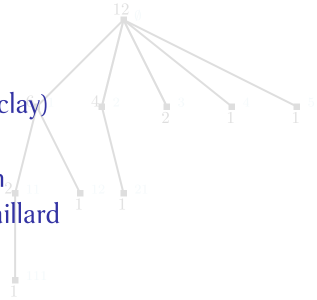
$O(n)$ model on random quadrangulations: the cascade of loop perimeters



Linxiao Chen

(Université Paris-Sud, CEA Saclay)

based on joint work with

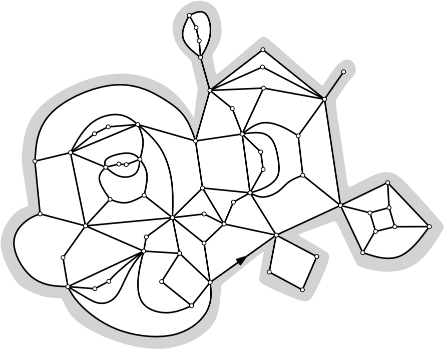
Nicolas Curien and Pascal Maillard

Random Trees and Maps conference, 9 June 2016, CIRM

Model and results

Definitions

A *bipartite map with a boundary* is a rooted bipartite map in which face on the right of the root edge is called the *external face*, and the other faces called *internal faces*.
A *quadrangulation with a boundary* is a bipartite map with a boundary whose internal faces are all quadrangles.



Remark

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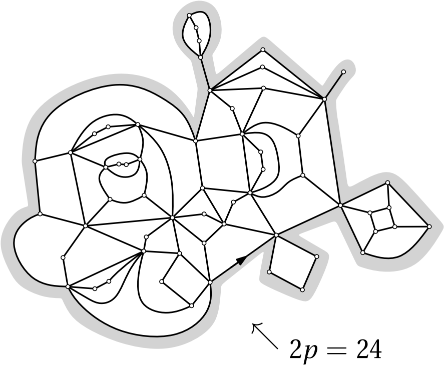

Remark

The boundary is not necessarily simple.

We denote by $2p$ the *perimeter* of the map (i.e. degree of the external face).

Enumeration results

Generating function of *pointed quadrangulations* with a boundary of length 2, with a weight g per face

$$Q_1^\bullet(g) := 1 + \sum_{\mathfrak{q} \in \mathcal{Q}^\bullet} g^{f(\mathfrak{q})}$$

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Generating function of *pointed bipartite maps* with a boundary of length $2p$, with a weight g_k per face of degree $2k$

$$B_p^\bullet(g_1, g_2, \dots) := \sum_{\mathfrak{m}\in\mathcal{B}_p^\bullet} \prod_{k=1}^{\infty} g_k^{f_k(\mathfrak{m})}$$

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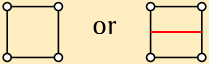
where $\quad x = 1 + \phi_{\mathbf{g}}(x) \quad$ with

$$\phi_{\mathbf{g}}(x) = \sum_{k=1}^{\infty} \frac{1}{2}\binom{2k}{k} g_k x^k$$

$O(n)$ model on quadrangulations

A *loop configuration* on a quadrangulation with boundary $\mathfrak{q}$ is a collection of *disjoint simple closed paths* on the dual of $\mathfrak{q}$ which do not visit the external face. We restrict ourselves to the so-called *rigid* loops, i.e. such that every internal face is of type



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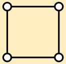
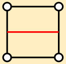
 or 

$$\mathcal{O}_p = \left\{ (\mathfrak{q}, \boldsymbol{\ell}) \middle| \begin{array}{c} \mathfrak{q} \text{ is a quadrangulation with a boundary of length } 2p, \\ \boldsymbol{\ell} \text{ is a rigid loop configuration on } \mathfrak{q}. \end{array} \right\}$$

For $n \in (0,2)$ and $g, h > 0$, let

$$F_p(n; g, h) = \sum_{(\mathfrak{q},\boldsymbol{\ell}) \in \mathcal{O}_p} g^{\#\square} h^{\#\boxminus} n^{\#\bigcirc}$$

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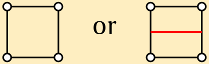

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A triple $(n; g, h)$ is *admissible* if $F_p(n; g, h) < \infty$. (This is independent of p).

