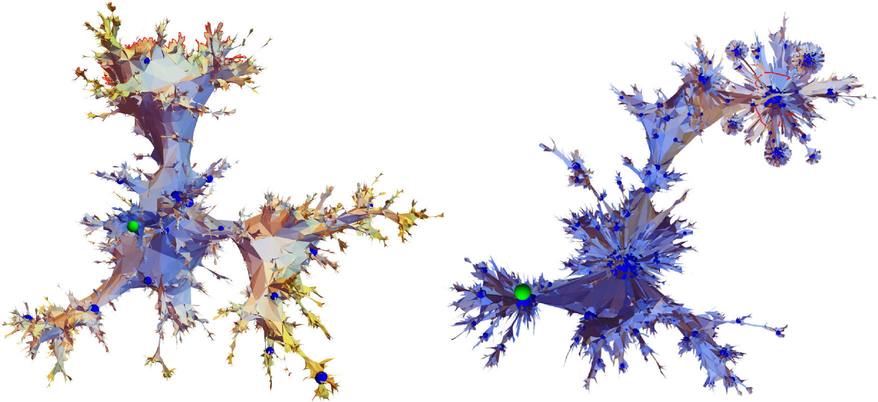
Random Trees and Maps, CIRM, 07-06-2016

Geometry of random planar maps with high degrees

Timothy Budd



Based on arXiv:1506.01590, arXiv:1602.01328 with N. Curien,
and arXiv:1605.00581 with J. Bertoin, N. Curien, I. Kortchemski.

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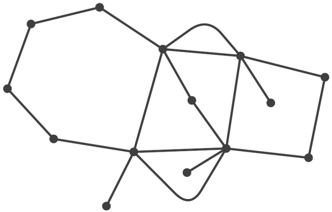
Boltzmann planar maps

- Let $\mathfrak{m} \in \mathcal{M}^{(l)}$ be a bipartite rooted planar map with root face degree $2l$.



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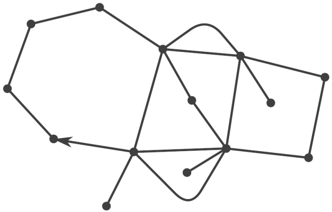

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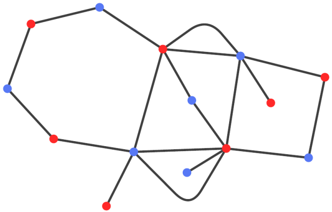

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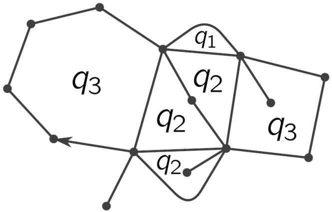
- Let $\mathfrak{m} \in \mathcal{M}^{(l)}$ be a bipartite rooted planar map with root face degree $2l$.
- Given a sequence $\mathbf{q} = (q_1, q_2, \ldots)$ in $[0, \infty)$, define *weight* of $\mathfrak{m}$ to be the product $w_{\mathbf{q}}(\mathfrak{m}) = \prod_f q_{\deg(f)/2}$ over non-root faces f.



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- $\mathbf{q}$ *admissible* iff $W^{(l)}(\mathbf{q}) := \sum_{\mathfrak{m} \in \mathcal{M}^{(l)}} w_{\mathbf{q}}(\mathfrak{m}) < \infty$. Then $w_{\mathbf{q}}$ gives rise to probability measure on $\mathcal{M}^{(l)}$: the $\mathbf{q}$-*Boltzmann planar map* (with *boundary* of length $2l$).
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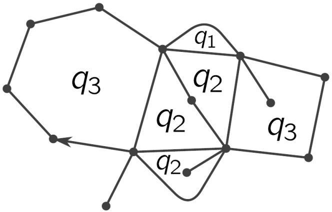

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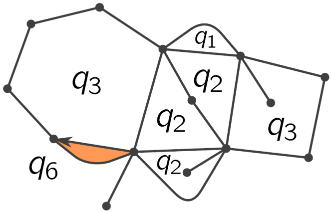

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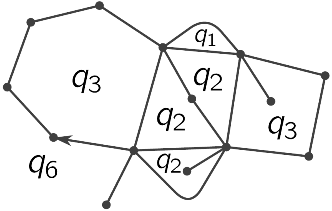

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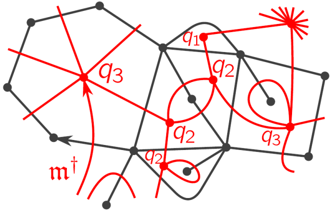
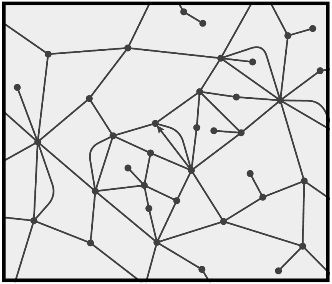
- Dual planar map denoted by $\mathfrak{m}^\dagger$.



Infinite Boltzmann planar maps

- *Local limit*: there exists a unique random infinite map, the **q**-*IBPM*, whose neighborhoods of the root are distributed as those of a **q**-BPM conditioned to have large number of vertices.

  [Björnberg, Stefánsson, '14] [Stephenson, '14]



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PRIMAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}_r}\right)$	$\sim r^4$	
	Scaling limit (Gromov-Hausdorff)	Brownian map [Le Gall, Miermont]	
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DUAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}_r^{\dagger}\right)$		
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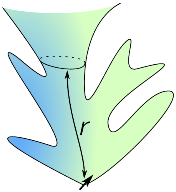

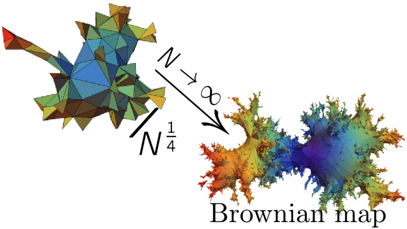

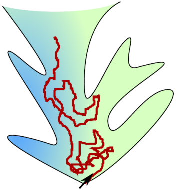

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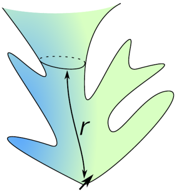

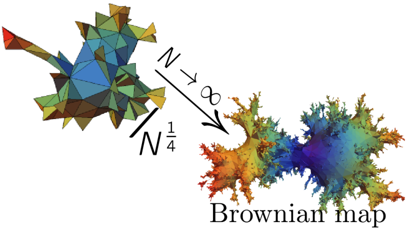

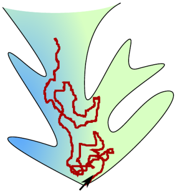

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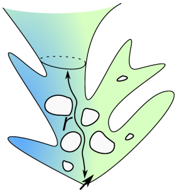

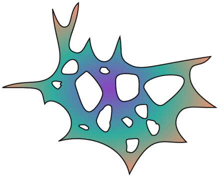

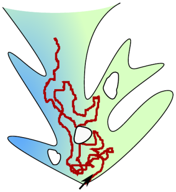

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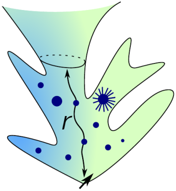

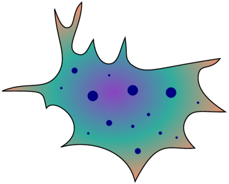

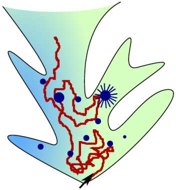

Submaps and peeling processes

- Two convenient representations of a *submap*:
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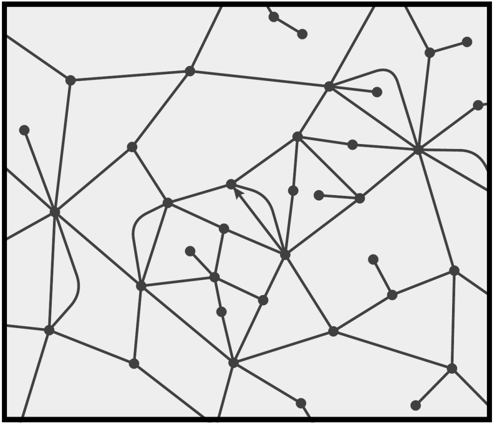

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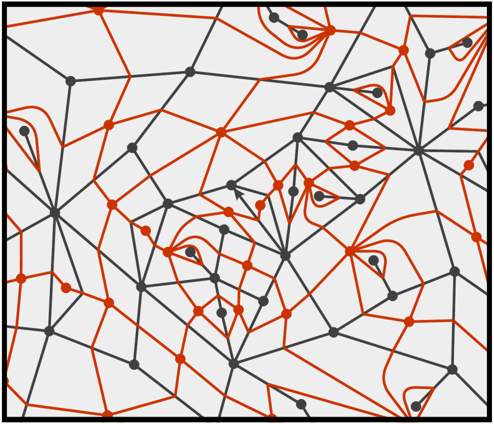

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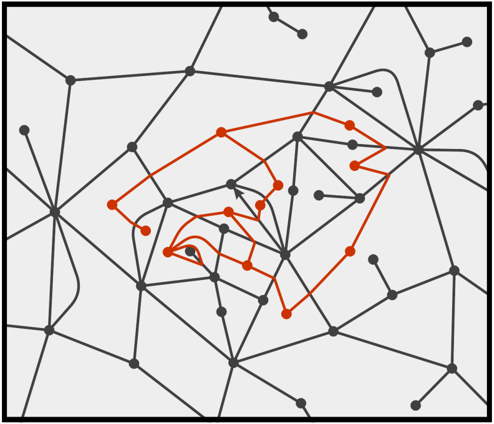

