Evolving Random Genealogies: Infinite Divisibility and Branching Property

Random Trees and Maps: Probabilistic and Combinatorial Aspects
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Goal

Develop framework for modelling of genealogical information of a *stochastically evolving population* as a strong **Markov process**, i.e. the *forward* picture is described.

Apply this to obtain properties of process, equilibria, longtime behaviour, parameter dependence, path properties.

In particular study

- infinite divisibility
- branching property, generator criterion

We want to extend the theory to spatial models.

Consider a finite population say a Moran-model or a branching model (Galton-Watson).

Moran:

- We have birth-death events
  a pair dies at rate d instantaneously one of them gives birth to two new individuals.

Galton-Watson:

- At rate b an individual gives birth to a new one
- At rate b an individual dies

The genealogical tree

Birth event
$\iota \longrightarrow \iota', \iota''$
ι', ι'' are the *descendents* of ι, ι is *ancestor* of ι', ι''.

This defines:
(1) a "genealogical tree", a rooted labelled tree.
(2) Genealogical distance: $2 \cdot$ time to MRCA

Of interest is a structure *abstracting from labels*.

We are interested to model the stochastic evolution of the genealogical structure as solution to a **martingale problem** *on a* **Polish space,** in particular for the diffusion limit.

Moran model size-N genealogies

Our model for a Moran population genealogy:

$\mathfrak{U} = \overline{(U, r, \mu)},$

U =set of individuals
$r : U \times U \longrightarrow [0, \infty)$, quasi ultrametric
$r(\iota, \iota')$ = genealogical distance of ι and ι' = twice time to MRCA
$\mu = \frac{1}{|U|} \cdot \sum_{\iota \in U} \delta_\iota \in \mathcal{M}_1(U)$, sampling measure
— : equivalence class under isometries and measure-preserving maps of supp(μ) : $\mathfrak{U}$.

$\mathfrak{U}$ is called an *ultrametric probability space*
$\mathbb{U} := \{\mathfrak{U} | \mathfrak{U}$ is *up*-space$\}$
$\mathbb{U}$ is a Polish space for topology introduced later.

Branching genealogies: extended up-space

Our model for a branching genealogy with varying population size *requires* $\mu \in \mathcal{M}_{\text{fin}}(U)$.

$\mathfrak{U} = (\overline{\bar{\mu}, (U, r, \widehat{\mu})),}$

$\widehat{\mu} \in \mathcal{M}_1(U), \quad \bar{\mu} \in \mathbb{R}^+,$
$\mu = \sum_{\iota \in U} \delta_\iota \quad , \quad \bar{\mu} = \mu(U) \quad , \quad \widehat{\mu}(\cdot) = \frac{\mu(\cdot)}{\mu(U)}.$

$\mathbb{U}^* = \mathbb{R}^+ \times \mathbb{U}$ (extended ultrametric probability measures)

$\mathbb{U}^*$ is a Polish space

Population with types and locations: marked up-space

Individuals with types and locations require:
$V = \mathbb{K} \times \mathbb{G}$, $\quad \mathbb{K} =$ type space, $\quad \mathbb{G} =$ geographic space.
Assume that $\mathbb{K}$ is a compact Polish space, $\mathbb{G}$ finite or compact.
Mark function is added to description $\kappa : U \longrightarrow V$.
Consider (U, r, κ, μ).
Form equivalence class including mark preservation.
Obtain (extended) V-marked ultrametric probability measure spaces:

$$\mathfrak{U} = \overline{(U, r, \kappa, \mu)}$$

Allow : κ : kernel $U \times V$, $\quad \nu = \mu \otimes \kappa : (U, r, \nu)$

State space: $\mathbb{U}_V$, respectively $\mathbb{U}_V^*$ ultrametric (probability) measure spaces.

$\mathbb{U}_V, U_V^*$ are Polish spaces.