$O(n)$ model on quadrangulations

Definition

Fix $p > 0$. For each admissible triple $(n; g, h)$, we define a probability distribution on $\mathcal{O}_p$ by

$$\mathbb{P}^{(p)}_{n;g,h}((\mathfrak{q}, \boldsymbol{\ell})) = \frac{g^{\#\square} h^{\#\boxminus} n^{\#\circlearrowleft}}{F_p(n; g, h)}$$

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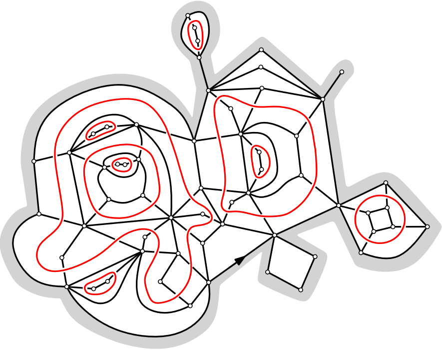

$$\rightsquigarrow \quad \mathbb{P}^{(12)}_{n;g,h}(\,\cdot\,) = \frac{g^{8}\, h^{38}\, n^{9}}{F_{12}(n; g, h)}$$

Theorem (Borot, Bouttier, Guitter '12)

For all admissible $(n; g, h)$, there exist $\bar{\kappa}(n; g, h)$ and $\alpha(n; g, h)$ such that

$$F_p(n; g, h) \underset{p \to \infty}{\sim} C \, \bar{\kappa}^{-p} \, p^{-\alpha - 1/2}$$

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For each $n(0, 2)$, there are *four* possible values of α

subcritical: $\alpha = 1$ generic critical: $\alpha = 2$

non-generic critical

dense phase: $$\alpha = \frac{3}{2} - \frac{1}{\pi}\arccos(n/2) \quad \in (1, 3/2)$$

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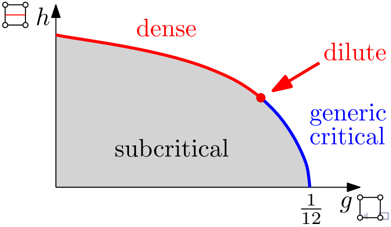
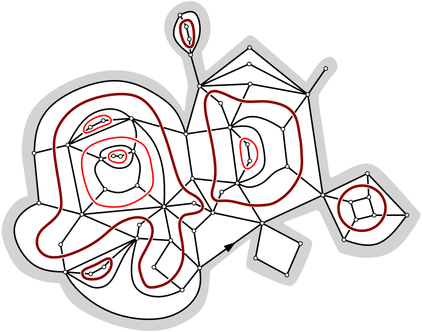
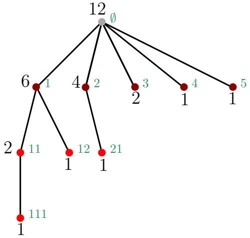


The perimeter cascade of loops

We focus on the *hierarchical structure* of the loops, which we represent by a tree labeled by the *half-perimeters* of the loops.





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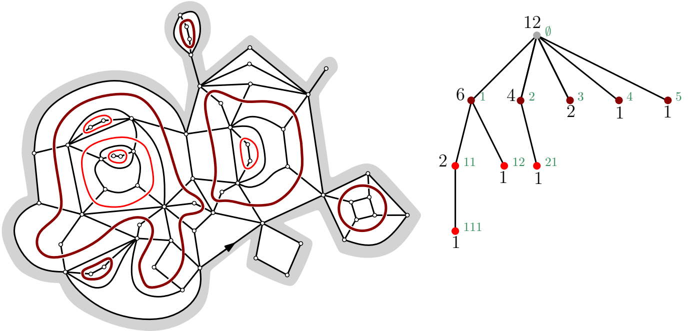


We complete the tree by vertices of label 0. This gives a random process $(\chi_u^{(p)})_{u\in\mathcal{U}}$ labeled by the *Ulam tree* $\mathcal{U} = \bigcup_{n\geq 0}(\mathbb{N}^*)^n$. We call this process the *(half-)perimeter cascade* of the rigid $O(n)$ model on quadrangulations.

Main results

Theorem (C., Curien, Maillard 2016+)

Let $(\chi_u^{(p)})_{u\in\mathcal{U}}$ be the previously defined perimeter cascade. Then, we have the following convergence in distribution in $\ell^\infty(\mathcal{U})$:

$$\left(p^{-1}\chi_u^{(p)}\right)_{u\in\mathcal{U}} \overset{p\to\infty}{\Longrightarrow} (Z_u^\alpha)_{u\in\mathcal{U}},$$

where $Z^\alpha = (Z_u^\alpha)_{u\in\mathcal{U}}$ is a multiplicative cascade to be defined later.

