Vertex Degrees in Planar Maps

Michael Drmota*

joint work with Gwendal Collet and Lukas Klausner

Institute of Discrete Mathematics and Geometry

Vienna University of Technology

A 1040 Wien, Austria

michael.drmota@tuwien.ac.at

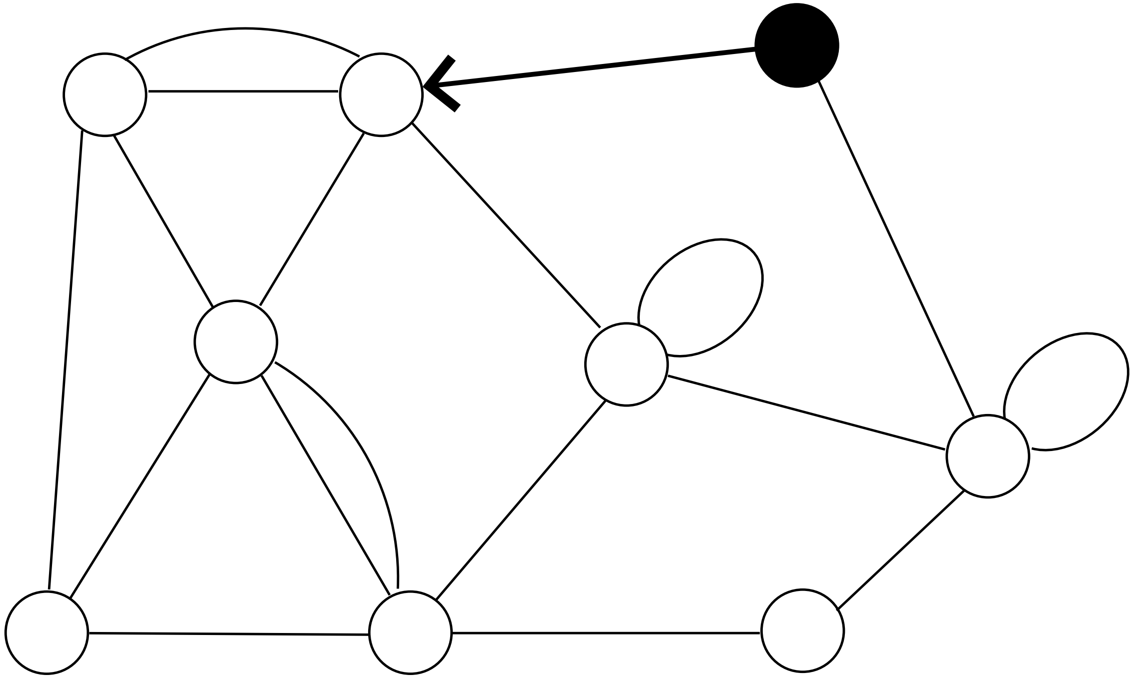
http://www.dmg.tuwien.ac.at/drmota/

* supported by the Austrian Science Foundation FWF, grant F50-02.

Random Trees and Maps: Probabilistic and Combinatorial Aspects

CIRM, Luminy, June 6–10, 2016.

Enumeration of Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

Enumeration of Planar Maps

M_n ... number of rooted maps with n edges [Tutte]

$$\boxed{M_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

Asymptotics:

$$M_n \sim c \cdot n^{-5/2} 12^n$$

Enumeration of Planar Maps

Root degree and degree distribution

$d_{n,k}$ ... probability that the root-vertex has degree k in a map with n edges (= probability that the outer face has valency k in a map with n edges – by duality)

$d_k = \lim_{n\to\infty} d_{n,k}$

$p_{n,k}$ ... probability that a randomly chosen vertex has degree k in a map with n edges (= probability that a randomly chosen face has valency k in a map with n edges)

$p_k = \lim_{n\to\infty} p_{n,k}$

$$\boxed{p_k = \frac{\mu}{k} d_k}$$

The expected number of vertices of degree k is $\sim \mu_k n$,
(where $\mu_k = v p_k$ for some constant $v > 0$).

Vertices of Given Degree

Theorem [D.+Panagiotou 2013]

$X_n^{(k)}$ ... **number of vertices of degree** k in a random planar map with n edges.

Then we have

$$\mathbb{E}[X_n^{(k)}] = \mu_k n + O(1), \quad \mathbb{V}ar[X_n^{(k)}] = \sigma_k^2 n + O(1)$$

and

$$\boxed{\frac{X_n^{(k)} - \mathbb{E}[X_n^{(k)}]}{(\mathbb{V}ar[X_n^{(k)}])^{1/2}} \to N(0,1)}.$$

Proof method. Analytic version of the quadratic method.

Enumeration of Planar Maps

Universal Property

$$M_n \sim c \cdot n^{-5/2} \gamma^n$$

Class of maps	γ
Arbitrary	12
Eulerian	8
3-connected	4
Loopless	256/27
2-connected	27/4
Bipartite	8

[Banderier, Flajolet, Schaeffer, Soria] (for the $n^{-5/2}$-property):
'This generic asymptotic form is "universal" in so far as it is valid for all known "natural families of maps".'

Enumeration of Planar Maps

Theorem 1

Ω ... an arbitrary set of positive integers, not a subset of $\{1, 2\}$

$\mathcal{M}_\Omega$ ... planar rooted maps such that all vertex degrees are in Ω

$M_{\Omega,n}$ ... number of maps in $\mathcal{M}_\Omega$ with n edges

Then there exist positive constants c_Ω and γ_Ω with

$$\boxed{M_{\Omega,n} \sim c_\Omega n^{-5/2} \gamma_\Omega^n, \qquad n \equiv 0 \bmod d},$$

where $d = \gcd\{i : 2i \in \Omega\}$ if Ω contains only even numbers, otherwise $d = 1$.

Remark 1. [Bender+Canfield 1993]: Ω subset of even integers

Remark 2. By duality the same theorem holds if *degree* is replaced by *face valency*.

Enumeration of Planar Maps

Examples.

$\Omega = \{3\}$ ... triangulations or 3-regular maps

$\Omega = \{4\}$ ... quadrangulations or 4-regular maps

$\Omega = 2\mathbb{N}$ ... bipartiite or Eulerian maps

$\Omega = \mathbb{N}$ ... all maps

$\Omega = \mathbb{P} = \{2, 3, 5, 7, \ldots\}$ ... all degrees are prime numbers

...