Polynomials

Polynomials are functions of the form:

$$\Phi : \mathbb{U} \longrightarrow \mathbb{R}, \quad \mathbb{U}^* \longrightarrow \mathbb{R}, \quad \mathbb{U}^*_V \longrightarrow \mathbb{R}$$

(1) $\Phi(\mathfrak{U}) = \int\limits_{U^n} \varphi((r(u_i, u_j))_{1 \leq i < j \leq n}) \mu(du_1) \cdots \mu(du_n)$

$\varphi \in C_b^1((\mathbb{R})^{\binom{n}{2}}, \mathbb{R}) \quad , \quad \mu \in \mathcal{M}_1(U).$

(2) $\Phi(\mathfrak{U}) = \bar{\Phi}(\bar{\mu}) \widehat{\Phi}((U, r, \widehat{\mu})) \quad , \quad \bar{\Phi} \in C_b(\mathbb{R}, \mathbb{R})$

(3)

$$\Phi(\mathfrak{U}) = \quad \bar{\Phi}(\bar{\nu}) \int_{(U\times V)^n} \varphi((r(u_i,u_j))_{1\le i<j\le n})\chi((v_1,v_2,\cdots,v_n))\widehat{\nu}^{\otimes n}(d(\underline{u},\underline{v}))$$

$$\nu = \mu \otimes \kappa$$

$$\chi \in C_b(V,\mathbb{R}) \quad (\chi \in C_{bb}(V,\mathbb{R}))$$

We call the generated algebra of functions by Π, Π^*, Π_V.

Topology

Define topology via convergence of sequences.

Idea: Convergence $\longleftrightarrow$ convergence of sampled (marked) subtrees and population sizes.

$\mathfrak{U}_n \Longrightarrow \mathfrak{U}$ as $n \to \infty$ if

$$\Phi(\mathfrak{U}_n) \underset{n\to\infty}{\longrightarrow} \Phi(\mathfrak{U}), \quad \forall\, \Phi \in \Pi \text{ resp. } \Pi^*, \Pi_V, \Pi_V^*.$$

In particular polynomials are bounded continuous functions on :

$$\mathbb{U}, \mathbb{U}_V, \mathbb{U}^*, \mathbb{U}_V^*.$$

Next Goal

Decompose the tree into subfamilies of depth h, for varying h.
Obtain the path

$$h \longrightarrow \text{ subfamily decomposition} \tag{2.1}$$

How to formalize this on the level of ultrametric measure spaces?

This leads to
- Lévy-Khintchine formula
- branching property

Definition (Forests, trees and concatenation)

Let $h > 0$.

(a) Define the subset of *h-forests*

$$\mathbb{U}(h)^{\sqcup} := \mathbb{U}([0,2h]) := \{[U,r,\mu] \in \mathbb{U} : \mu^{\otimes 2}(\{(u,v) \in U^2 : r(u,v) \in (2h,\infty)\}) = 0\}. \tag{2.2}$$

(b) For $\mathfrak{u}, \mathfrak{v} \in \mathbb{U}(h)^{\sqcup}$ with $\mathfrak{u} = [U, r_U, \mu], \mathfrak{v} = [V, r_V, \nu]$ define the *h-concatenation*:

$$\mathfrak{u} \sqcup^h \mathfrak{v} := [U \uplus V, r_U \sqcup^h r_V, \mu + \nu], \tag{2.3}$$

where $\uplus$ is the disjoint union of sets and $r_U \sqcup^h r_V|_{U \times U} = r_U$, $r_U \sqcup^h r_V|_{V \times V} = r_V$ and for $x \in U$, $y \in V$:

$$\left(r_U \sqcup^h r_V\right)(x,y) = 2h. \text{ We write } \sqcup \text{ if } h > 0 \text{ has been fixed.} \tag{2.4}$$

(c) Define the subset of *h-trees*

$$\mathbb{U}(h) := \mathbb{U}([0,2h)) := \{[U,r,\mu] \in \mathbb{U} : \mu^{\otimes 2}(\{(u,v) \in U^2 : r(u,v) \in [2h,\infty) = 0\}. \tag{2.5}$$

We define the *h*-truncation of an h'-top for $h' > h$ as

$$\mathbb{U}^{\sqcup}(h') \to \mathbb{U}^{\sqcup}(h) \tag{2.6}$$
$$\tau_{h',h}(\mathfrak{u}) = (\mathfrak{u}, r \wedge 2h, \mu), \mathfrak{u} \in \mathbb{U}^{\sqcup}(h')$$

Theorem (Semigroup structure)