Multiplicative cascades

Multiplicative cascades

Definition

A *multiplicative cascade* is a random process $Z = (Z(u))_{u \in \mathcal{U}}$ such that

$$Z_\emptyset = 1, \quad \forall u \in \mathcal{U},\ i \geq 1 : Z_{ui} = Z_u \cdot \xi_u(i),$$

where $(\boldsymbol{\xi}_u)_{u \in \mathcal{U}} = (\xi_u(i), i \geq 1)_{u \in \mathcal{U}}$ is an i.i.d. family of random vectors in $(\mathbb{R}_+)^{\mathbb{N}^*}$. The law of $\boldsymbol{\xi} = \boldsymbol{\xi}_\emptyset$ is the *offspring distribution* of the cascade Z.

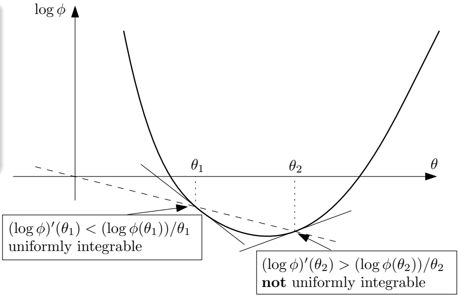
Remark: $X = \log Z = (\log Z_u)_{u \in \mathcal{U}}$ is a *branching random walk*.

Mellin transform and martigales of multiplicative cascades

Definition (Mellin transform)

$$\phi(\theta) := \mathbb{E}\left[\sum_{i\in\mathbb{N}^*} \xi(i)^\theta\right] \in (-\infty, +\infty]$$

- $\log\phi$ is convex
- $W_n^{(\theta)} := \phi(\theta)^{-n}\sum_{|u|=n}(Z_u)^\theta$ is a martingale.



Theorem (Biggins, Lyons)

$W_n^{(\theta)}$ is uniformly integrable if and only if
$\mathbb{E}[W_1^{(\theta)}\log^+ W_1^{(\theta)}] < \infty$ and $(\log\phi)'(\theta) < (\log\phi(\theta))/\theta$

The multiplicative cascade Z^α

- $(\zeta_t)_{t\geq 0}$: α-stable Lévy process without negative jumps, started from 0.
- τ: the hitting time of -1 by ζ.
- $(\Delta\zeta)^{\downarrow}_{\tau}$: the jumps of ζ before τ, sorted in $\downarrow$ order.
- $\mathrm{d}\,\nu_\alpha := \frac{1/\tau}{\mathbb{E}[1/\tau]}\mathrm{d}\,\widetilde{\nu}_\alpha$, where $\tilde{\nu}_\alpha$ is the law of $(\Delta\zeta)^{\downarrow}_{\tau}$

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Theorem (C., Curien, Maillard 2016+)

Let $(\chi^{(p)}(u))_{u\in\mathcal{U}}$ be the perimeter cascade of the rigid $O(n)$ model on quadrangulations. Then we have the convergence in distribution in $\ell^\infty(\mathcal{U})$:

$$\left(p^{-1}\chi_u^{(p)}\right)_{u\in\mathcal{U}} \overset{p\to\infty}{\Longrightarrow} (Z_u^\alpha)_{u\in\mathcal{U}},$$

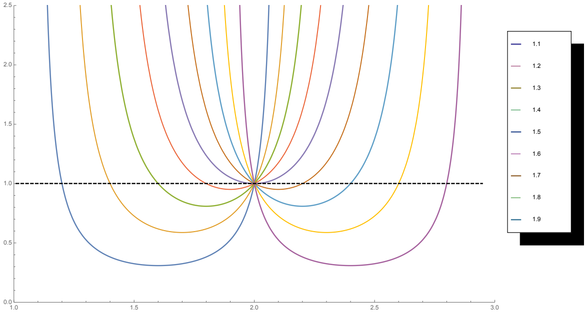
where $(Z_u^\alpha)_{u\in\mathcal{U}}$ is a multiplicative cascade of offspring distribution ν_α.

Properties of Z^α

Theorem (C., Curien, Maillard 2016+)

The Mellin transform of the multiplicative cascade Z^α is

$$\phi_\alpha(\theta) = \frac{\sin(\pi(2-\alpha))}{\sin(\pi(\theta-\alpha))} \quad \textit{pour } \theta \in (\alpha, \alpha+1) \quad \textit{et} \quad \phi_\alpha(\theta) = \infty \textit{ sinon.}$$



Proofs

The gasket decomposition

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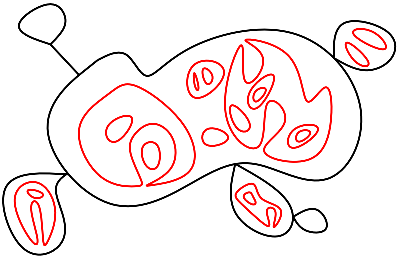
A gasket.