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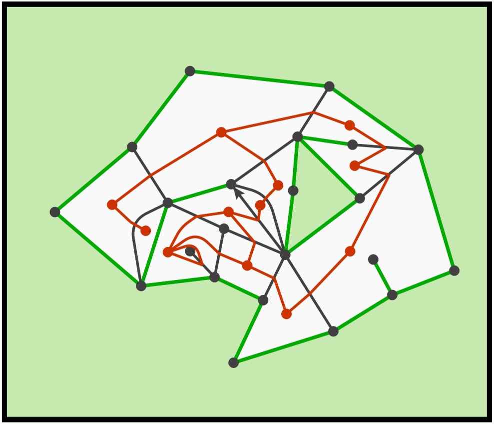

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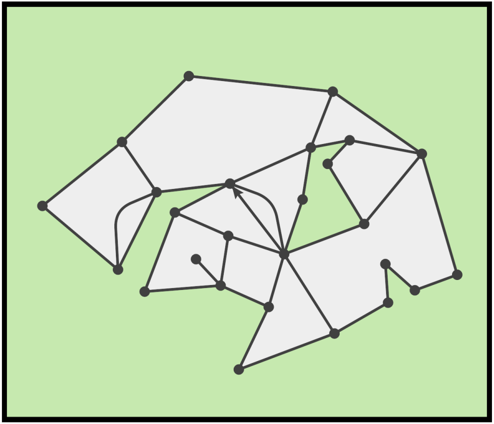

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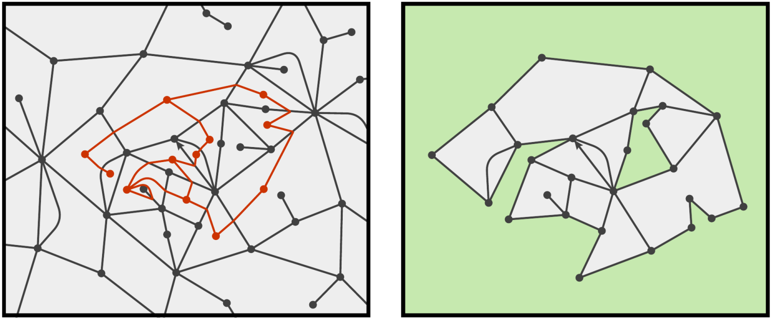

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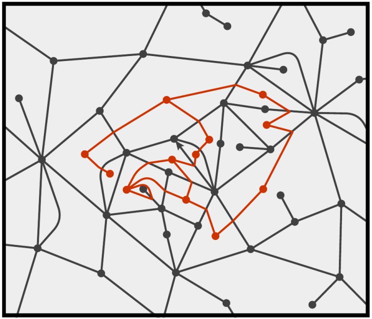

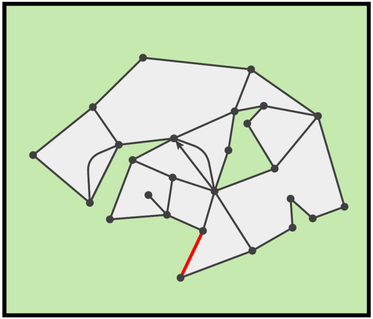

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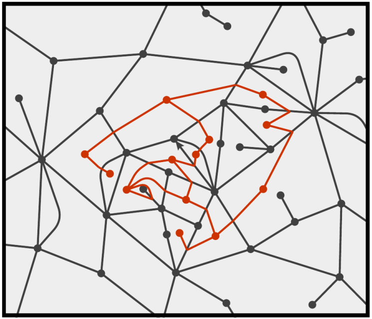

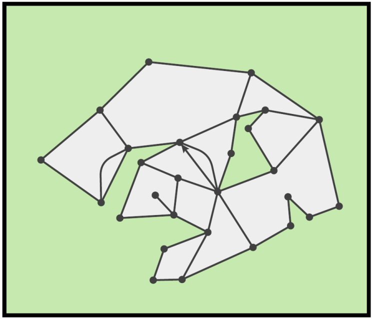

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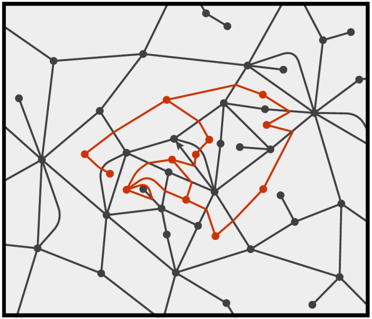

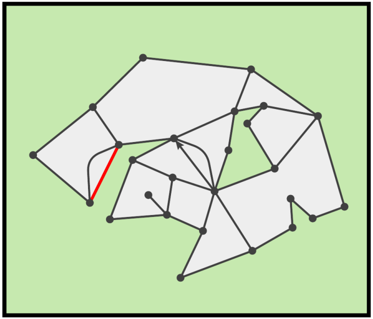

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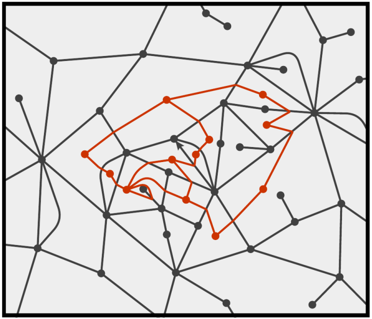
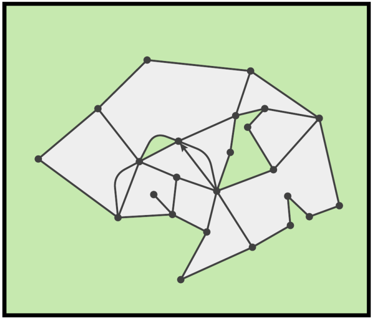

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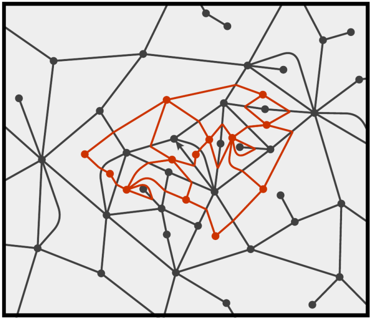
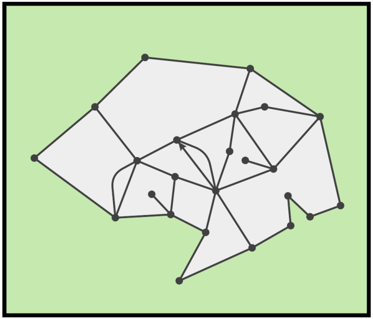
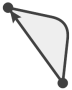
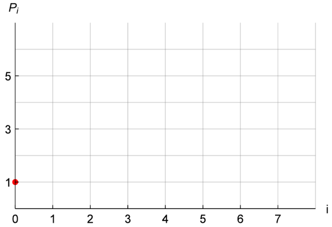




Peeling process of **q**-IBPM (non-branching)





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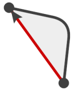

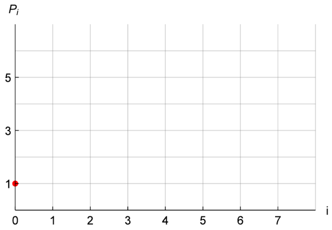

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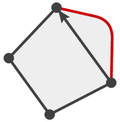

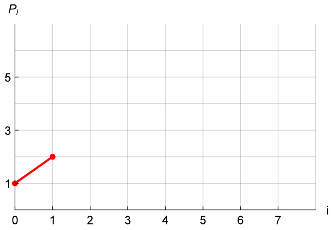

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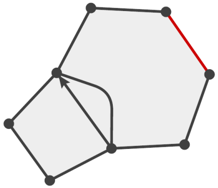

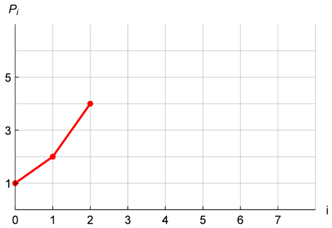

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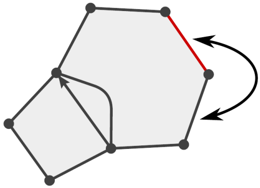

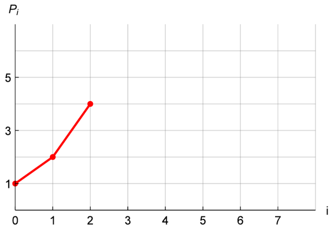

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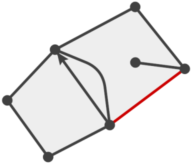

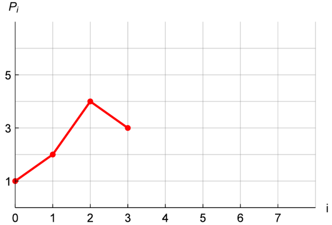

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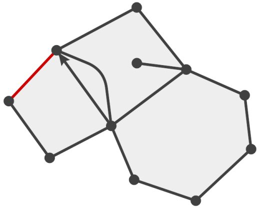

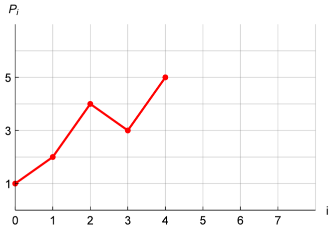

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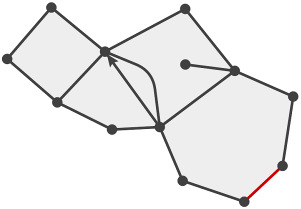

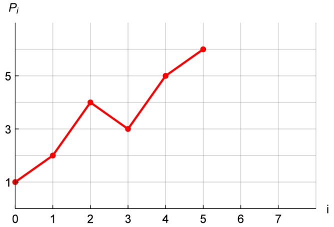

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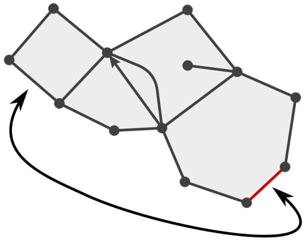