(a)The algebraic structure $(\mathbb{U}(h)^{\sqcup}, \sqcup^h)$ *is a Delphic semigroup. The set of irreducible elements is* $\mathbb{U}(h)$ *and any* $\mathfrak{u} \in \mathbb{U}(h)^{\sqcup}$ *has a unique (up to order) decomposition:*

$$\mathfrak{u} = \bigsqcup_{i \in I}^{h} \mathfrak{u}_i \,, \tag{2.7}$$

where I *is a countable index set and* $\mathfrak{u}_i \in \mathbb{U}(h) \setminus \{0\}$ *associating with* $\mathfrak{u}$ *a collection of decompositions for* $h' \in [0, h]$.

(b)The h-decompositions are consistent, i.e.

$$\tau_{h',h}\,(\mathfrak{u}_1^{h'} \sqcup \ldots \sqcup \mathfrak{u}_\ell^{h'}) = \mathfrak{u}_1^{h} \sqcup \ldots \sqcup \mathfrak{u}_m^{h}, \tag{2.8}$$
$$\text{where} \quad \{\mathfrak{u}_2^{h'}, \ldots \mathfrak{u}_\ell^{h'}\}, \{\mathfrak{u}_1^{h}, \ldots \mathfrak{u}_m^{h}\}$$

are the h' *respectively h-decomposition of* $\mathfrak{u}$. □

We obtain a path of consistent decompositions

$$h \longrightarrow \{u(h)_1, \cdots, u(h)_n\} \tag{2.9}$$

which consistent w.r.t. to truncation.

Values are taken in semigroups $(U(h)^{\sqcup}, \sqcup^h)$.

Definition (Tops and trunks)

Let $h > 0$ and $\mathfrak{u} = [U, r, \mu] \in \mathbb{U}$:

(a) Define the *h-top* $\mathfrak{u}(h) := [U, r \wedge 2h, \mu] \in \mathbb{U}(h)^{\sqcup}$.

(b) Suppose $\mathfrak{u}(h) = \bigsqcup_{i \in I}^{h} \mathfrak{u}_i$ as in (2.7) with at most countable index set I and $\mathfrak{u}_i \in \mathbb{U}(h) \setminus \{0\}$ and write $\mathfrak{u}_i = [U_i, r_i, \mu_i]$ for $i, i' \in I$. The *h-trunk* of $\mathfrak{u}$ is defined as the ultrametric space

$$\mathfrak{u}(\underline{h}) = [I, r^*, \mu^*], \tag{2.10}$$

with

$$r^*(i, i') = \inf \{ r(u, v) - 2h \mid u \in U_i, v \in U_{i'} \} \tag{2.11}$$

and the weights

$$\mu^*(\{i\}) = \mu_i(U_i). \tag{2.12}$$

Proposition

The mapping $\mathbb{U} \mapsto \mathbb{U}(h)^{\sqcup}$, $\mathfrak{u} \mapsto \mathfrak{u}(h)$*and* $(0,t] \mapsto \mathbb{U}(h)^{\sqcup}$, $h \mapsto \mathfrak{u}(h)$ *is continuous.*

Definition (number of h-balls)

If $\mathfrak{u} \in \mathbb{U}$ and let I be the index set belonging to the decomposition of $\mathfrak{u}(h)$ as given in Theorem 2. Then we set $\#_h \mathfrak{u} := \# I$.

Proposition (Lemma 2.4(a) in[EM14])

The number of open 2h-balls $\#_h$ *is measurable. It is an additive functional on* $(\mathbb{U}(h)^{\sqcup}, \sqcup^h)$*, that is*

$$\#_h(\mathfrak{u} \sqcup \mathfrak{v}) = \#_h(\mathfrak{u}) + \#_h(\mathfrak{v}), \tag{2.13}$$

for all $\mathfrak{u}, \mathfrak{v} \in \mathbb{U}(h)^{\sqcup}$*, where we interpret* $\infty + a = a + \infty = \infty$ *for* $a \in \mathbb{N}_0 \cup \{\infty\}$.

Definition (ultrametric distance matrices)

(a) Define the set of ultrametric distance matrices of order $m \geq 2$ by

$$\mathbb{D}_m := \{(r_{ij})_{i \leq i < j \leq m} : r_{ij} \geq 0\,,\ r_{ij} \leq r_{ik} \vee r_{kj}\ \ \forall 1 \leq i < k < j \leq m\} \tag{3.1}$$

and $\mathbb{D}_1 = \{0\}$.