The gasket decomposition [Borot, Bouttier, Guitter '12]



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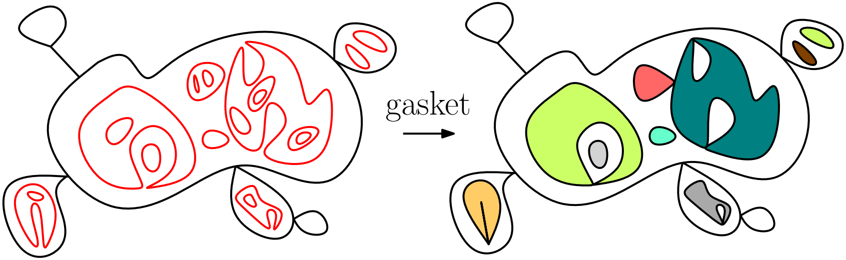

gasket: a bipartite map

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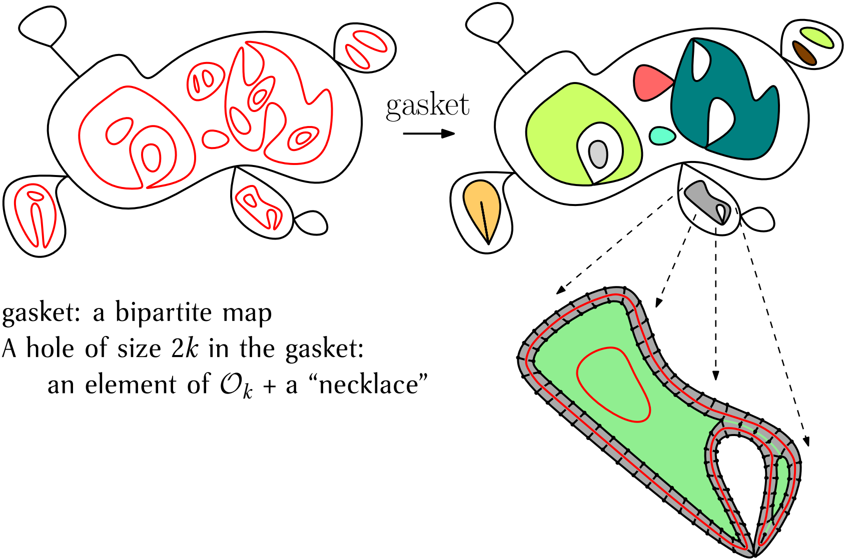

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A hole of size $2k$ in the gasket:
an element of $\mathcal{O}_k$ + a "necklace"

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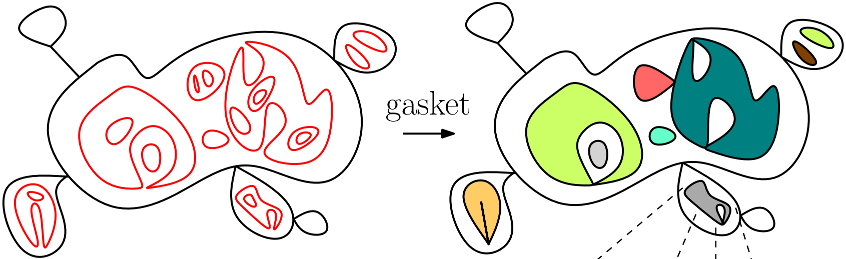

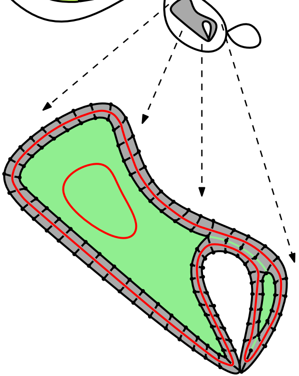

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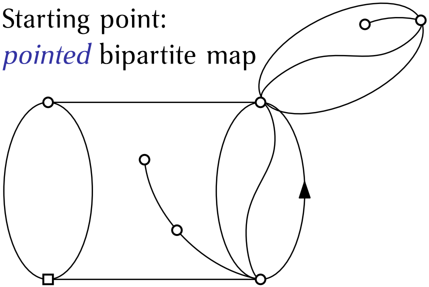
$\Rightarrow$ fixed point condition

$$\begin{cases} F_p(n; g, h) = B_p(g_1, g_2, \ldots) \\ g_k = g\delta_{k,2} + n\, h^{2k} F_k(n; g, h) \qquad (k \geq 1) \end{cases}$$

Encoding the gasket: the BDG and JS bijections

[Bouttier, Di Francesco, Guitter '04, Janson-Stefánsson '15]