Vertices of Given Degree

Theorem 2

$X_n^{(k)}$ ... number of vertices of degree k ($\in \Omega$) of maps in $\mathcal{M}_\Omega$.

Then

$$\mathbb{E}[X_n^{(k)}] = \mu_k n + O(1), \quad \mathbb{V}ar[X_n^{(k)}] = \sigma_k^2 n + O(1) \qquad n \equiv 0 \bmod d$$

and the random vector $\boxed{\mathbf{X}_n = (X_n^{(k)})_{k\in\Omega}}$ satisfies a **central limit theorem**:

$$\boxed{\frac{1}{\sqrt{n}}(\mathbf{X}_n - \mathbb{E}(\mathbf{X}_n)) \to \mathbf{Z}}, \qquad n \equiv 0 \bmod d$$

where $\mathbf{Z}$ (in ℓ^2) is a centered Gaussian random variable.

Remark. By duality the same theorem holds if *degree* is replaced by *face valency*.

Vertices of Given Degree

Weak convergence to a Gaussian random variable.

B ... separable Banach space (e.g. ℓ^2)

$\mathbf{Z}_n$, $\mathbf{Z}$ ... B-valued random variables

P_n, P ... corresponding laws on B

Prohorovs theorem. $\mathbf{Z}_n \to \mathbf{Z}$ *weakly* if and only if

1. $\ell^*(\mathbf{Z}_n) \to \ell^*(\mathbf{Z})$ weakly for all $\ell^* \in B^*$

2. $\{P_n : n \in \mathbb{N}\}$ is tight.

$\mathbf{Z}$ is *Gaussian* if $\ell^*(\mathbf{Z})$ is Gaussian for all $\ell^* \in B^*$.

Mobiles

Definition.

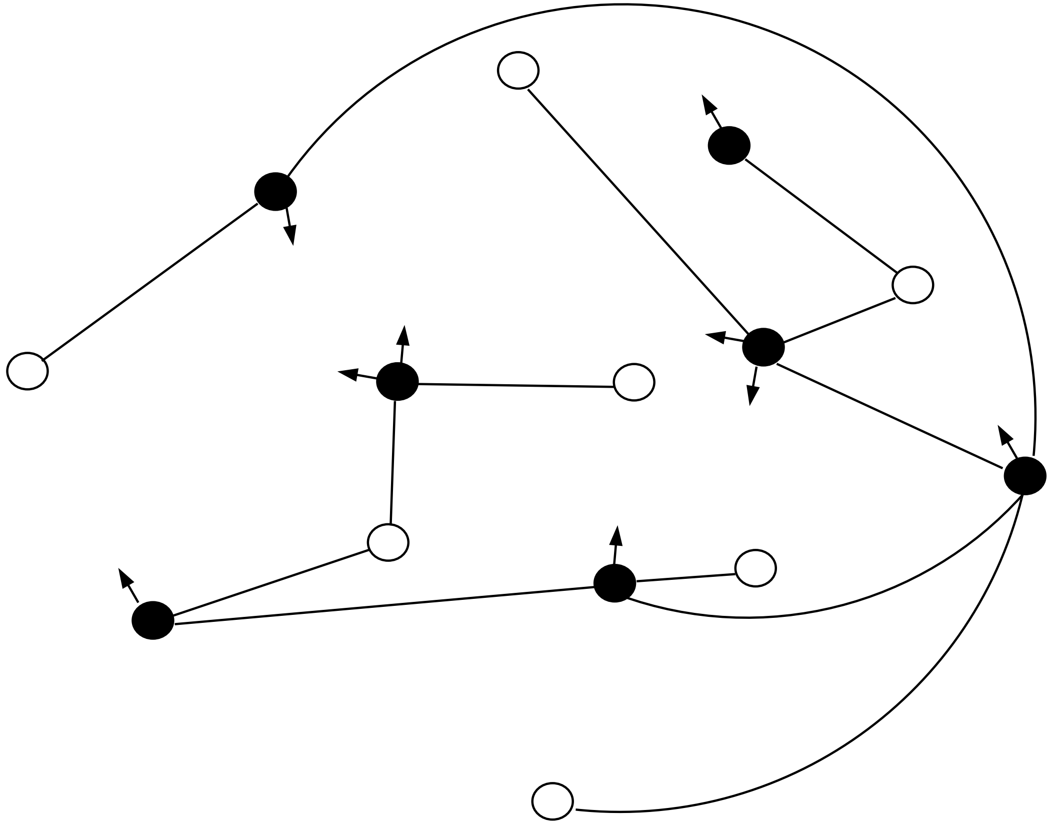
A **mobile** is a **planar tree** – that is, a map with a single face – such that there are **two kinds of vertices** (black and white) with **no white-white edges**, and black vertices additionally have so-called **“legs”** attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.

A **bipartite mobile** is a mobile without black–black edges.