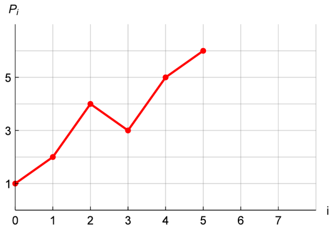

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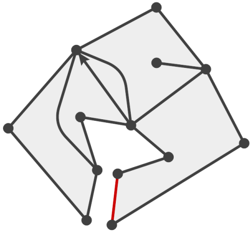

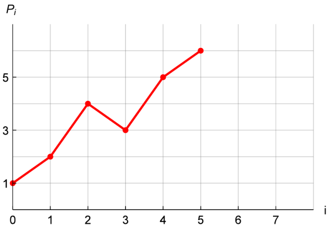

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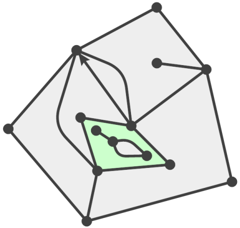

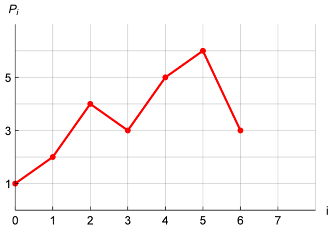

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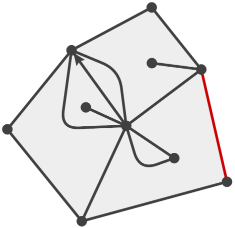

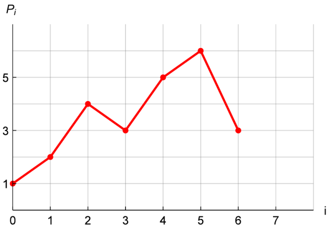

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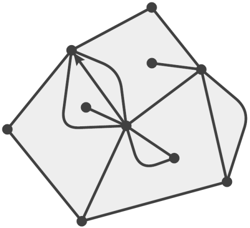

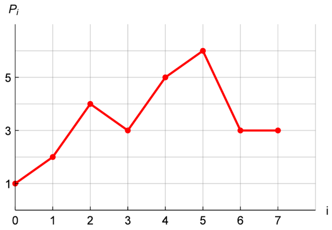

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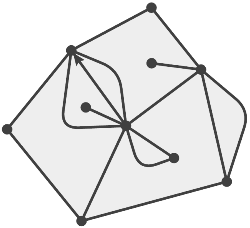

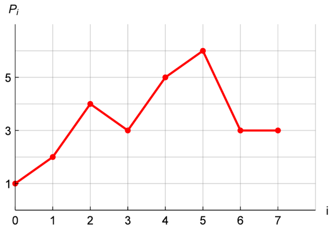

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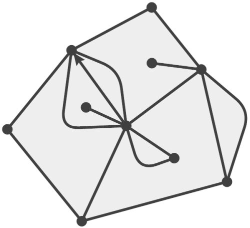

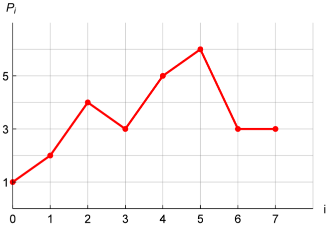

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Using $$W_N^{(l)} := \sum_{\substack{\text{maps } \mathfrak{m} \\ N \text{ vertices} \\ \text{boundary } 2l}} w_{\mathbf{q}}(\mathfrak{m}) \overset{N\to\infty}{\sim} C\, N^{-\gamma} h^{\uparrow}(l) \kappa^{-l}, \qquad h^{\uparrow}(l) := 2l\, 2^{-2l} \binom{2l}{l}$$

$$\mathbb{P}(P_{i+1} = P_i + k) = \frac{h^{\uparrow}(P_i + k)}{h^{\uparrow}(P_i)} \begin{cases} q_{k+1} \kappa^{-k} & k \geq 0 \\ 2 W^{(-k-1)} \kappa^{-k} & k < 0 \end{cases}$$

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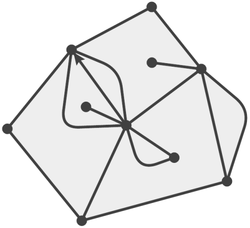

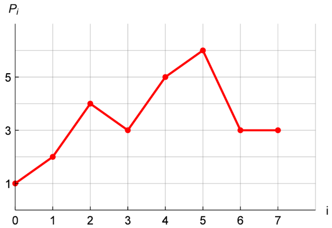

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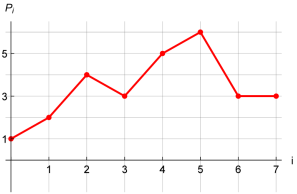
- defines probability measure on $\mathbb{Z}$



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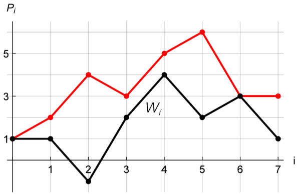

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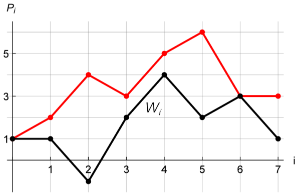

Proposition (TB, '15)

- $(P_i)_i \overset{(d)}{=} (W_i^\uparrow)_i$, *i.e.* $(W_i)_i$ *started at* 1 *and conditioned to stay positive.*
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$$\sum_{k=-\infty}^{\infty} h^\uparrow(l+k)\nu(k) \overset{l\geq 0}{=} h^\uparrow(l)$$

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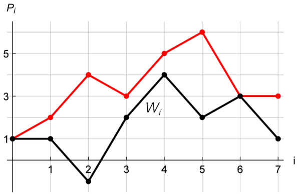

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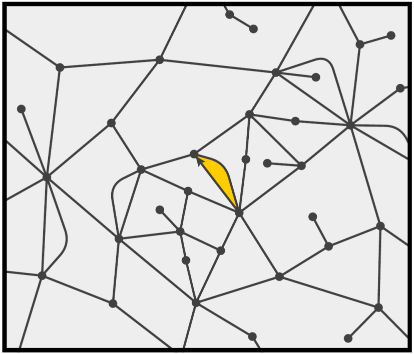
- $\mathbf{q} \to \nu$ *defines a bijection*

$$\{\mathbf{q} \textit{ critical}\} \longleftrightarrow \left\{ \nu : \sum_{k=-\infty}^{\infty} h^{\uparrow}(l+k)\nu(k) \stackrel{l \geq 0}{=} h^{\uparrow}(l) \right\}$$

Dual graph distance

- $\mathrm{Ball}_r^\dagger(\mathfrak{m}_\infty)$ is the submap of $\mathfrak{m}_\infty$ determined by all dual edges with at least one end at $d_{\mathrm{gr}}^\dagger < r$ from root. $\overline{\mathrm{Ball}}_r^\dagger(\mathfrak{m}_\infty)$ is its *hull*.



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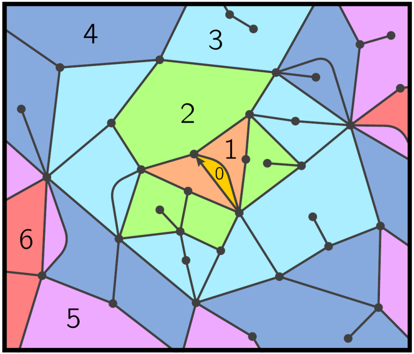

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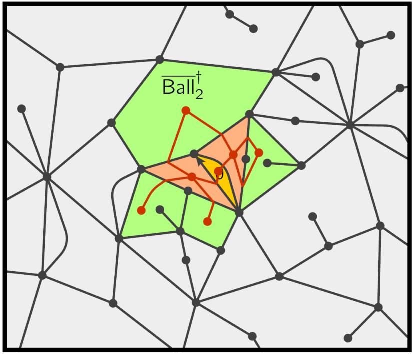

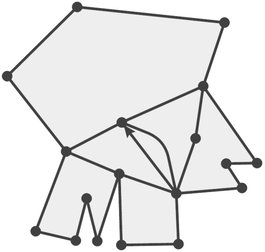

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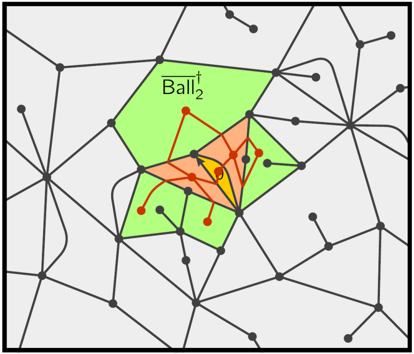

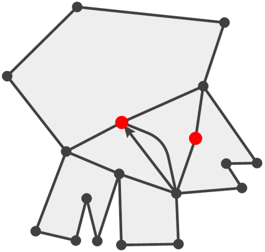

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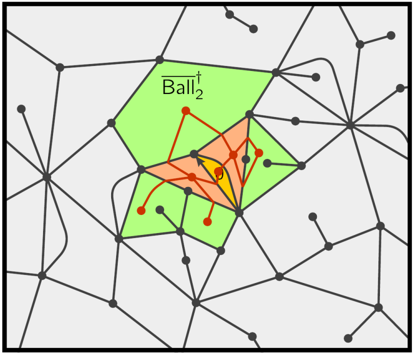

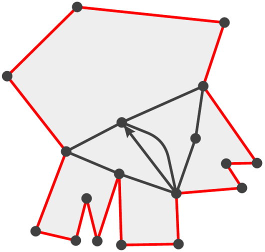

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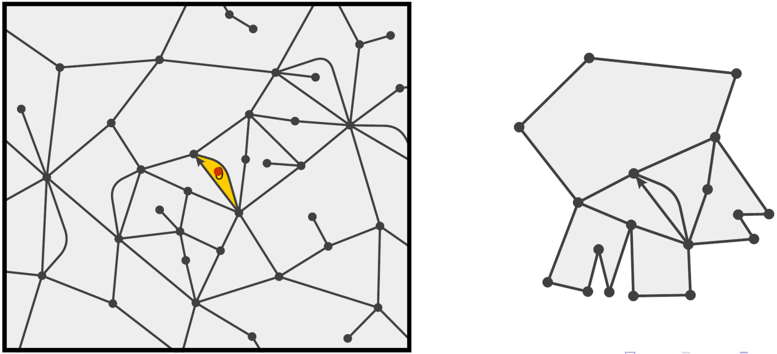