(b) For an ultrametric space $\mathfrak{u} = [U, r, \mu] \in \mathbb{U}$ and $m \geq 2$ we define the *distance matrix map of order m*

$$R^{m,(U,r)} : U^m \to \mathbb{D}_m\,, \quad (u_i)_{i=1,\ldots,m} \mapsto (r(u_i, u_j))_{1 \leq i < j \leq m} \tag{3.2}$$

and the *distance matrix measure of order m*

$$\begin{aligned} \nu^{m,\mathfrak{u}}(\mathrm{d}\underline{r}) &:= \mu^{\otimes m} \circ (R^{m,(U,r)})^{-1} \qquad (3.3)\\ &= \mu^{\otimes m}(\{(u_1, \ldots, u_m) \in U^m : (r(u_i, u_j))_{1 \leq i < j \leq m} \in \mathrm{d}\underline{r}\})\,. \end{aligned}$$

For $m = 1$ we set $\nu^{1,\mathfrak{u}} := \bar{\mathfrak{u}} := \mu(U)$ the *total mass*.

Definition (Polynomials)

For $m \geq 1$ and $\phi \in C_b(\mathbb{D}_m)$, define the *monomial*

$$\Phi = \Phi^{m,\phi} : \mathbb{U} \to \mathbb{R}, \quad \mathfrak{u} \mapsto \langle \phi, \nu^{m,\mathfrak{u}} \rangle = \int \nu^{m,\mathfrak{u}}(d\underline{r})\, \phi(\underline{r})\,. \tag{3.4}$$

The elements of the algebra generated by Π are called *polynomials*, the corresponding set $\mathcal{A}(\Pi)$. We denote special classes of monomials for $h > 0$ as follows:

$$\Pi_h := \{\Phi^{m,\phi} \in \Pi : \operatorname{supp}(\phi) \subseteq [0, 2h)^{\binom{m}{2}}\}, \Pi_+ := \{\Phi^{m,\phi} \in \Pi : \phi \geq 0\} \tag{3.5}$$

$$\text{and } \Pi_{h,+} = \Pi_h \cap \Pi_+\,.$$

Definition (truncation)

Let $m \in \mathbb{N}$ and $\phi : \mathbb{D}_m \to \mathbb{R}$. Define the *upper h-truncation of* ϕ:

$$\phi_h(\underline{r}) := \phi(\underline{r}) \cdot \prod_{1 \le i < j \le m} \mathbb{1}_{[0,2h)}(r_{ij}), \tag{3.6}$$

For the monomial $\Phi^{m,\phi} \in \Pi$ define

$$\Phi_h^{m,\phi}(\mathfrak{u}) := \langle \phi_h, \nu^{m,\mathfrak{u}} \rangle. \tag{3.7}$$

This extends to polynomials by linearity.

Definition (Laplace transform)

The *Laplace functional* $L_{\mathfrak{U}} : \Pi_+ \to [0,1]$ of a random um-space $\mathfrak{U}$ by

$$L_{\mathfrak{U}}(\Phi) := \mathbb{E}\left[\exp(-\Phi^{m,\phi}(\mathfrak{U}))\right], \quad \Phi = \Phi^{m,\phi} \in \Pi_+ , \tag{3.8}$$

the *truncated Laplace functional* $L_{\mathfrak{U}} : \mathcal{A}(\Pi_{h,+}) \to [0,1]$ by restriction.

The next result tells us that Laplace transforms on $\mathbb{U}(h)^{\sqcup}$ share an important property with Laplace transforms on $[0,\infty)$: they well-define a probability measure on that space.