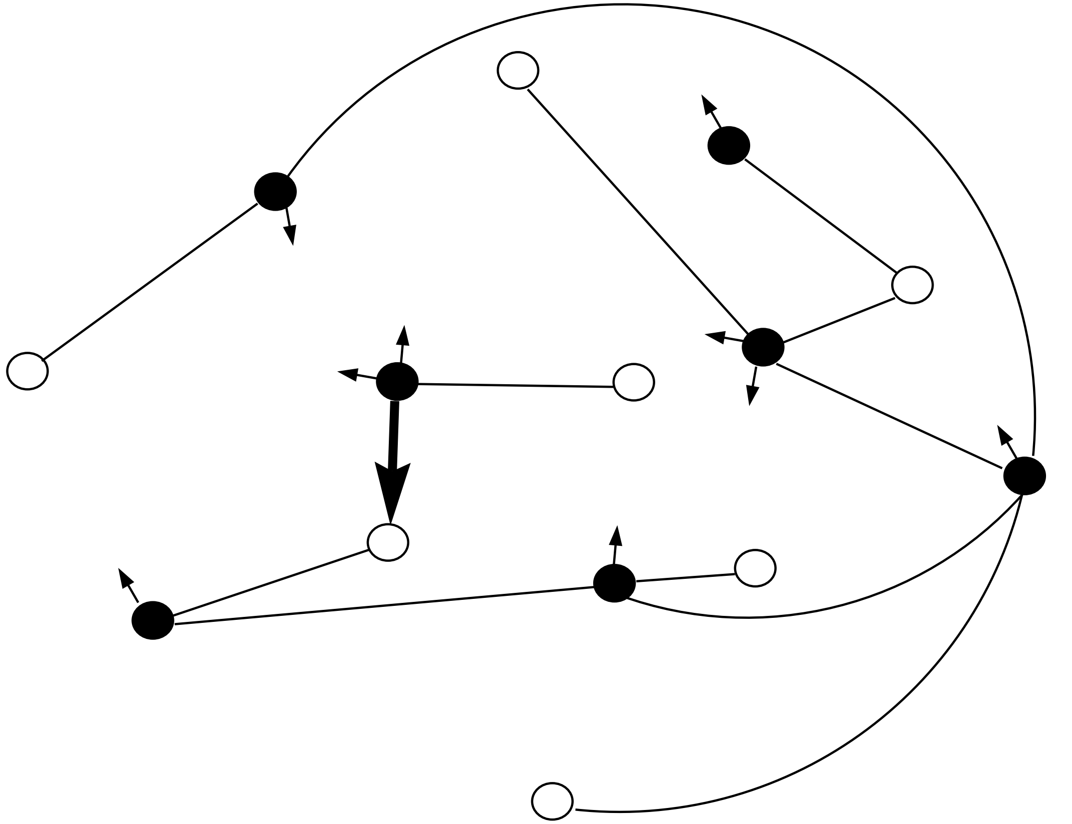
The **degree** of a black vertex is the number of half-edges plus the number of legs that are attached to it.

A mobile is called **rooted** if an edge is distinguished and oriented.

Mobiles



Mobiles



Mobiles

Theorem [Cori+Vauquelin, Schaeffer, Bouttier+Di Francesco+Guitter, Bernardi+Fusy, Collet+Fusy, ...]

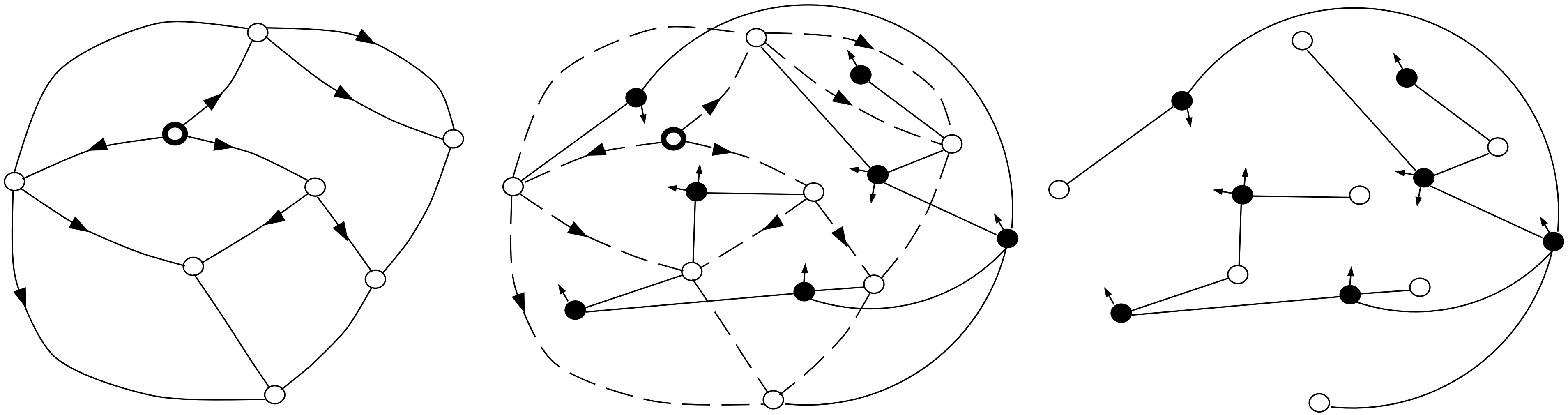
There is a **bijection** between **mobiles** that contain at least one black vertex and **pointed planar maps**, where *white vertices* in the mobile correspond to *non-pointed vertices* in the equivalent planar map, *black vertices* correspond to *faces* of the map, and the *degrees of the black vertices* correspond to the *face valencies*.

This bijection induces a *bijection on the edge sets* so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)

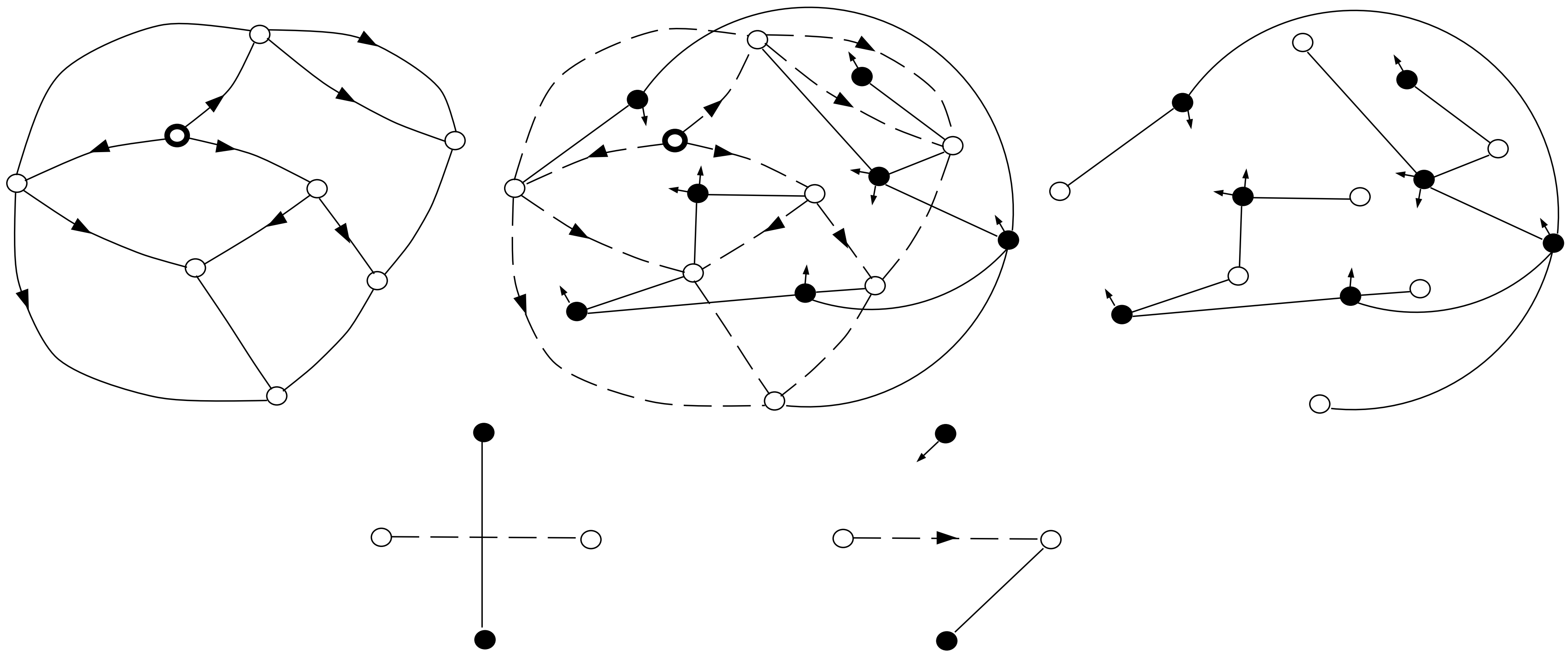
Similarly, *rooted mobiles* that contain at least one black vertex are in bijection to *rooted and vertex-pointed planar maps*.