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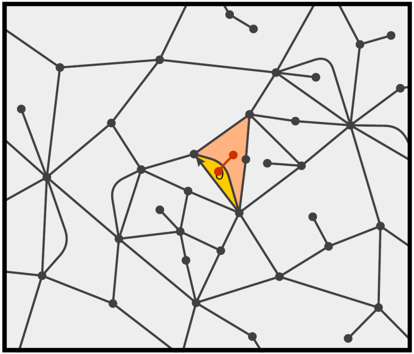

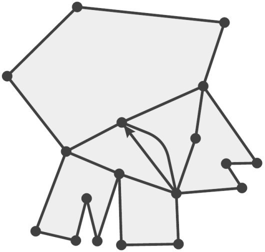

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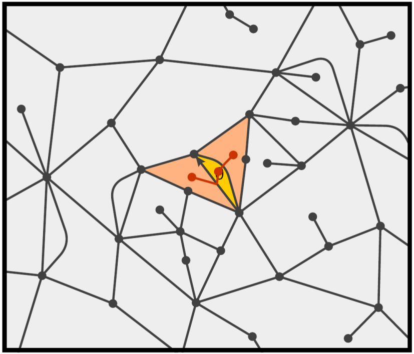

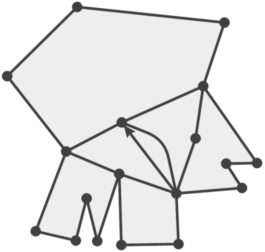

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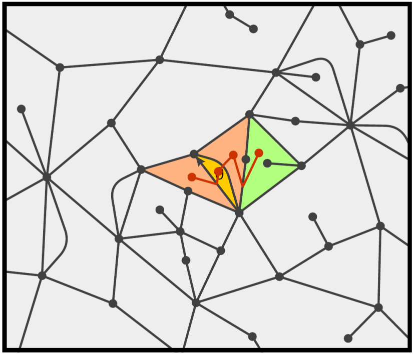

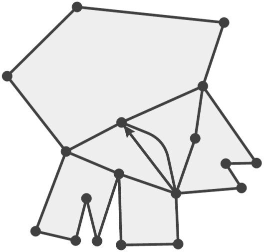

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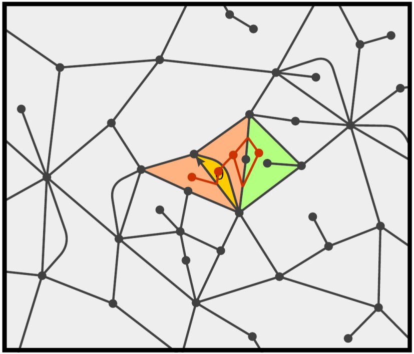

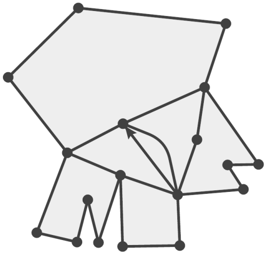

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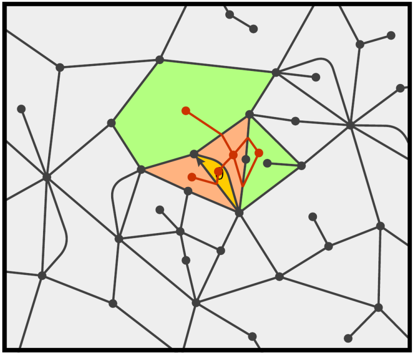

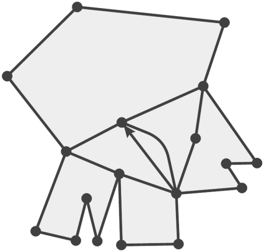

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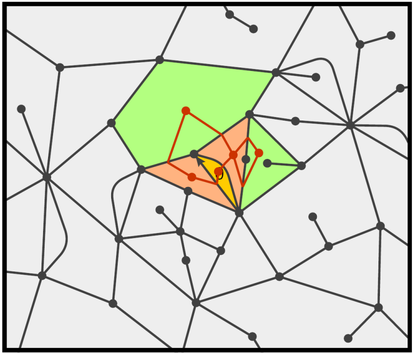

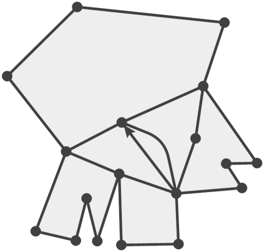

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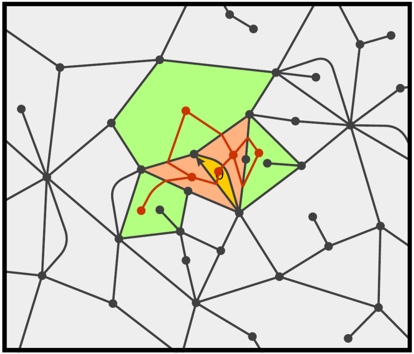

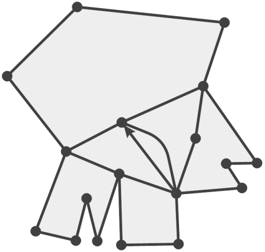

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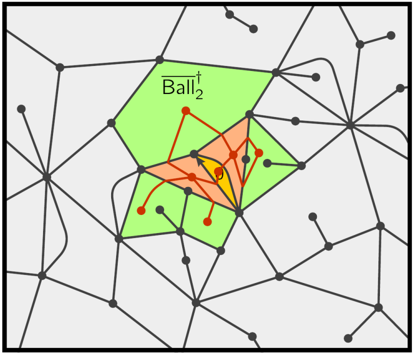
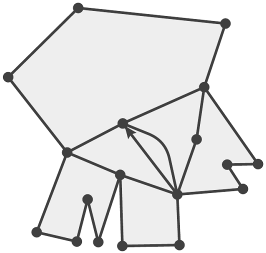
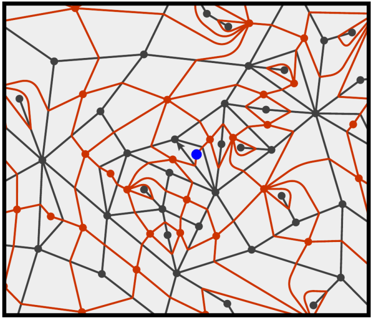




First passage percolation

- Equip each dual edge with i.i.d. Exp(1) random length, and view $\mathfrak{m}_\infty^\dagger$ as a length metric space.



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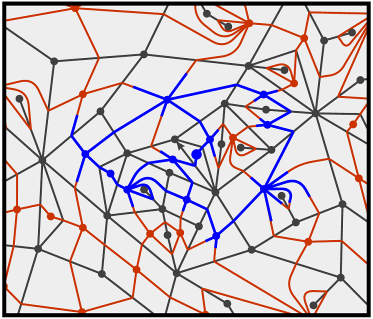

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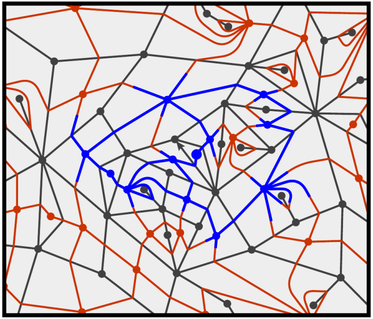

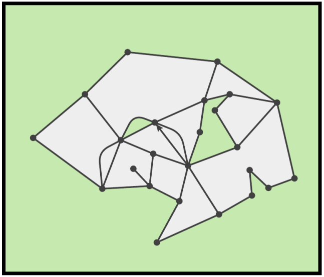

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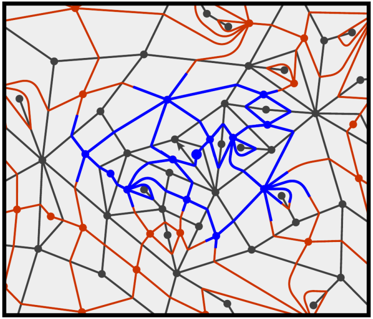

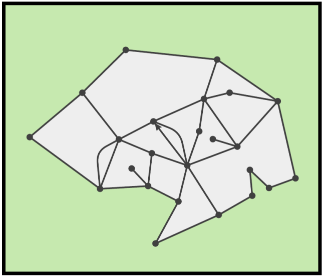

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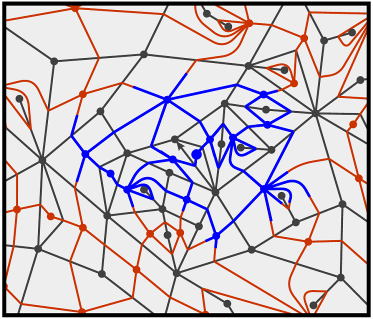

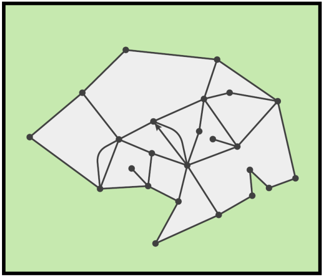

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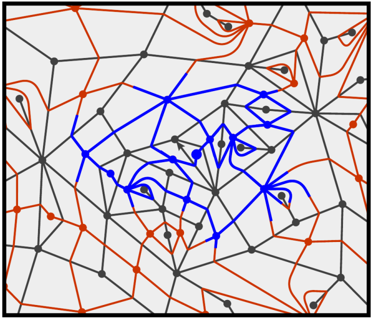
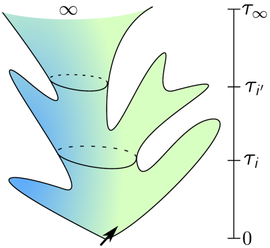
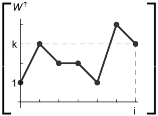