Starting point:
pointed bipartite map



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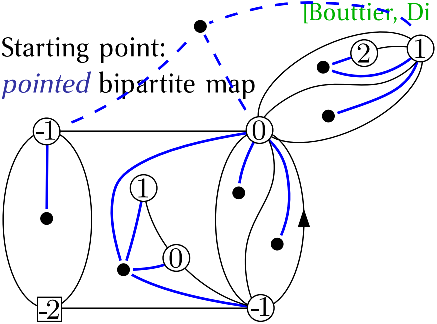

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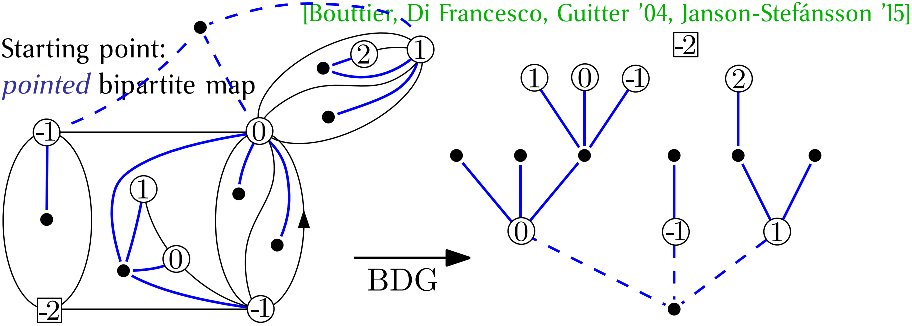

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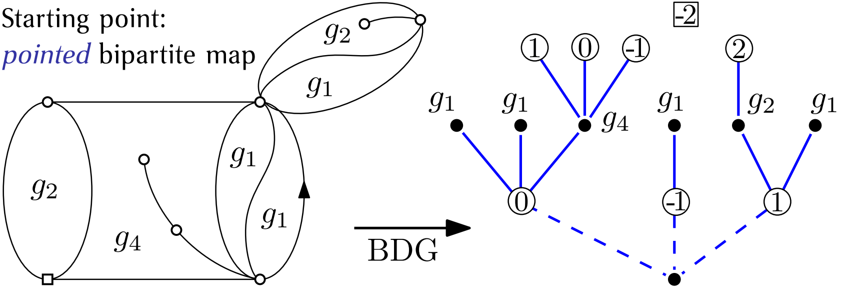

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$$\xrightarrow{BDG} \quad g_k \rightsquigarrow \bullet \text{ of degree } k$$

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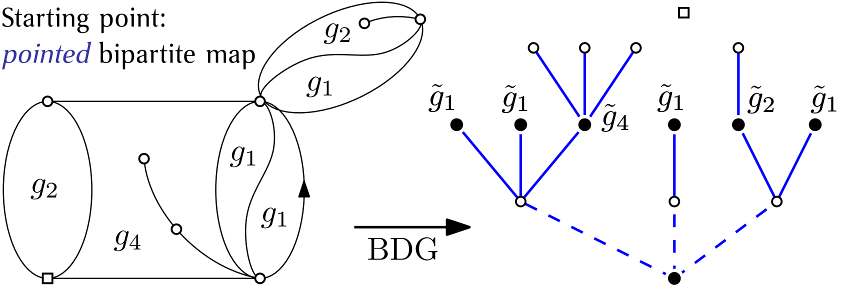

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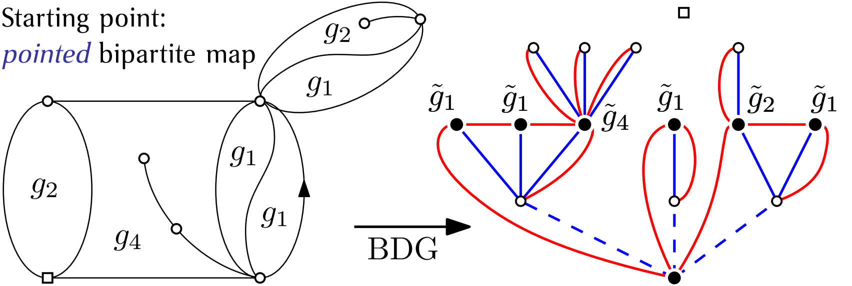

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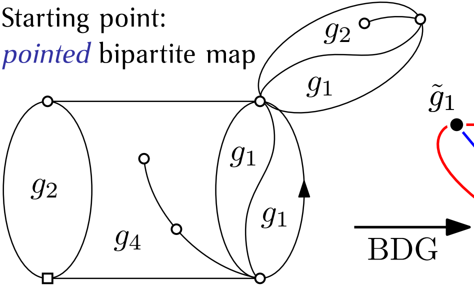