Finally, *bipartite mobiles* with at least two vertices correspond to *bipartite maps* with at least two vertices, in the unrooted as well as in the rooted case.

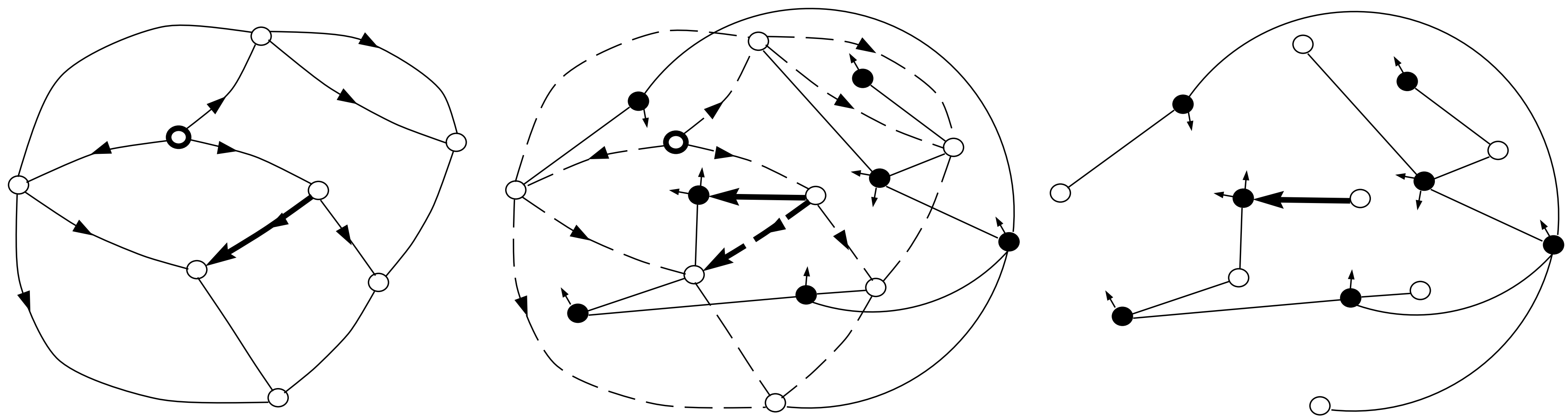
Mobiles



Mobiles



Mobiles



Bipartite Mobiles and Maps

- $R(t, z, x_2, x_4, \ldots)$ ... mobiles rooted at a white vertex and where an additional edge is attached to the root vertex.

Lemma

$R = R(t, z, x_2, x_4, \ldots)$ satisfies the equation

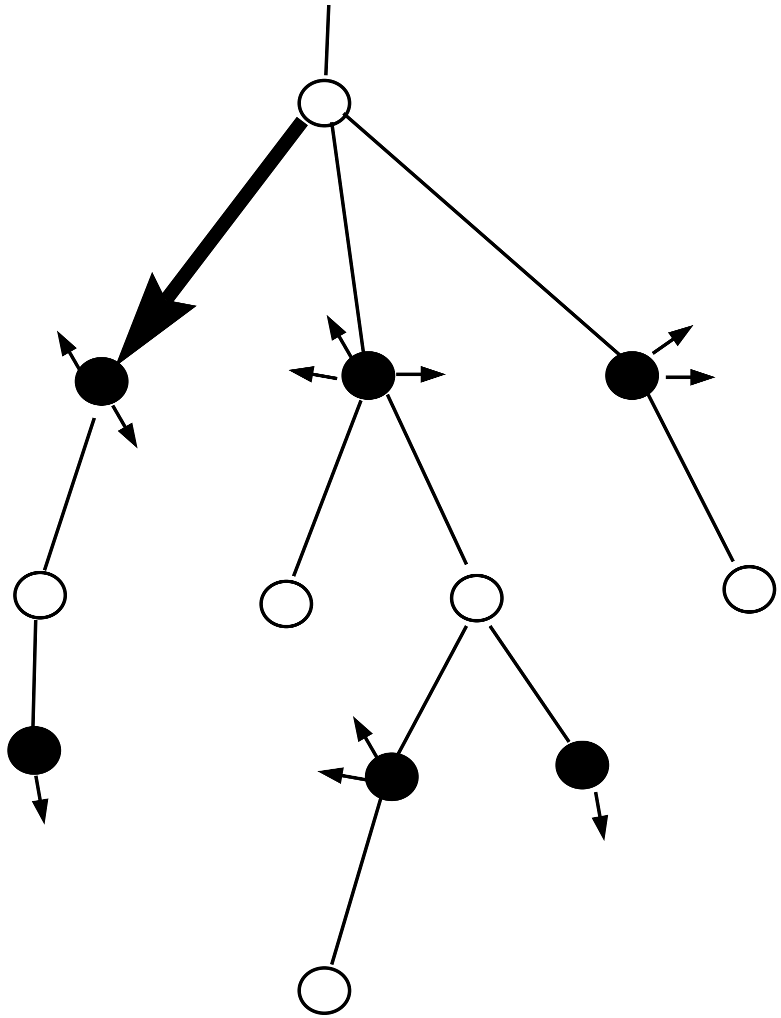
$$\boxed{R = tz + z \sum_{i \geq 1} x_{2i} \binom{2i-1}{i} R^i}.$$