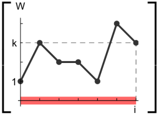


Back of the envelope: does $\tau_i \to \infty$?

$$\mathbb{E}\tau_\infty = \sum_{i=0}^{\infty} \mathbb{E}\left[\mathrm{Exp}(2P_i)\right] = \sum_{i=0}^{\infty} \mathbb{E}_1\left[\frac{1}{2W_i^{\uparrow}}\right] = \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2k}\,\mathbb{P}$$



$$= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \frac{h^{\uparrow}(k)}{2k}\,\mathbb{P}$$



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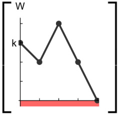
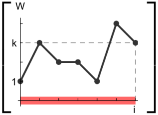


$$\mathbb{P}$$



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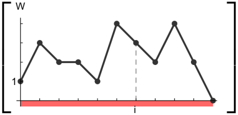
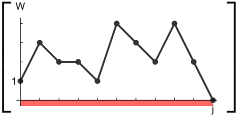


$$= \sum_{j=1}^{\infty} j\,\mathbb{P}$$



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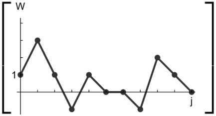
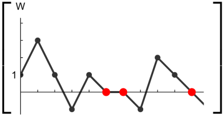


$$= \mathbb{E}$$



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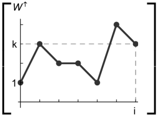

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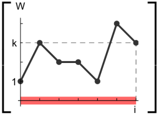

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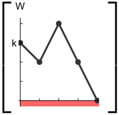
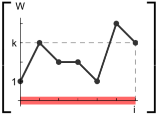


$$\mathbb{P}$$



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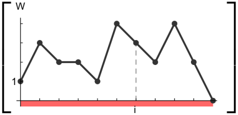

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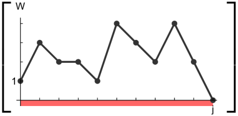

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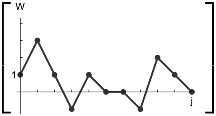

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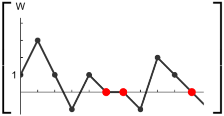
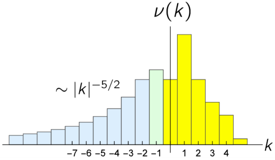
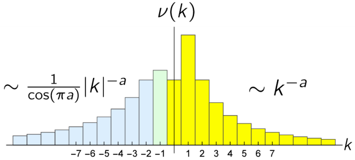


$\mathbb{E}\tau_\infty = \infty$ iff (W_i) is recurrent on $\mathbb{Z}$!

		Regular critical $\mathbf{q}$	Non-generic $q_k \sim c\,\kappa^{k-1} k^{-a}$ $\quad a \in \left(\frac{3}{2}, \frac{5}{2}\right)$
PRIMAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}_r\right)$	$\sim r^4$	$\sim r^{2a-1}$
	Scaling limit (Gromov-Hausdorff)	Brownian map [Le Gall, Miermont]	Stable maps [Le Gall, Miermont]
	Simple random walk	Recurrent [Gurel-Gurevich, Nachmias]	Recurrent [Björnberg, Stefánsson]
DUAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}^{\dagger}_r\right)$	$\sim r^4$	?
	Scaling limit (Gromov-Hausdorff)	Probably Brownian map Triangulations: [Curien, Le Gall]	?
	Simple random walk	Recurrent	?





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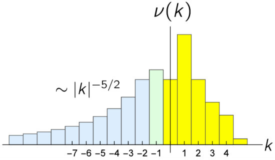
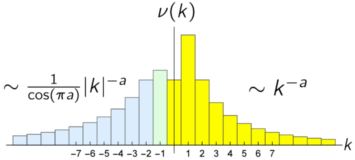
PRIMAL	Simple random walk	Recurrent [Gurel-Gurevich, Nachmias]	Recurrent [Björnberg, Stefánsson]	
DUAL			"Dilute" $a \in \left(2, \frac{5}{2}\right)$	"Dense" $a \in \left(\frac{3}{2}, 2\right)$
DUAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}^{\dagger}_r\right)$	$\sim r^4$		
DUAL	Scaling limit (Gromov-Hausdorff)	Probably Brownian map. Triangulations: [Curien, Le Gall]		
DUAL	Simple random walk	Recurrent		
	τ_∞	∞ a.s.	∞ a.s.	Finite a.s.





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	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}_r^{\dagger}\right)$	$\sim r^4$		
	Scaling limit (Gromov-Hausdorff)	Probably Brownian map Triangulations: [Curien, Le Gall]		
	Simple random walk	Recurrent		Transient
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Proposition (TB, Curien, '16)

Any infinite graph with $\mathbb{E}\tau_\infty < \infty$ is transient.

		Regular critical $\mathbf{q}$	Non-generic $q_k \sim c\,\kappa^{k-1}k^{-a}$ $\quad a \in \left(\frac{3}{2}, \frac{5}{2}\right)$	
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DUAL			"Dilute" $a \in \left(2, \frac{5}{2}\right)$	"Dense" $a \in \left(\frac{3}{2}, 2\right)$
	$\mathrm{Vol}\left(\overline{\mathrm{Ball}^\dagger_r}\right)$	$\sim r^4$		$\sim \exp(r)$
	Scaling limit (Gromov-Hausdorff)	Probably Brownian map Triangulations: [Curien, Le Gall]		
	Simple random walk	Recurrent		Transient
	τ_∞	∞ a.s.	∞ a.s.	Finite a.s.

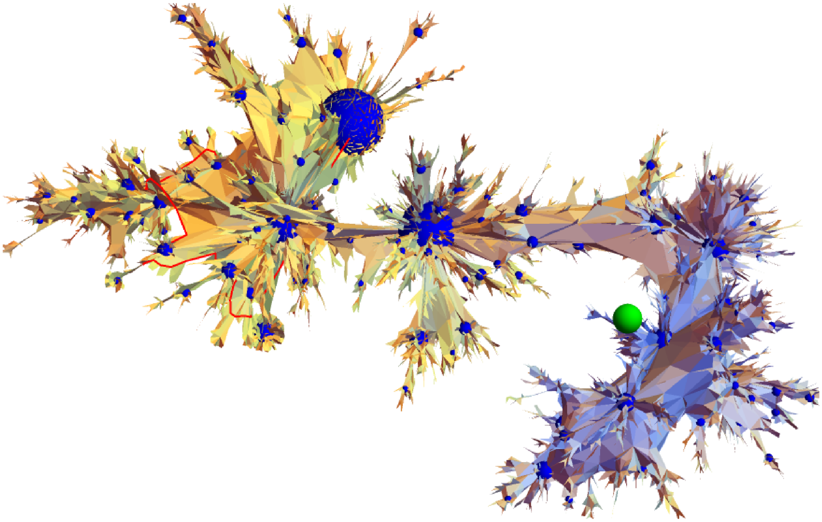
Theorem (TB, Curien, '16)

In the dense case $a \in \left(\frac{3}{2}, 2\right)$ there exists $c_a > 0$ such that

$$r^{-1}\log\left(\left|\partial \overline{Ball}^\dagger_r\right|\right) \xrightarrow[r\to\infty]{(\mathrm{p})} c_a, \qquad r^{-1}\log\left(\left|\overline{Ball}^\dagger_r\right|\right) \xrightarrow[r\to\infty]{(\mathrm{p})} \left(a - \tfrac{1}{2}\right)c_a$$

Simulations: dense case

$a = 1.8$



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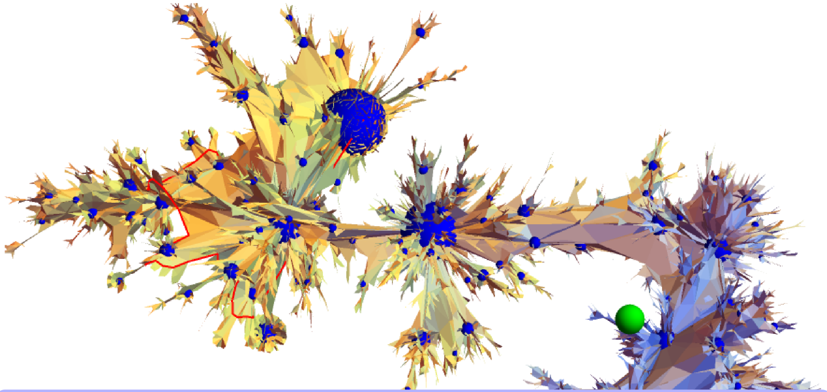
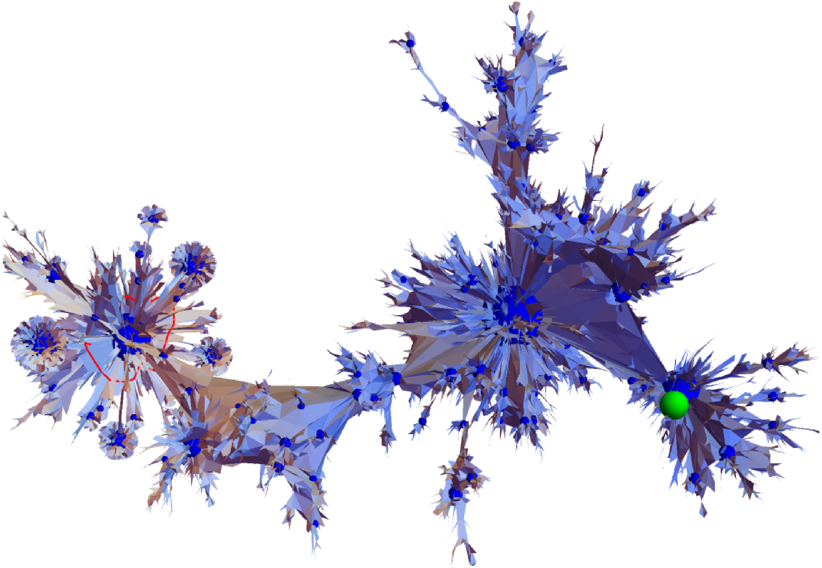