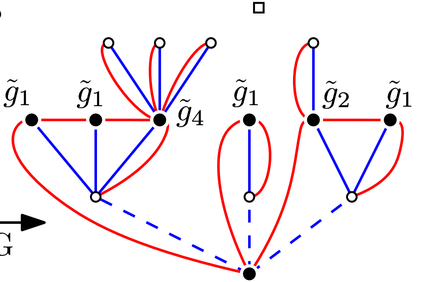

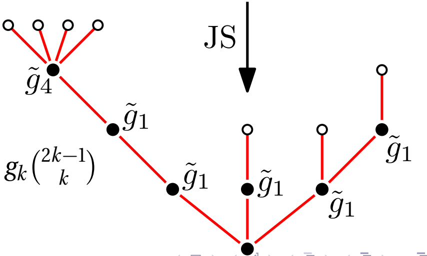

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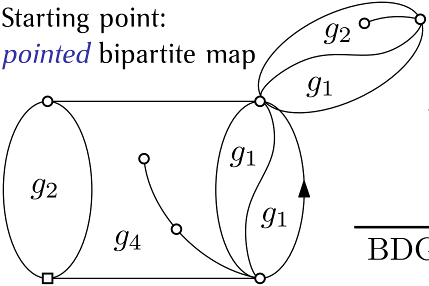

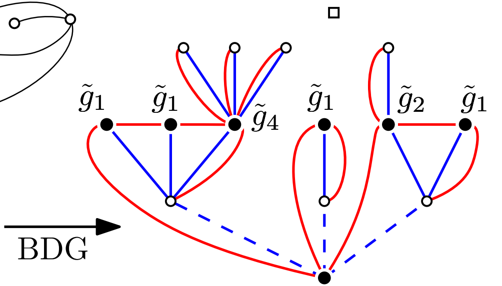

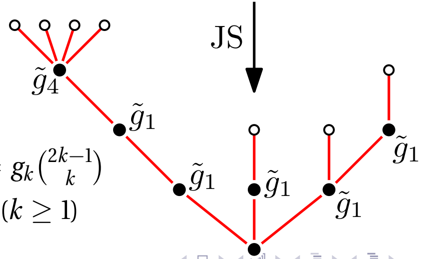

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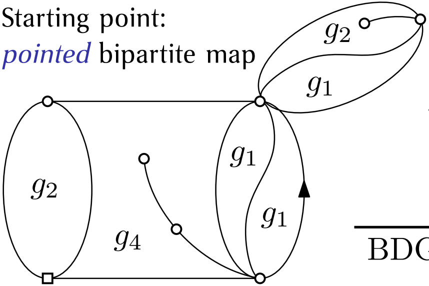
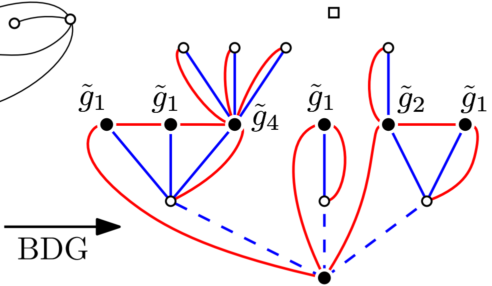
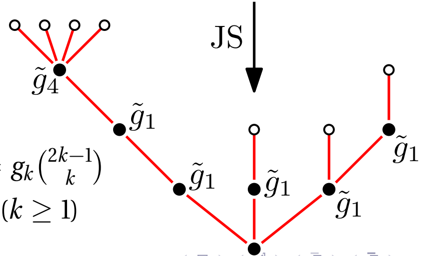
pointed bipartite map







$g_k \rightsquigarrow$ face of degree $2k$

$\xrightarrow{BDG}$ $g_k \rightsquigarrow \bullet$ of degree k

$\xrightarrow{\cancel{labels}}$ $\tilde{g}_k \rightsquigarrow \bullet$ of degree k $\qquad \tilde{g}_k = g_k \binom{2k-1}{k}$

$\xrightarrow{JS}$ $\tilde{g}_k \rightsquigarrow \bullet$ with k descendants ($k \geq 1$)

($1 \rightsquigarrow \circ$ with 0 descendant)

Encoding the gasket: the BDG-JS bijection

pointed bipartite maps under the Boltzmann distribution

$$\mathbb{P}^{\bullet}_{p,\mathbf{g}}(M = \mathfrak{m}^{\bullet}) = \frac{\prod_{k=1}^{\infty} g_k^{f_k(\mathfrak{m}^{\bullet})}}{B_p^{\bullet}(\mathbf{g})}$$