Furthermore the generating function $M = M(t, z, x_2, x_4, \ldots)$ of **bipartite rooted maps** satisfies

$$\boxed{\frac{\partial M}{\partial t} = 2\,(R/z - t) = \sum_{i \geq 1} x_{2i} \binom{2i-1}{i} R^i},$$

where the variable t corresponds to the number of vertices, z to the number of edges, and x_{2i}, $i \geq 1$, to the number of faces of valency $2i$.

Bipartite Mobiles and Maps



Bipartite Mobiles and Maps

Proof

Splitting around the root edge gives

$$R = \frac{tz}{1 - z\sum_{i\geq 1} x_{2i}\binom{2i-1}{i} R^{i-1}}$$

Hence, the generating function of rooted mobiles that are rooted by a white vertex is and where we discount the mobile that consists just of one (white) vertex is given by

$$R/z - t = \sum_{i\geq 1} x_{2i}\binom{2i-1}{i} R^{i}.$$

The right hand side is precisely the generating function of rooted mobiles that are rooted at a black vertex.

Thus, by the above bijection

$$\boxed{\frac{\partial M}{\partial t} = 2\,(R/z - t)}.$$

Mobiles and Maps

- $L(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a black vertex and where an additional edge is attached to the black vertex.

- $Q(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a univalent white vertex, which is not counted,

- $R(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a white vertex and where an additional edge is attached to the root vertex.

$$B_{\ell,m} = \binom{l+2m}{l,m,m}$$

$$B_{\ell,m}^{(+1)} = \binom{l+2m+1}{l,m,m+1}$$

$$\overline{B}_{\ell,m} = \frac{l+m}{l+2m}\binom{l+2m}{l,m,m}$$

Mobiles and Maps

Lemma

The generating functions $L = L(t, z, x_1, x_2, \ldots)$, $Q = Q(t, z, x_1, x_2, \ldots)$, and $R = R(t, z, x_1, x_2, \ldots)$ satisfy the system of equations

$$\boxed{\begin{array}{rcl} L & = & z \sum_{\ell,m} x_{2m+\ell+1} B_{\ell,m} L^\ell R^m, \\ Q & = & z \sum_{\ell,m} x_{\ell+2m+2} B^{(+1)}_{\ell,m} L^\ell R^m, \\ R & = & \frac{tz}{1-Q}. \end{array}}$$

Let $T = T(t, z, x_1, x_2, \ldots)$ be given by

$$T = 1 + \sum_{\ell,m} x_{2m+\ell} \overline{B}_{\ell,m} L^\ell R^m,$$

Then the generating function $M = M(t, z, x_1, x_2, \ldots)$ of **rooted maps** satisfies

$$\boxed{\frac{\partial M}{\partial t} = R/z - t + T},$$

where the variable t corresponds to the number of vertices, z to the number of edges, and x_i, $i \geq 1$, to the number of faces of valency i.

Ω-Mobiles and Maps

Set $\boxed{x_i = 1}$ for $\boxed{i \in \Omega}$ and $\boxed{x_i = 0}$ for $\boxed{i \notin \Omega}$.

Then $M = M(t, z, x_1, x_2, \ldots)$ is the corresponding generating function of Ω-maps (that is, all face valencies are contained in Ω).

Similarly we deal with Ω-mobiles by choosind x_i in this way for L, Q, and R.

In particular in the **bipartite case** we have to discuss the equation

$$R = \frac{tz}{1 - z \sum_{2i \in \Omega} \binom{2i-1}{i} R^{i-1}}$$

or equivalently the equation

$$\boxed{R = tz + z \sum_{2i \in \Omega} \binom{2i-1}{i} R^i}.$$

Squareroot Singularity

(Folkore) Lemma [Bender, Canfield, Meir+Moon, ...]

Suppose that where $F(z,y)$ has a power series expansion at $(0,0)$ with **non-negative coefficients** and $F(0,0)=0$, $F_{yy}(z,y)\neq 0$, $F_z(z,y)\neq 0$ and suppose that a power series $y(z)=\sum y_n z^n$ satisfies the functional equation

$$\boxed{y(z) = F(z, y(z))}$$

Let $y_0 > 0$ and $\rho > 0$ be defined by

$$y_0 = F(\rho, y_0), \quad 1 = F_y(\rho, y_0)$$

such that (ρ, y_0) is **inside the region of convergence of** F.

Then there exists analytic function $g(z), h(z) \neq 0$ such that locally

$$\boxed{y(z) = g(z) - h(z)\sqrt{1 - \frac{z}{\rho}}.}$$

Squareroot Singularity

Proof of Theoram 1

$R = R_\Omega$ satisfies an equation of the form

$$R = tz + z \sum_{2i \in \Omega} \binom{2i-1}{i} R^i = F(t,z,R).$$

We have to show that there exist solutions $\rho > 0$, $R_0 > 0$ of the system of equations

$$R_0 = F(1,\rho,R_0), \qquad 1 = F_R(1,\rho,R_0),$$

which are **inside the range of convergence of** F:

$$R_0 = \rho + \rho \sum_{2i \in \Omega} \binom{2i-1}{i} R_0^i, \quad 1 = \rho \sum_{2i \in \Omega} i \binom{2i-1}{i} R_0^{i-1},$$

Squareroot Singularity

Proof of Theoram 1 (continued)

After eliminating ρ we have to solve the equation

$$\sum_{2i\in\Omega} i\binom{2i-1}{i}R_0^i = 1 + \sum_{2i\in\Omega}\binom{2i-1}{i}R_0^i$$

which we can rewrite to

$$\boxed{\sum_{2i\in\Omega}(i-1)\binom{2i-1}{i}R_0^i = 1}$$

If Ω is finite there is a unique $R_0 > 0$ (and all functions are polynomials).