Theorem (Truncated Laplace transform)

(a) *Let $\mathfrak{U}, \mathfrak{U}' \in \mathbb{U}(h)^{\sqcup}$ be random h-forests. Then,*

$$\mathfrak{U} \stackrel{d}{=} \mathfrak{U}' \quad \Longleftrightarrow \quad L_{\mathfrak{U}}(\Phi) = L_{\mathfrak{U}'}(\Phi) \;\; \forall \Phi \in \mathcal{A}(\Pi_{h,+}) . \tag{3.9}$$

(b) *Let $\mathfrak{U}, \mathfrak{U}_n$, $n \in \mathbb{N}$, be random h-forests. Then,*

$$\mathfrak{U}_n \underset{n\to\infty}{\Longrightarrow} \mathfrak{U} \quad \Longleftrightarrow \quad L_{\mathfrak{U}_n}(\Phi) \to L_{\mathfrak{U}}(\Phi) \;\; \forall \Phi \in \mathcal{A}(\Pi_{h,+}) . \tag{3.10}$$

We have a $\mathbb{R}^+$ indexed *collection* of nested semigroups for which we want to decompose our law. The key concept is the following.

Definition (Infinite divisibility)

Suppose $t > 0$. A random um-space $\mathfrak{U}$ taking values in $\mathbb{U}(t)^\sqcup$ which is not identically 0 is called *infinitely divisible* if for all $h > 0$ (or t-infinitely divisible if for all $h \in (0,t]$) and $n \in \mathbb{N}$
we find i.i.d. $\mathfrak{U}_1^{(h,n)}, \dots, \mathfrak{U}_n^{(h,n)} \in \mathbb{U}(h)^\sqcup$,
s.t.the h-top of $\mathfrak{U}$ is a concatenation of these random forests:

$$\mathfrak{U}(h) \stackrel{\text{d}}{=} \mathfrak{U}_1^{(h,n)} \sqcup \cdots \sqcup \mathfrak{U}_n^{(h,n)} . \tag{4.1}$$

Note that by Theorem (9) this is equivalent to saying that for all $h > 0$ the Laplace functional of the h-treetop factorizes for any $n \in \mathbb{N}$:

$$\exists \mathfrak{U}^{(h,n)} \in \mathfrak{U}(h)^\sqcup \text{ with } L_{\mathfrak{U}}(\Phi) = (L_{\mathfrak{U}^{(h,n)}}(\Phi))^n, \quad \Phi \in \mathcal{A}(\Pi_{h,+}) . \tag{4.2}$$

Recall that $\mathcal{M}^{\#}(E)$ denotes the set of boundedly finite measures on E, which we will consider here for the space $E = \mathbb{U}(h)^{\sqcup} \setminus \{0\})$ with the point 0 infinitely far away.

Theorem (Lévy-Khintchine representation)

An infinitely divisible random ultrametric measure space $\mathfrak{U}$ allows for a Lévy-Khintchine representation of its Laplace functional; more precisely, there exists a unique $\lambda_\infty \in \mathcal{M}^{\#}(\mathbb{U} \setminus \{0\})$ with $\int(\bar{\mathfrak{u}} \wedge 1)\lambda_\infty(\mathrm{d}\mathfrak{u}) < \infty$ such that for any $h \in (0, \infty)$:

$$-\log L_{\mathfrak{U}}(\Phi_h) = \int_{\mathbb{U}(h)^{\sqcup}\setminus\{0\}} (1 - e^{-\Phi_h(\mathfrak{u})})\lambda_h(\mathrm{d}\mathfrak{u}) \quad \forall\, \Phi \in \Pi_+ \,, \qquad (4.3)$$

for

$$\lambda_h(\mathrm{d}\mathfrak{u}) = \int_{\mathbb{U}\setminus\{0\}} \lambda_\infty(\mathrm{d}\mathfrak{v})\, \mathbb{1}(\mathfrak{v}(h) \in \mathrm{d}\mathfrak{u}) \in \mathcal{M}^{\#}(\mathbb{U}(h)^{\sqcup} \setminus \{0\})\,. \qquad (4.4)$$

Theorem

If $\mathfrak{U}$ is merely t-infinitely divisible, there is a unique $\lambda_t \in \mathcal{M}^{\#}(\mathbb{U}(t)^{\sqcup} \setminus \{0\})$ such that $\mathfrak{u} \mapsto (\bar{\mathfrak{u}} \wedge 1)$ is also integrable, (4.3) *holds for $h \in (0, t]$ and* (4.4) *holds with λ_t instead of λ_∞ for $h \in (0, t]$. In either case,*

$$\lambda_h(\mathbb{U}(h)^{\sqcup} \setminus \{0\}) = -\log \mathbb{P}(\bar{\mathfrak{U}} = 0) \in [0, \infty] \text{ for any } h. \tag{4.5}$$

We refer to λ_h as the h-Lévy measure and to λ_∞ as the Lévy measure.