Theorem (TB, Curien, '16)

When $a < 2$ the $\mathbf{q}$*-IBPM and its dual both contain infinitely many cut vertices separating root from ∞.*

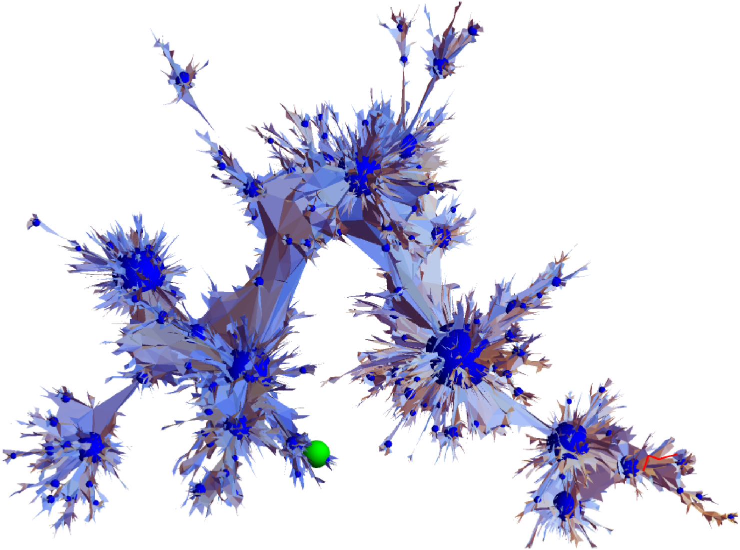
Simulations: dense case

$a = 1.8$



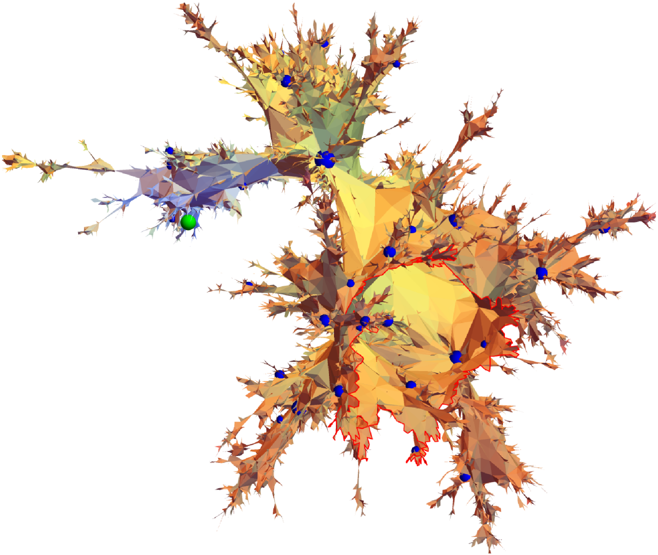
Simulations: dense case

$a = 1.7$



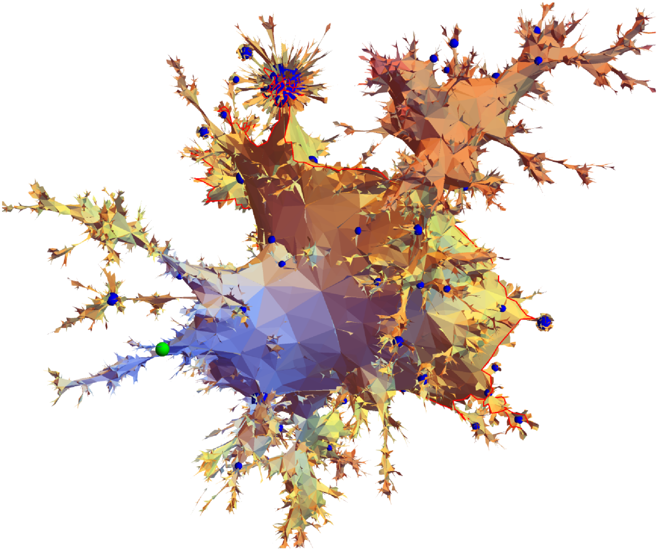
Simulations: dilute case

$a = 2.3$



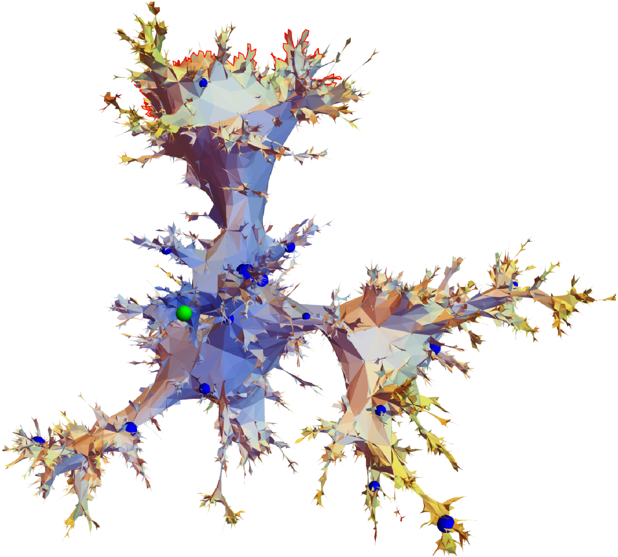
Simulations: dilute case

$a = 2.3$



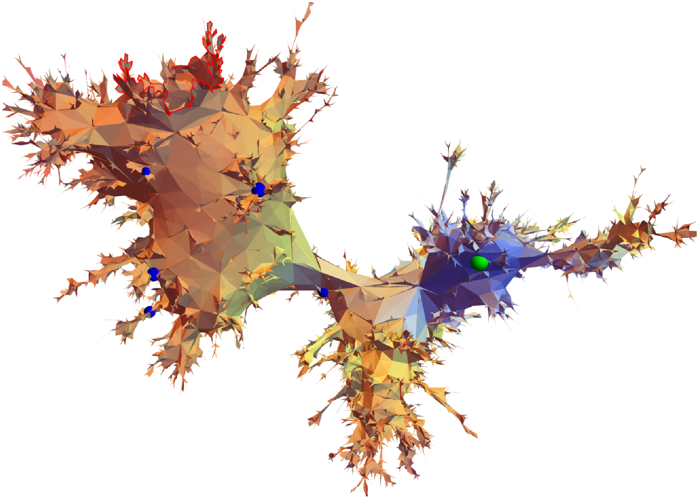
Simulations: dilute case

$a = 2.35$



Simulations: dilute case

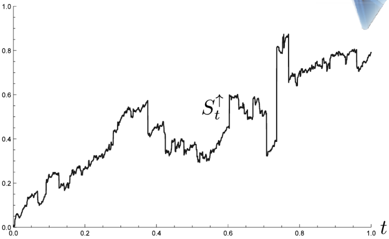
$a = 2.45$



Scaling limit in dilute case

- As $\nu(k) \overset{|k|\to\infty}{\sim} |k|^{-a}$ we have convergence to a $(a-1)$-stable process (S_t) with $\mathbb{P}(S_t \leq 0) = \frac{1}{2(a-1)}$.
- Since $(P_i) \overset{\text{(d)}}{=} (W_i^{\uparrow})$, we have [Caravenna, Chaumont]

$$\left(\frac{P_{\lfloor nt \rfloor}}{n^{\frac{1}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \mathbf{p_q}\,(S_t^{\uparrow})_{t\geq 0}$$



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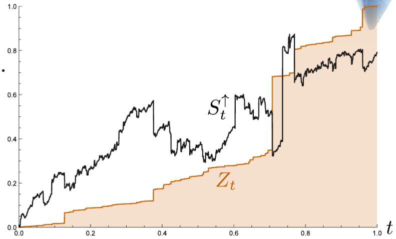

Theorem (TB, Curien, '16)

The peeling process on a dilute **q***-IBPM satisfies*

$$\left(\frac{P_{\lfloor nt \rfloor}}{n^{\frac{1}{a-1}}}, \frac{V_{\lfloor nt \rfloor}}{n^{\frac{a-1/2}{a-1}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q} \cdot S_t^{\uparrow}, \mathbf{v_q} \cdot Z_t\right)_{t\geq 0}$$

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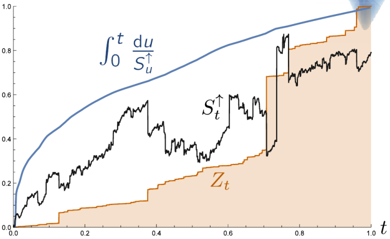

Theorem (TB, Curien, '16)

The *uniform* *peeling process on a dilute* $\mathbf{q}$*-IBPM satisfies*

$$\left(\frac{P_{\lfloor nt \rfloor}}{n^{\frac{1}{a-1}}}, \frac{V_{\lfloor nt \rfloor}}{n^{\frac{a-1/2}{a-1}}}, \frac{\mathcal{T}_{\lfloor nt \rfloor}}{n^{\frac{a-2}{a-1}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q}\cdot S_t^{\uparrow}, \mathbf{v_q}\cdot Z_t, \frac{1}{2\mathbf{p_q}}\int_0^t \frac{\mathrm{d}u}{S_u^{\uparrow}}\right)_{t\geq 0}$$