$\xrightarrow[\text{JS}]{\text{BDG}}$

Galton-Watson tree of offspring distribution

$$\mu_{\mathtt{JS}}(k) = \tilde{g}_k \kappa^{k-1} \sim Ck^{-\alpha}$$

face of degree $2k$ $\longrightarrow$ interal vertex with k children

vertices $\longrightarrow$ leaves

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The BDG and JS bijections applies naturally to *pointed* bipartite maps. To recover a *non-pointed* Boltzmann map, we need to bias the law of the Galton-Watson tree by $1/\{$its number of leaves$\}$.

$$\mathbb{E}_{p,\mathbf{g}}[F(M)] = \frac{\mathbb{E}^{\bullet}_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}} F(M)\right]}{\mathbb{E}^{\bullet}_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}}\right]} = \frac{\mathbb{E}_{\mathrm{GW}}\left[\frac{1}{\#\text{leaf}} F(T)\right]}{\mathbb{E}_{\mathrm{GW}}\left[\frac{1}{\#\text{leaf}}\right]}$$

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$$\mathbb{E}_{p,\mathbf{g}}[F((\chi_k))] = \frac{\mathbb{E}^\bullet_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}}F((\chi_k))\right]}{\mathbb{E}^\bullet_{p,\mathbf{g}}\left[\frac{1}{\#\text{vertex}}\right]} = \frac{\mathbb{E}_{\text{GW}}\left[\frac{1}{\#\text{leaf}}F((D_i)^\downarrow)\right]}{\mathbb{E}_{\text{GW}}\left[\frac{1}{\#\text{leaf}}\right]}$$

Encoding the gasket: scaling limit of the hole sizes

Conclusion

Let $(\chi_i^{(p)})_{i\geq 1}$ be the half-degrees of faces of the gasket, sorted in $\downarrow$ order and completed with zeros. Then for all bound function F,

$$\mathbb{E}[F(\chi_i^{(p)})] = \frac{\mathbb{E}\left[\frac{1}{\#\{i\leq T_p : X_i=-1\}} F((X_i+1)^{\downarrow}_{i\leq T_p})\right]}{\mathbb{E}\left[\frac{1}{\#\{i\leq T_p : X_i=-1\}}\right]}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ is a random walk with step distribution $\mu(k) = \mu_{\mathrm{JS}}(k+1)$ $(k \geq -1)$ and T_p its hitting time of -1.

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$$\mathbb{E}[F(\chi_i^{(p)})] \approx \frac{\mathbb{E}\left[\frac{1}{T_p}F((X_i+1)^{\downarrow}_{i\leq T_p})\right]}{\mathbb{E}\left[\frac{1}{T_p}\right]} \approx \frac{\mathbb{E}\left[\frac{1}{\tau}F((\Delta\zeta)^{\downarrow}_{\tau})\right]}{\mathbb{E}\left[\frac{1}{\tau}\right]}$$

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When p is large, $\#\{i \leq T_p : X_i = -1\} \approx \mu(-1)T_p$.

Proposition

$(p^{-1}\chi_i^{(p)})_{i\geq 1} \underset{p\to\infty}{\Longrightarrow} \nu_\alpha$ as $p \to \infty$ in the sense of finite dimension marginals.

An identity on random walks

Theorem (CCM)

Let $S_n = X_1 + \cdots + X_n$ be a random walk with steps $X_i \in \{-1, 0, 1, \cdots\}$. Let T_p be its hitting time of $-p$. Then, for all $f : \mathbb{Z} \to \mathbb{R}_+$ and all $p \geq 2$,

$$\mathbb{E}\left[\frac{1}{T_p - 1}\sum_{i=1}^{T_p} f(X_i)\right] = \mathbb{E}\left[f(X_1)\frac{p}{p + X_1}\right].$$

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Theorem (CCM)

Let $(\eta_t)_{t\geq 0}$ *be a Lévy process without negative jumps and of Lévy measure* π*. Let* τ *be its hitting time at* -1*. Then, for all measurable* $f : \mathbb{R}_+^* \to \mathbb{R}_+$

$$\mathbb{E}\left[\frac{1}{\tau}\sum_{t\leq\tau} f(\Delta\eta_t)\right] = \int f(x)\frac{1}{1+x}\pi(\mathrm{d}x).$$

Proof of the discrete identity

Kemperman's formula (/cyclic lemma /ballot theorem . . .)