Corollary (Poisson cluster representation)

Let $\mathfrak{U}$ be infinitely divisible. Then for every $h > 0$ there exists a Poisson point process N^{λ_h} on $\mathbb{U}(h)^{\sqcup}$ with intensity measure $\lambda_h \in \mathcal{M}^{\#}(\mathbb{U}(h)^{\sqcup} \setminus \{0\})$ such that

$$\mathfrak{U}(h) \stackrel{\mathrm{d}}{=} \bigsqcup_{\mathfrak{u} \in N^{\lambda_h}} \mathfrak{u} \,. \tag{4.6}$$

If $\mathfrak{U}$ is t-infinitely divisible, then there exists a Poisson point process on $\mathbb{U}(t)^{\sqcup}$ such that the h-truncations of the points form a Poisson point process N^h with Lévy measure λ_h with (4.6).

Branching property:

- Different individuals grow independent trees of descendents.
- State of the process however contains information about also the early ancestral relations, creates dependence.

How to reconcile this and how to formalize this for $\mathbb{U}-$valued processes?

Setup

Let S be a Polish space. We use $\mathcal{B}(S)$ for the Borel σ-field on S and $b\mathcal{B}(S)$ to denote the bounded measurable functions on S.
Assume there are $S_t \subset S$, $t \geq 0$ with the following properties.

- $S_s \subseteq S_t$ *for* $0 \leq s \leq t$.
- S_t *is closed in* S.
- *There is a continuous mapping* $T_t : S \mapsto S_t$, *which is the identity if restricted to* S_t*:* $T_t(x) = x$ *for any* $x \in S_t$.
- *There is a binary operation* $\sqcup^t : S_t \times S_t \to S_t$ *such that* $(S_t, \sqcup^t)$ *is a commutative topological semigroup for all* $t \geq 0$ *with neutral element* $0 \in S_0$.
- *The extension of* $\sqcup^t$ *to all elements of* S *is defined via*

$$\sqcup^t : S \times S \to S_t, \ (x, y) \mapsto (T_t(x)) \sqcup^t (T_t(y)) . \qquad (5.1)$$

For simplicity, we drop the index t *at* $\sqcup^t$ *if it's clear from the context.*

Example (um-spaces)

Recall the setting of ultrametric spaces and the semigroup of t-forests. The evolving genealogy of the population was described via the genealogical distance between individuals and a sampling measure giving an ultrametric measure space and element of $\mathbb{U}$.
Trees were $h-$truncated by truncating the metric at $2h$ and two such objects were $h-$ concatenated, $\sqcup^h$, by taking the disjoint union of the sets of individuals, keeping the metric in each population and setting the distance between individuals from different subpopulations equal to $2h$ and by adding the measures. Then $S = \mathbb{U}$, $S_t = \mathbb{U}(t)^{\sqcup}$ ultrametric measure spaces of diameter at most t and $T_t\mathfrak{u} = \mathfrak{u}(t)$ the classical truncation, $t \geq 0$, $\mathfrak{u} \in \mathbb{U}$.

Definition (t-multiplicativity and t-additivity)

Let $f : S \to \mathbb{R}$ measurable and $t \geq 0$. We say that f is *t-multiplicative on S_t* if

$$f(x_1 \sqcup^t \cdots \sqcup^t x_n) = f(x_1) \cdots f(x_n)\,, \quad n \in \mathbb{N}\,,\, x_1, \ldots, x_n \in S_t\,. \tag{5.2}$$

We say that f is *$\sqcup^t$-additive on S_t* if

$$f(x_1 \sqcup^t \cdots \sqcup^t x_n) = f(x_1) + \cdots + f(x_n)\,, \quad n \in \mathbb{N}\,,\, x_1, \ldots, x_n \in S. \tag{5.3}$$

Definition (Convolution)

Suppose $t \geq 0$ and $n \in \mathbb{N}$. If Q_1, Q_2 are probability measures on $\mathcal{B}(S_t)$, then define the *t-convolution*
$Q_1 *^t Q_2 : \mathcal{B}(S) \to [0,1]$, $A \mapsto (Q_1 *^t Q_2)(A)$ via

$$(Q_1 *^t Q_2)(A) := \int_{\mathbb{S}_t} Q_1(\mathrm{d}x_1) \int_{S_t} Q_2(\mathrm{d}x_2)\, \mathbb{1}(x_1 \sqcup^t x_2 \in A)\,. \qquad (5.4)$$

Certain functions on the semigroup will play an important role.