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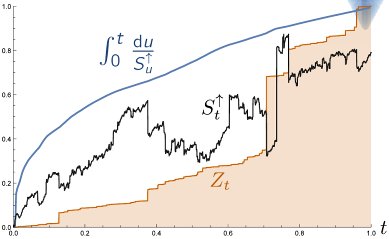

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$$\left(\frac{P_{\lfloor nt \rfloor}}{n^{\frac{1}{a-1}}}, \frac{V_{\lfloor nt \rfloor}}{n^{\frac{a-1/2}{a-1}}}, \frac{\tau_{\lfloor nt \rfloor}}{n^{\frac{a-2}{a-1}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q}\cdot S_t^{\uparrow}, \mathbf{v_q}\cdot Z_t, \frac{1}{2\mathbf{p_q}}\int_0^t \frac{\mathrm{d}u}{S_u^{\uparrow}}\right)_{t\ge 0}$$

$$\left(\frac{|\partial \overline{Ball}^{fpp}_{\lfloor nt \rfloor}(\mathfrak{m}_\infty)|}{n^{\frac{1}{a-2}}}, \frac{|\overline{Ball}^{fpp}_{\lfloor nt \rfloor}(\mathfrak{m}_\infty)|}{n^{\frac{a-1/2}{a-2}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q}\cdot S^{\uparrow}_{\theta_{2\mathbf{p_q}}t}, \mathbf{v_q}\cdot Z_{\theta_{2\mathbf{p_q}}t}\right)_{t\ge 0}$$

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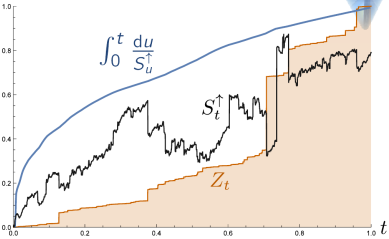

Theorem (TB, Curien, '16)

The "by layers" peeling process on a dilute $\mathbf{q}$*-IBPM satisfies*

$$\left(\frac{P_{\lfloor nt \rfloor}}{n^{\frac{1}{a-1}}}, \frac{V_{\lfloor nt \rfloor}}{n^{\frac{a-1/2}{a-1}}}, \frac{r_{\lfloor nt \rfloor}}{n^{\frac{a-2}{a-1}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q}\cdot S_t^{\uparrow}, \mathbf{v_q}\cdot Z_t, \mathbf{h_q}\int_0^t \frac{\mathrm{d}u}{S_u^{\uparrow}}\right)_{t\ge 0}$$

$$\left(\frac{|\partial \overline{Ball}^{\dagger}_{\lfloor nt \rfloor}(\mathfrak{m}_\infty)|}{n^{\frac{1}{a-2}}}, \frac{|\overline{Ball}^{\dagger}_{\lfloor nt \rfloor}(\mathfrak{m}_\infty)|}{n^{\frac{a-1/2}{a-2}}}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{p_q}\cdot S^{\uparrow}_{\theta_{t/\mathbf{h_q}}}, \mathbf{v_q}\cdot Z_{\theta_{t/\mathbf{h_q}}}\right)_{t\ge 0}$$

		Regular critical $\mathbf{q}$	Non-generic $q_k \sim c\,\kappa^{k-1}k^{-a}$ $\quad a \in \left(\frac{3}{2}, \frac{5}{2}\right)$	
PRIMAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}_r\right)$	$\sim r^4$	$\sim r^{2a-1}$	
PRIMAL	Scaling limit (Gromov-Hausdorff)	Brownian map [Le Gall, Miermont]	Stable maps [Le Gall, Miermont]	
PRIMAL	Simple random walk	Recurrent [Gurel-Gurevich, Nachmias]	Recurrent [Björnberg, Stefánsson]	
DUAL			"Dilute" $a \in \left(2, \frac{5}{2}\right)$	"Dense" $a \in \left(\frac{3}{2}, 2\right)$
DUAL	$\mathrm{Vol}\left(\overline{\mathrm{Ball}}_r^{\dagger}\right)$	$\sim r^4$	$\sim r^{\frac{a-1/2}{a-2}}$	$\sim \exp(r)$
DUAL	Scaling limit (Gromov-Hausdorff)	Probably Brownian map Triangulations: [Curien, Le Gall]		✗
DUAL	Simple random walk	Recurrent		Transient

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DUAL			$\left(\frac{3}{2}, 2\right)$
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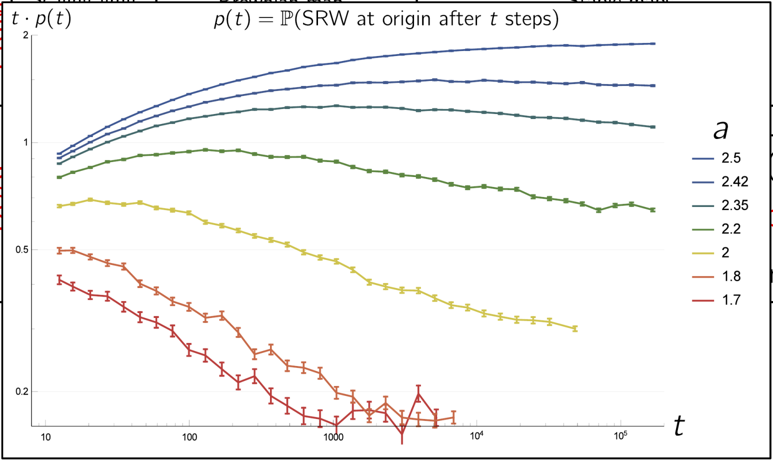

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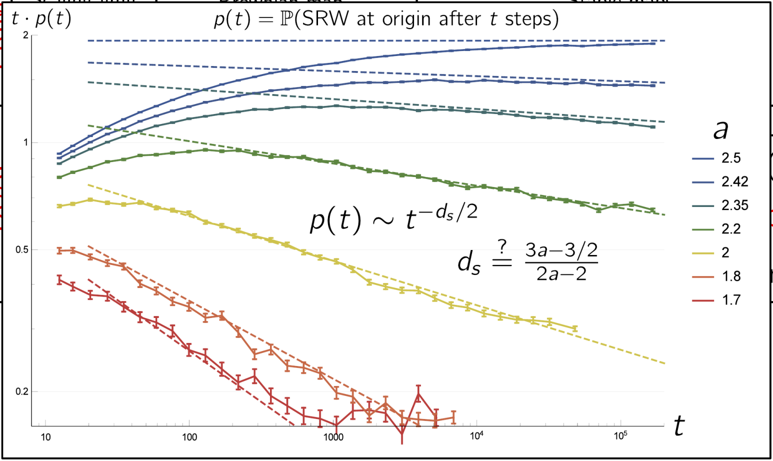

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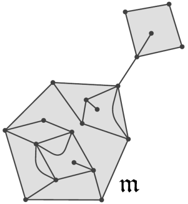
Slicing at heights (using branched peeling)

- Consider $\mathrm{Ball}^\dagger_r(\mathfrak{m})$ of a (finite) $\mathbf{q}$-BPM $\mathfrak{m}$ with boundary length $2l$.



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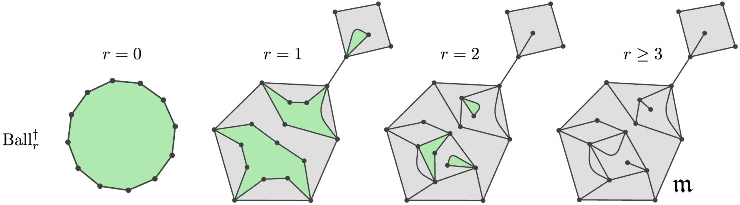

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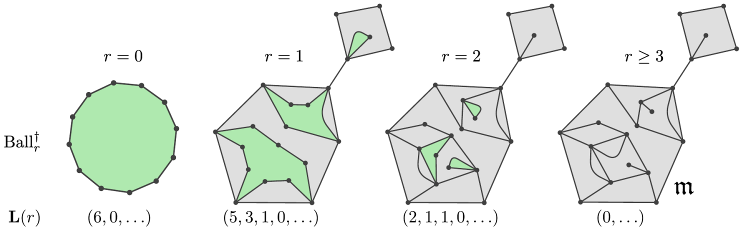

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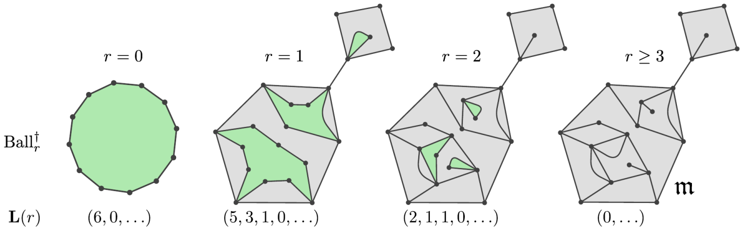


Theorem (Bertoin, TB, Curien, Kortchemski, '16)

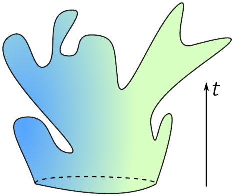
If $\mathbf{q}$ *is dilute critical,* $a \in (2, \frac{5}{2})$*, then* $\left(\frac{\mathbf{L}(\lfloor l^{a-2} \cdot t \rfloor)}{l}\right)_{t \geq 0} \xrightarrow[l \to \infty]{(\mathrm{d})} \left(c\mathbf{X}_t^{(a)}\right)_{t \geq 0}$,

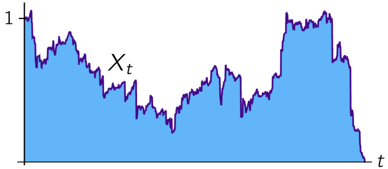
where $\mathbf{X}_t^{(a)}$ *is a self-similar growth-fragmentation process, taking values in*

$$\ell^{\downarrow}_{a+1/2} := \left\{ (x_i)_{i \in \mathbb{N}} : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i^{a+1/2} < \infty \right\}.$$

Growth-fragmentation process

- There exists a self-similar Markov process (X_t) closely related to $(S^{\uparrow}_{\theta_t})$ describing perimeter of *locally largest* cycle.