If Ω is infinite we show that there is a (unique) positive solution $R_0 < 1/4$.

Squareroot Singularity

Proof of Theoram 1 (continued)

From

$$(i-1)\binom{2i-1}{i} \sim \frac{4^i\sqrt{i}}{2\sqrt{\pi}}.$$

it follows that the the power series

$$x \mapsto H(x) = \sum_{2i\in\Omega} (i-1)\binom{2i-1}{i} x^i$$

has radius of convergence 1/4 and we also have

$$H(x) \to \infty \qquad \text{as } x \to 1/4-.$$

Hence there is $R_0 < 1/4$ with

$$H(R_0) = \sum_{2i\in\Omega} (i-1)\binom{2i-1}{i} R_0^i = 1$$

Finally, we set $\rho = \left(\sum_{2i\in\Omega} i\binom{2i-1}{i} R_0^{i-1}\right)^{-1}.$

Squareroot Singularity

Proof of Theoram 1 (continued)

We have

$$\boxed{R(1,z) = g(z) - h(z)\sqrt{1 - \frac{z}{\rho}}}$$

and similary we get

$$\boxed{R(t,z) = g(t,z) - h(t,z)\sqrt{1 - \frac{z}{\rho(t)}}}$$

for proper analystic function $g(t,z)$, $h(t,z)$, $\rho(t)$.

Squareroot Singularity

Proof of Theoram 1 (continued)

Since $\frac{\partial M}{\partial t} = 2\,(R/z - t)$ this implies a local representaton of $M(t,z)$ of the form

$$\boxed{M(t,z) = \overline{g}(t,z) + \overline{h}(t,z)\left(1 - \frac{z}{\rho(t)}\right)^{3/2}}$$

and consequently (by a proper **transfer theorem**)

$$\boxed{M_{\Omega,n} = [z^n]M(1,z) \sim c_\Omega\, n^{-5/2}\, \gamma_\Omega^n}$$

where $\gamma_\Omega = 1/\rho(1)$.

Central Limit Theorem

$\mathbf{X}_n = (X_n^{(k)})_{k\in\Omega}$

$$\Longrightarrow \quad \ell^*(\mathbf{X}_n) = \sum_{k\in\Omega} \ell_k X_n^{(k)}$$

with $\ell^* = (\ell_k) \in B^*$.

$$\Longrightarrow \quad e^{is\ell^*(\mathbf{X}_n)} = \prod_{k\in\Omega} e^{is\ell_k X_n^{(k)}}$$

and consequently

$$\boxed{\sum_{n\geq 0} M_{\Omega,n}\mathbb{E}[e^{is\ell^*(\mathbf{X}_n)}]\, z^n = M\left(1, z, (e^{is\ell_k})_{k\in\Omega}\right)}$$

Squareroot Singularity

Proof of Theorem 2: weak convergence

For every fixed $\ell^* = (\ell_k) \in B^*$ we get (similarly to the above)

$$M\left(1, z, (e^{is\ell_k})_{k\in\Omega}\right) = \tilde{g}(s,z) + \tilde{h}(t,z)\left(1 - \frac{z}{\tilde{\rho}(s)}\right)^{3/2}$$

which implies

$$M_{\Omega,n}\mathbb{E}[e^{is\ell^*(\mathbf{X}_n)}] \sim c(s)\, n^{-5/2}\, \tilde{\rho}(s)^{-n}$$

and consequently

$$\boxed{\mathbb{E}[e^{is\ell^*(\mathbf{X}_n)}] \sim \frac{c(s)}{c(1)}\left(\frac{\tilde{\rho}(1)}{\tilde{\rho}(s)}\right)^n}$$

which proves a **central limit theorem**.

Squareroot Singularity

Theorem[D.+Gittenberger+Morgenbesser] (**tightness**)

Suppose that $y(z,(x_k)_{k\ge 0})$ is the unique solution of a single functional equation

$$\boxed{y = F(z,y,(x_k)_{k\ge 0})},$$

where $F : B\times U\times V \to \mathbb{C}$ is a positive analytic function on $B\times U\times V \subseteq \mathbb{C}^2\times \ell^2$ such that there exist positive real $(z_0,y_0)\in B\times U$ with $y_0 = F(z_0,y_0,\mathbf{1})$ and $1 = F_y(z_0,y_0,\mathbf{1})$ such that $F_z(z_0,y_0,\mathbf{1}) \ne 0$ and $F_{yy}(z_0,y_0,\mathbf{1}) \ne 0$. Furthermore assume that the corresponding random variables $X_n^{(k)}$ have the property that $\boxed{X_n^{(k)} = 0}$ if $\boxed{k > cn}$ for some constant $c>0$ and that the following conditions are satisfied:

$$\sum_{k\ge 0} F_{x_k} < \infty, \quad \sum_{k\ge 0} F_{yx_k}^2 < \infty, \quad \sum_{k\ge 0} F_{x_k x_k} < \infty,$$

$$F_{zx_k} = o(1), \quad F_{zx_kx_k} = o(1), \quad F_{yyx_k} = o(1), \quad F_{yyx_kx_k} = o(1),$$

$$F_{zzx_k} = O(1), \quad F_{zyx_k} = O(1), \quad F_{zyyx_k} = O(1), \quad F_{yyyx_k} = O(1), \quad (k\to\infty)$$

where all derivatives are evaluated at $(z_0,y_0,\mathbf{1})$.

Then the the set of laws of $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$, $n \ge 1$, where $\mathbf{X}_n = (X_n^{(k)})_{k\ge 0}$ is **tight** and has a **Gaussian limit**.

Extension to Genus-g-Maps

Asymptotic enumeration of maps embedded on (orientable) surfaces of genus $g \ge 0$:

[Bender+Canfield 1986]

$$\boxed{M_n^{(g)} \sim t_g \, n^{5(g-1)/2} \, 12^n},$$

where $M_n^{(g)}$ denotes the number of maps with n edges on a surface of genus g.

It seems that the kind of asymptotic behavior, in particular the exponent $5(g-1)/2$ is (again) **universal**.