Definition (Branching property for semigroups)

A family $(P_t)_{t \geq 0}$ of probability kernels $P_t : S \times \mathcal{B}(S) \to [0,1]$ has the *branching property* if

$$P_t(x_1 \sqcup^s x_2, h_t) = (P_t(x_1, \cdot) *^s P_t(x_2, \cdot))(h_t), \quad x_1, x_2 \in S_s\,, \qquad (5.5)$$

for any $s, t \geq 0$ and $h_t \in b\mathcal{B}(S)$ t-multiplicative on S_t.

We need a criterion easy to check to decide whether the solutions of a martingale problem

$$(A, \mathcal{F}, \delta_{\mathfrak{u}}) \tag{6.1}$$

has the branching property.

This replaces the classical argument with values in linear spaces based on the Log-Laplace equation, in the context of genealogical processes.

Theorem (Branching generators)

Let $D_t \subset C_b(S)$ be t-multiplicative functions on S_t, $t \geq 0$, and

$$\tilde{D} \subset \{(x,t) \mapsto \psi(t)h_t(x) : \psi \in C_b^1(\mathbb{R},\mathbb{R}),\ h_t \in D_t,\ (t \mapsto h_t) \in C^1(\mathbb{R}, C_b(S))\} \tag{6.2}$$

on which we have a map $\tilde{A} = A + \partial_t : \tilde{D} \to B(S)$. Assume that for any $(x,0) \in S \times \mathbb{R}$ the following holds.

For any two solutions $(X_t,t)_{t\geq 0}$ and $(X'_t,t)_{t\geq 0}$ of the martingale problem for $(\tilde{A}, \tilde{D}, \delta_{(x,0)})$ one has $T_t X_t \stackrel{d}{=} T_t X'_t$ for every $t > 0$, and a solution $(X_t,t)_{t\geq 0}$ has a stochastically continuous version.

Theorem

Then the family $(P_t)_{t\geq 0}$ *defined via* $P_t(x,f) = \mathbb{E}[f(X_t)|X_0 = x]$, $x \in S, f \in b\mathcal{B}(S), t \geq 0$ *has the branching property if and only if either of the following conditions is satisfied:*

(a) *For* $x_1, x_2 \in S_t$, $\psi h \in \tilde{D}$, $t \geq 0$:

$$\tilde{A}\psi(t)h_t(x_1 \sqcup x_2) = \psi'(t)h_t(x_1 \sqcup x_2) + \psi(t)[\tilde{A}h_t(x_1)h_t(x_2) + h_t(x_1)\tilde{A}h_t(x_2)], \quad (6.3)$$

(b) *For each* $\psi h \in \tilde{D}$ *there exists a function* $g : \mathbb{R}_+ \times S \to \mathbb{R}$ *such that* $g(t,\cdot)$ *is* $\sqcup$*-additive for each* $t \geq 0$ *and, for all* $(t,x) \in \mathbb{R}_+ \times S_t$,

$$\tilde{A}\psi(t)h_t(x) = \psi'(t)h_t(x) + \psi(t)g(t,x)h_t(x)\,. \quad (6.4)$$

Consider the case where $S_t \equiv S, t \geqslant 0$.

Corollary (branching generator, classical case)

Assume S is a Polish space and $D \subset b\mathcal{B}(S)$ is multiplicative on S. Assume that the (A, D)-martingale problem is well-posed and has a stochastically continuous solution $(X_t)_{t \geq 0}$. Then the semigroup associated to $(X_t)_{t \geq 0}$ has the branching property if and only if either of the following conditions is satisfied.