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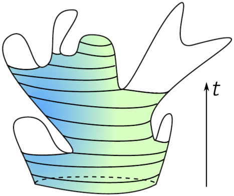

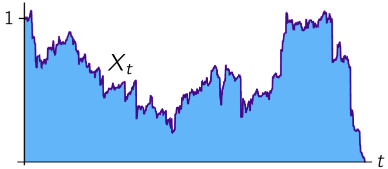

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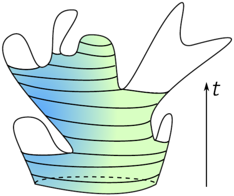

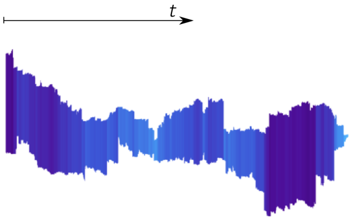

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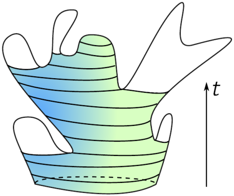

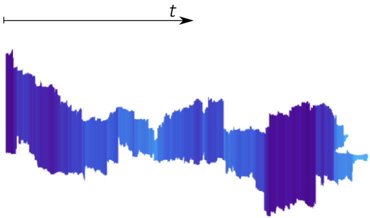

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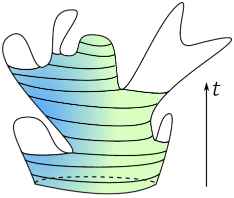

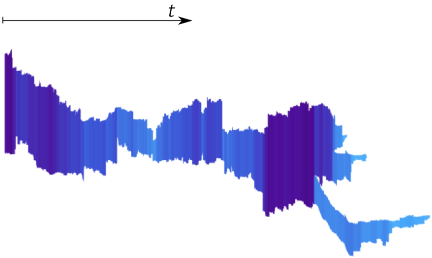

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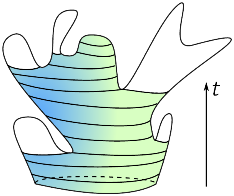

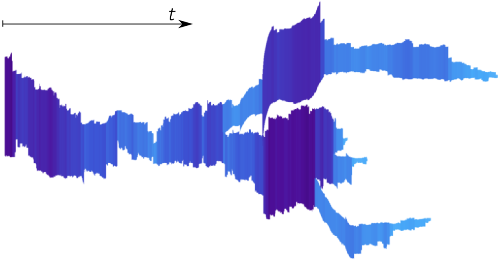

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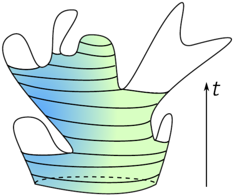

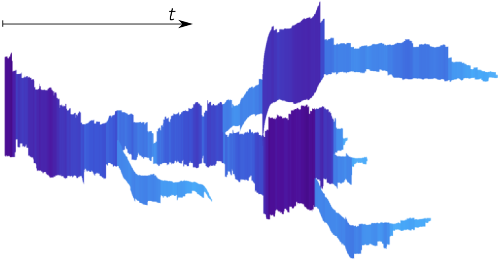

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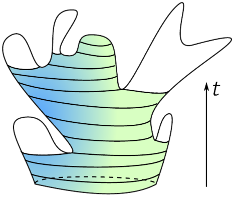

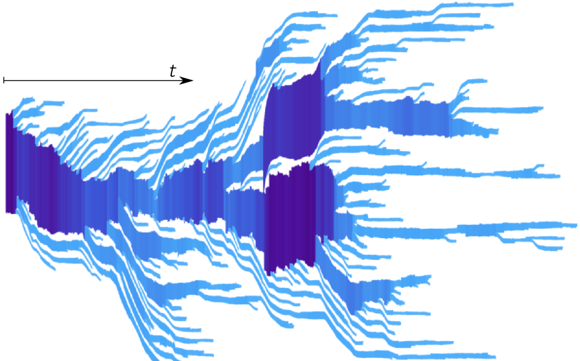

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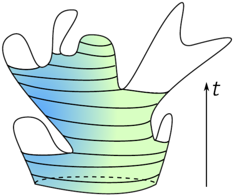

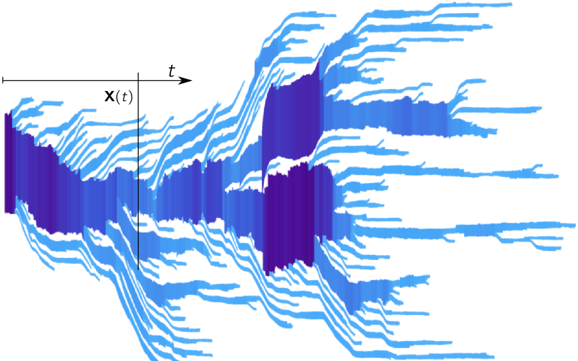

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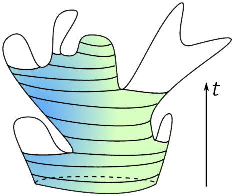

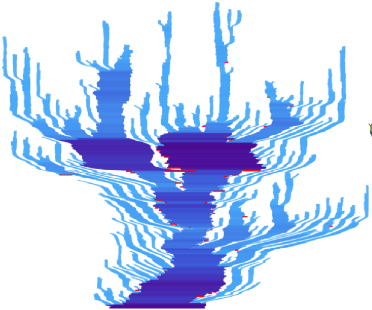

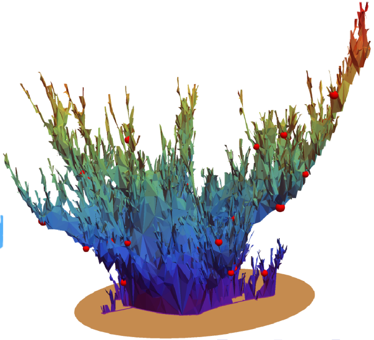

Summary/Outlook

- Dilute critical Boltzmann planar maps equipped with the dual graph distance may possess scaling limits with fractal dimensions $\frac{a-1/2}{a-2} > 4$, different from Brownian map and stable maps.
- The peeling process is tool of choice to study these distances and its scaling limits support the belief.

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- Next steps: critical case $a = 2$, $O(n)$ loop model, geodesics.

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- Dilute critical Boltzmann planar maps equipped with the dual graph distance may possess scaling limits with fractal dimensions $\frac{a-1/2}{a-2} > 4$, different from Brownian map and stable maps.
- The peeling process is tool of choice to study these distances and its scaling limits support the belief.
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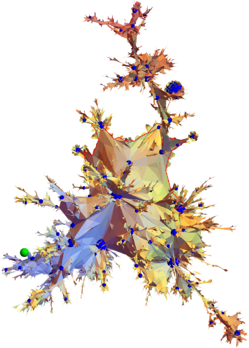

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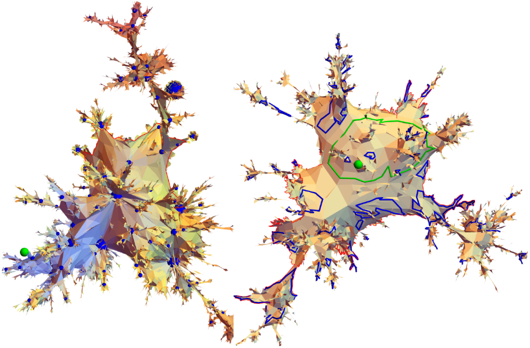

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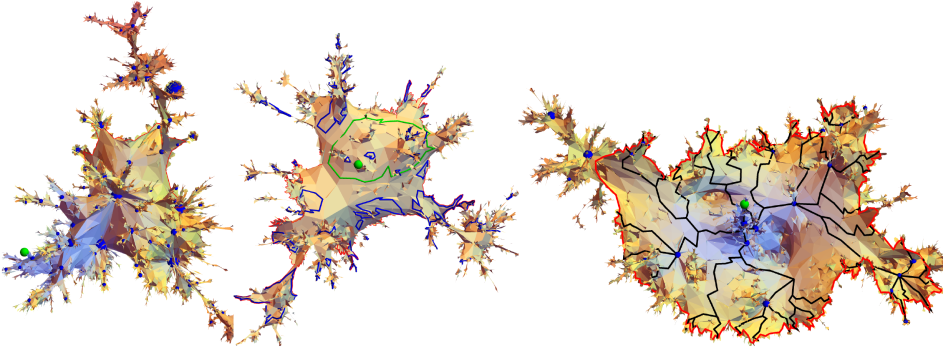

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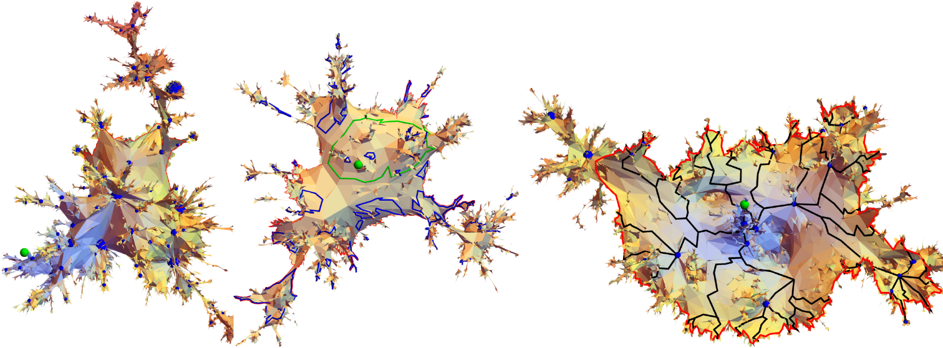

Thanks for your attention!