Extension to Genus-g-Maps

Definition

Given $g \geq 0$, a g-**mobile** is a one-faced map of genus g – embedded on the g-torus – such that there are **two kinds of vertices (black and white)** with **no white-white edges**, and black vertices additionally have so-called **"legs"** attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.

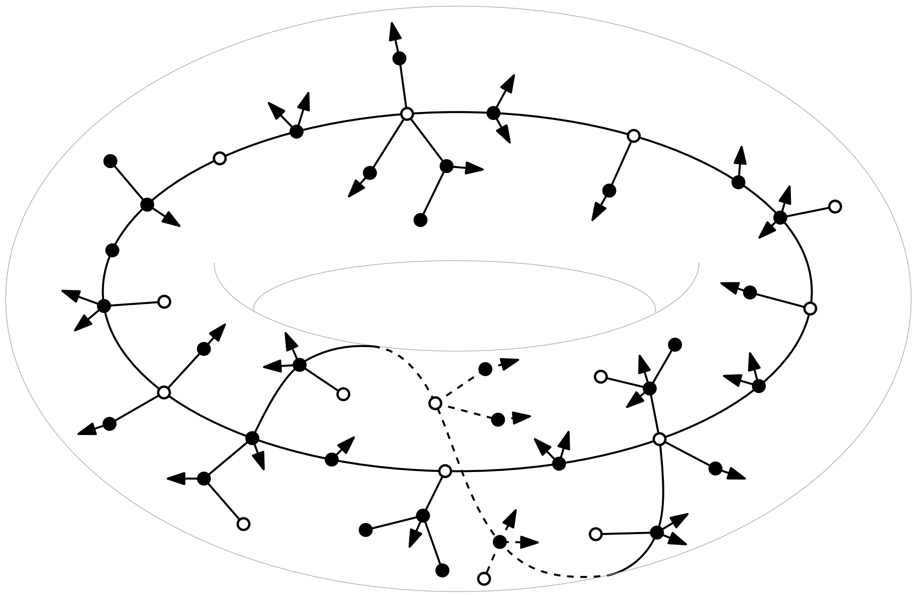
Furthermore, for each cycle c of the g-mobile, let $n_\circ$, $n_\rightarrow$ and $n_{\multimap}$ respectively be the numbers of white vertices on c, of legs dangling to the left of c and of white neighbours to the left of c. One has the following constraint: $\boxed{n_\rightarrow = n_\circ + n_{\multimap}}$

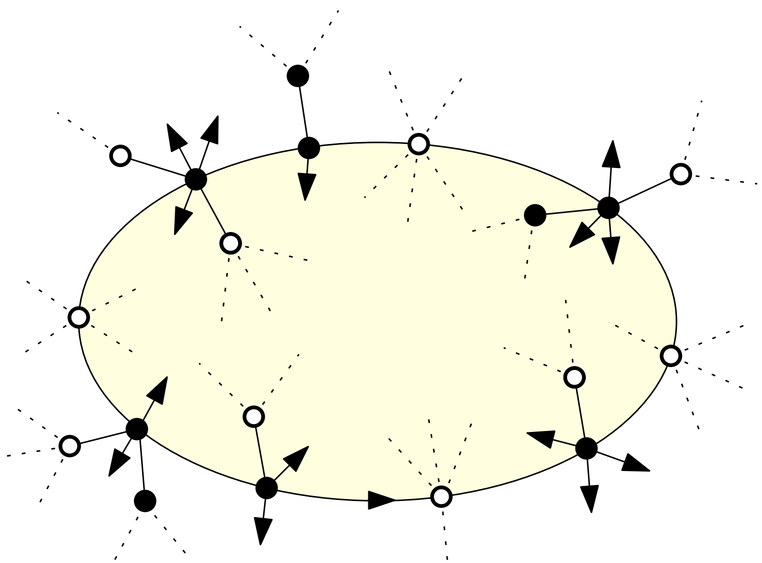
The *degree* of a black vertex is the number of half-edges plus the number of legs that are attached to it.

A *bipartite* g-mobile is a g-mobile without black–black edges.

A g-mobile is called *rooted* if an edge is distinguished and oriented.

Extension to Genus-g-Maps





$n_{\rightarrow} = 7$

$n_{\circ} = 4$

$n_{\multimap} = 3$

$n_{\rightarrow} - n_{\circ} - n_{\multimap} = 0$

Extension to Genus-g-Maps

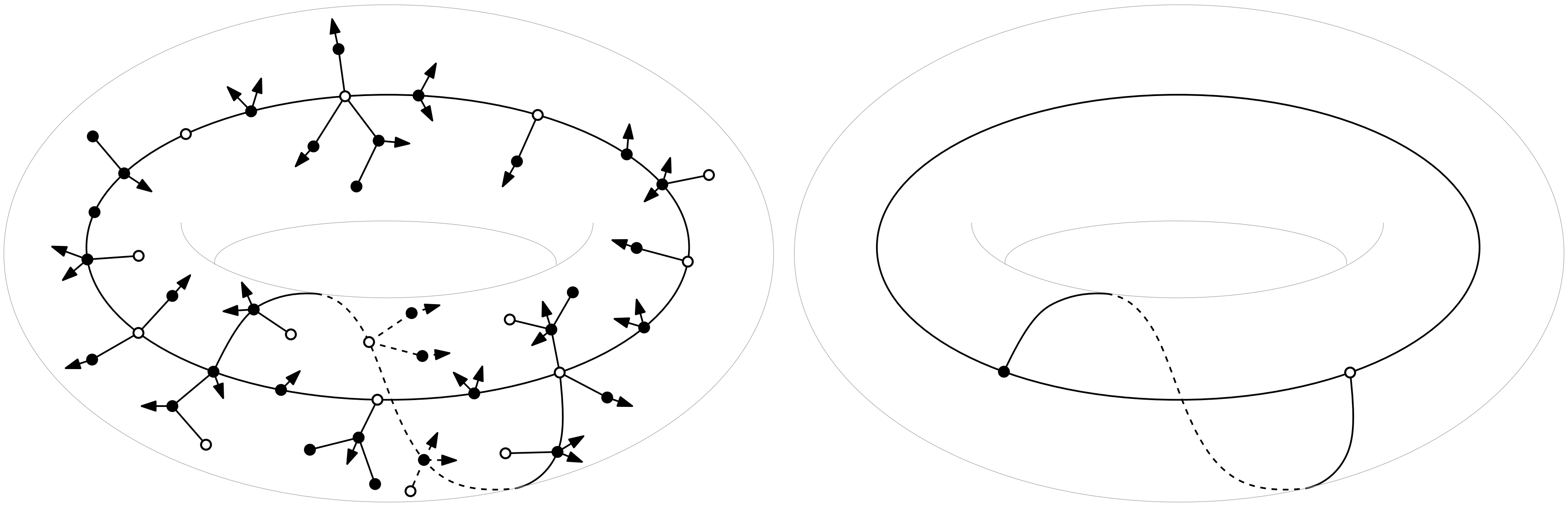
Theorem [Extension of Chapuy and Chapuy+Marcus+Schaeffer]