(a) *For all $x_1, x_2 \in S$, $h \in D$:*

$$Ah(x_1 \sqcup x_2) = Ah(x_1)h(x_2) + h(x_1)Ah(x_2), \qquad (6.5)$$

(b) *There exists a $\sqcup$-additive function $g : S \to \mathbb{R}$, i.e. $g(x_1 \sqcup x_2) = g(x_1) + g(x_2)$ for any $x_1, x_2 \in S$ with*

$$Ah(x) = g(x)h(x), \quad x \in S, h \in D. \qquad (6.6)$$

We next want to define the evolving genealogy of a Feller branching process, as the simplest example of the setting we proposed.

This $\mathbb{U}-$valued Markov processes solves the HP for the following operator.

$$\Phi^{n,\phi}([U,r,\mu]) = \int_{U^n} \phi((r(x_i,x_j))_1 \leq i < j \in n)\ \mu(dx_1)\dots\mu(dx_n) \quad (7.1)$$

$$\Omega^{\uparrow}\Phi^{n,\phi}(\mathfrak{u}) = \Omega^{\uparrow,\mathrm{grow}}\Phi^{n,\phi}(\mathfrak{u}) + \Omega^{\uparrow,\mathrm{bran}}\Phi^{n,\phi}(\mathfrak{u}) \quad (7.2)$$

and $\Omega^{\uparrow}\Phi^{n,\phi}(0) = 0$.

The operators are given by

$$\Omega^{\uparrow,\mathrm{grow}}\Phi^{n,\phi}(\mathfrak{u}) = \Phi^{n,2\overline{\nabla}\phi}(\mathfrak{u}), \quad \overline{\nabla}\phi = \sum_{1\le i<j\le n} \frac{\partial\phi}{\partial r_{i,j}}, \tag{7.3}$$

$$\Omega^{\uparrow,\mathrm{bran}}\Phi^{n,\phi}(\mathfrak{u}) = bn\Phi^{n,\phi}(\mathfrak{u}) + \frac{2a}{\bar{\mathfrak{u}}} \sum_{1\le k<l\le n} \Phi^{n,\phi\circ\theta_{k,l}}(\mathfrak{u}), \tag{7.4}$$

where

$$\left(\theta_{k,l}(\underline{\underline{r}})\right)_{i,j} := r_{i,j}\mathbb{1}_{\{i\neq l, j\neq l\}} + r_{k,j}\mathbb{1}_{\{i=l\}} + r_{i,k}\mathbb{1}_{\{j=l\}}, \quad 1 \le i < j\,. \tag{7.5}$$

$$\mathbb{U}(h)^{\sqcup} = \{U \in \mathbb{U} \mid^{\mu^{\otimes 2}} (\{(x,y) \in U^2 \mid r(x,y) \geqslant 2h\}) = 0 \tag{7.6}$$

$$\mathbb{U}(h) = \{U \in \mathbb{U} \mid^{\mu^{\otimes 2}} (\{(x,y) \in U^2 \mid r(x,y) > 2h\}) = 0 \tag{7.7}$$

Then define for $u, v \in \mathbb{U}(h)^{\sqcup}$ the *concatenation*:

$$\mathfrak{u} \sqcup \mathfrak{v} = [U \uplus V, r_U \sqcup^h r_V, \mu + v], \text{ with} \tag{7.8}$$

$$r_U \sqcup^h r_V \mid_{U \times U} = r_U, r_U \sqcup^h r_V \mid_{V \times V} = r_V, \tag{7.9}$$

$$r_U \sqcup r_V(x,y) = 2h, \; x \in U, y \in V. \tag{7.10}$$

The *h-top* $\mathfrak{u}(h)$ of $\mathfrak{u} \in \mathbb{U}$ is defined:

$$\mathfrak{u}(h) = [U, r \wedge 2h, \mu] \in \mathbb{U}(h)^{\sqcup}. \tag{7.11}$$

Then the truncation operation is :

$$T_h(\mathfrak{u}) = \mathfrak{u}(h), \; S_h = \mathbb{U}(h)^{\sqcup}. \tag{7.12}$$

Theorem (Branching property: tree-valued Feller)

The tree-valued Feller diffusion $\mathfrak{U}$ *has the branching property.* □

This result extends to spatial processes for example:

- tree-valued super random walk
- historical process of the above
- ancestral path marked tree-valued super random walk

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