parameters

$$\lambda_j = \gamma 2^{-j}, \quad p_K = 2^{-K}, \quad \text{and} \quad T_{i,j} = 2^{j-i-1}.$$

A (critical) hierarchical branching process with the above is distributionally equivalent to the critical binary Galton-Watson process with edge lengths distributed exponentially with rate $\gamma/2$.

• Critical binary Galton-Watson with exponential edge lengths is both invariant under pruning (from the leafs) and satisfies the forest invariance property (when cut from below). It is also self-similar under *continuous pruning*.