Given $g \geq 0$, there is a **bijection** between g-**mobiles** that contain at least one black vertex and **pointed maps of genus** g, where *white vertices* in the mobile correspond to *non-pointed vertices* in the equivalent map, *black vertices* correspond to *faces* of the map, and the *degrees of the black vertices* correspond to the *face valencies*. This bijection induces a bijection on the edge sets so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)

Similarly, rooted g-mobiles that contain at least one black vertex are in bijection to rooted and vertex-pointed maps of genus g.

Extension to Genus-g-Maps

The g-**scheme** of a g-mobile is what remains when we apply the following operations: first remove all legs, then remove iteratively all vertices of degree 1 and finally replace any maximal path of degree-2-vertices by a single edge.



Bipartite Genus-g-Maps

A doubly-rooted bipartite planar mobile can be decomposed along a sequence of elementary cells forming the path between its two roots.

Definition

An **elementary cell** is a half-edge connected to a black vertex itself connected to a white vertex with a dangling half-edge. The white vertex has a sequence of black-rooted mobiles attached on each side. The black vertex has $j \geq 0$ legs and $k \geq 0$ white-rooted mobiles on its left, $l \geq 0$ and $k+l-j+2$ legs on its right, and its degree is $2(k+l+1)$. The *increment* of the cell is then $k-j-1$.

Bipartite Genus-g-Maps

Generating functions

The generating series $P := P(t, z, (x_{2i}), s)$ of a cell, where s marks the increment, is:

$$\begin{aligned} P(t, z, (x_{2i}), s) &= \frac{z^2 R^2}{t} \sum_{j,k,l \geq 0} \binom{j+k}{j} \binom{k+2l-j+2}{l} s^{j-k-1} x_{2(k+l+2)} R^{k+l} \\ &= \frac{z^2 R^2}{st} \widehat{P}, \end{aligned}$$

where R is the generating function of 0-mobiles rooted at a white vertex (plus an extra edge).

The generating series $S := S(t, z, (x_{2i}), s)$ of a doubly-rooted mobile depends on the color of its roots (u, v) is given by:

$$S_{(u,v)}(t, z, (x_{2i}), s) = \begin{cases} \frac{1}{1-P} & \text{if } (u, v) = (\circ, \bullet) \text{ or } (\bullet, \circ), \\ \frac{z\widehat{P}}{1-P} & \text{if } (u, v) = (\circ, \circ), \\ \frac{zR^2}{st(1-P)} & \text{if } (u, v) = (\bullet, \bullet). \end{cases}$$

Bipartite Genus-g-Maps

Lemma

The generating series $R_S := R_S(t,z,(x_{2i}))$ of rooted bipartite g-mobiles with scheme S:

$$R_S(t,z,(x_{2i})) = 2\frac{z\partial}{\partial z}\frac{1}{2|E|} z^{|E|} t^{|V_\circ|} \left(\frac{R}{tz}\right)^{|C_\circ|}$$
$$\times \sum_{(l_c)\text{ labelling}} \Bigg[\prod_{e\in E} [s^{incr(e)}] S_{(e_-,e_+)} \prod_{v\in V_\bullet}$$
$$\sum_{i_1,\ldots,i_{\deg(v)}\geq 0} \left(\prod_{k=1}^{\deg(v)} \binom{2i_k + l_{c_{k+1}} - l_{c_k} + 1}{i_k} \right) x_{2(\deg(v)+\sum i_k)} \Bigg].$$

Consequently then generating series $M^{(g)} := M^{(g)}(t,z,(x_{2i}))$ for the family of rooted bipartite maps of genus g satisfies

$$\frac{\partial M^{(g)}}{\partial t} = \frac{2}{z} \sum_{\substack{S \text{ scheme} \\ \text{of genus } g}} R_S(t,z,(x_{2i})).$$

Extension to Genus-g-Maps

Theorem

Ω ... an arbitrary set of positive **even** integers,

$\mathcal{M}_\Omega^{(g)}$ ... rooted g-maps such that all vertex degrees are in Ω

$M_{\Omega,n}^{(g)}$ ... number of g-maps in $\mathcal{M}_\Omega$ with n edges

Then there exist positive constants $c_\Omega^{(g)}$ and $\gamma_\Omega^{(g)}$ with

$$\boxed{M_{\Omega,n}^{(g)} \sim c_\Omega^{(g)}\, n^{-5/2} (\gamma_\Omega^{(g)})^n, \qquad n \equiv 0 \bmod d},$$

where $d = \gcd\{i : 2i \in \Omega\}$.

Furthermore let $X_n^{(k)}$ denote number of vertices of degree k ($\in \Omega$) of maps in $\mathcal{M}_\Omega^{(g)}$. Then

$$\mathbb{E}[X_n^{(k)}] = \mu_k n + O(1), \quad \mathbb{V}ar[X_n^{(k)}] = \sigma_k^2 n + O(1) \qquad n \equiv 0 \bmod d$$

and the random vector $\boxed{\mathbf{X}_n = (X_n^{(k)})_{k\in\Omega}}$ satisfies a **central limit theorem**:

$$\boxed{\frac{1}{\sqrt{n}}\left(\mathbf{X}_n - \mathbb{E}(\mathbf{X}_n)\right) \to \mathbf{Z}}, \qquad n \equiv 0 \text{ mod } d$$

where $\mathbf{Z}$ (in ℓ^2) is a centered Gaussian random variable.

Thank You!