If the F is invariant under cyclic permutation of its arguments, then

$$\mathbb{E}\left[F(X_1, \cdots, X_n)\mathbf{1}_{\{T_p=n\}}\right] = \frac{p}{n}\mathbb{E}\left[F(X_1, \cdots, X_n)\mathbf{1}_{\{S_n=-p\}}\right]$$

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Proof.

$$\begin{aligned}
A_n := \mathbb{E}\left[\sum_{i=1}^{n} f(X_i)\mathbf{1}_{\{T_p=n\}}\right] &= \frac{p}{n}\,\mathbb{E}\left[\sum_{i=1}^{n} f(X_i)\mathbf{1}_{\{S_n=-p\}}\right] && \text{by Kemperman's formula}\\
&= p\,\mathbb{E}\left[f(X_1)\mathbf{1}_{\{S_n=-p\}}\right] && \text{by cyclic symmetry}\\
&= p\,\mathbb{E}\left[f(X_1)\mathbf{1}_{\{\tilde{S}_{n-1}=-p-X_1\}}\right] && \text{by Markov property}\\
&= p\,\mathbb{E}\left[f(X_1)\frac{n-1}{p+X_1}\mathbf{1}_{\{\tilde{T}_{p+X_1}=n-1\}}\right] && \text{by Kemperman's formula.}
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\end{aligned}$$

For $p \geq 2$ we have always $T_p \geq 2$, hence

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{T_p-1}\sum_{i=1}^{T_p} f(X_i)\right] = \sum_{n=2}^{\infty}\frac{A_n}{n-1} &= p\sum_{n=2}^{\infty}\mathbb{E}\left[f(X_1)\frac{1}{p+X_1}\mathbf{1}_{\{\tilde{T}_{p+X_1}=n-1\}}\right] \\
&= \mathbb{E}\left[f(X_1)\frac{p}{p+X_1}\right].
\end{aligned}$$

Consequences of the identities

- The Mellin transform of the continuous cascade Z^α: for $\theta \in (\alpha, \alpha+1)$,

$$\frac{\mathbb{E}\left[\frac{1}{\tau}\sum_{t\leq\tau}(\Delta\eta_t)^\theta\right]}{\mathbb{E}\left[\frac{1}{\tau}\right]} = \frac{\int \frac{x^\theta}{1+x}\pi(\mathrm{d}x)}{\int \frac{1}{1+x}\pi(\mathrm{d}x)} = \frac{\sin(\pi(2-\alpha))}{\sin(\pi(\theta-\alpha))}$$

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- Convergence of moments of the offspring distribution

$$\mathbb{E}\left[\sum_{i=1}^{\infty}\left(p^{-1}\chi_i^{(p)}\right)^\theta\right] \xrightarrow[p\to\infty]{} \mathbb{E}\left[\sum_{i=1}^{\infty}(Z_i^\alpha)^\theta\right]$$

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- Convergence of moments of the offspring distribution

$$\mathbb{E}\left[\sum_{|u|=k}\left(p^{-1}\chi_u^{(p)}\right)^\theta\right] \xrightarrow[p\to\infty]{} \mathbb{E}\left[\sum_{|u|=k}(Z_u^\alpha)^\theta\right]$$

- Convergence in $\ell^\infty(\mathcal{U}_k)$ of the perimeter cascade: for all $\epsilon > 0$,

$$\mathbb{P}\left(\exists u \in \mathcal{U}_k \setminus [\mathcal{U}_k]_n, p^{-1}\chi_u^{(p)} > \epsilon\right) \xrightarrow[n\to\infty]{} 0$$

Convergence in $\ell^\infty(\mathcal{U})$

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Two ingredients

There exists $\theta > 0$ such that

$$\mathbb{E}\left[\sum_{i=1}^{\infty}\left(\frac{\chi_i^{(p)}}{p}\right)^{\theta}\right] \underset{p\to\infty}{\longrightarrow} \mathbb{E}\left[\sum_{i=1}^{\infty}(Z_i^{\alpha})^{\theta}\right] < 1$$

$\forall p$, $\mathbb{E}\left[\sum_{i=1}^{\infty} \bar{V}(\chi_i^{(p)})\right] < \bar{V}(p)$

where $\bar{V}(p)$ is the expected volume of a rigid $O(n)$-quadrangulation of perimeter $2p$.

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For some p_0 and $c < 1$, we have

$$\forall p > p_0,\ \mathbb{E}\left[\sum_{i=1}^{\infty}\left(p^{-1}\chi_i^{(p)}\right)^\theta\right] \le c$$

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$$\mathbb{E}\left[\sum_{|u|>k}\left(\frac{\chi_i^{(p)}}{p}\right)^{\theta}\mathbf{1}_{\{N_{p_0}(u)\leq m\}}\right] \underset{k\to\infty}{\longrightarrow} 0$$

$$\mathbb{P}\left(\exists u \notin \mathcal{U}_k\ :\ N_{p_0}(u) \leq m\,,\ \chi_u^{(p)} \geq \epsilon p\right) \underset{k\to\infty}{\longrightarrow} 0$$

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Thank you for your attention !