Nesting statistics in the $O(n)$ loop model on random planar maps

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Introduction

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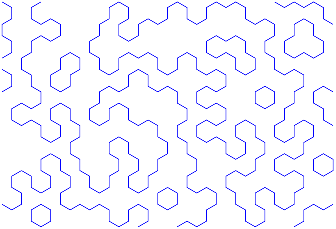
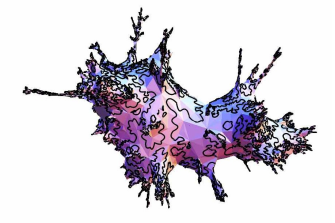
It is also believed that there is a one-parameter family of possible scaling limits (or universality classes) indexed by the so-called central charge $\mathfrak{c} \in (-\infty, 1]$ or equivalently by the SLE parameter κ.
In particular this whole family should be obtained as the scaling limits of random maps decorated by loops, as the weight per loop n varies in the range $[0, 2]$.





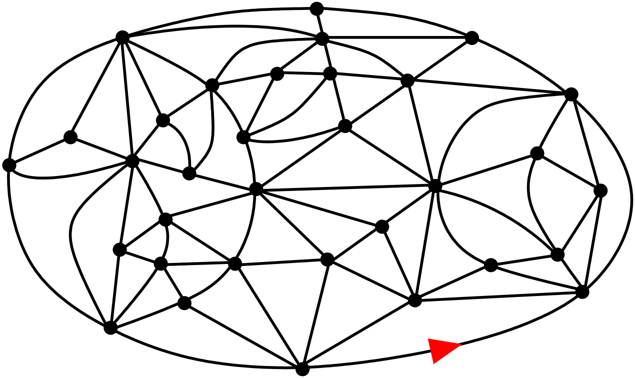
Picture by N. Curien

The $O(n)$ model: a partial history

- Originally introduced by H. E. Stanley (1968) as a statistical physics model for n-dimensional classical spins (unit vectors) on a lattice with Hamiltonian $H = -J \sum_{\langle i,j \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j$.
- De Gennes (1972) observed that self-avoiding walks are obtained as a suitably defined $n \to 0$ limit of this model.
- Domany *et al.* (1981) introduced a variant which, on a trivalent lattice, admits a simple exact representation in terms of loops for all n.
- 1980's: $O(n)$-type models on a regular 2D lattice were thoroughly analyzed using various techniques (Coulomb gas, integrability, CFT) [Nienhuis, Blöte, Batchelor...]
- 1990's: the $O(n)$ model on random maps was introduced and studied by matrix integral and quantum gravity techniques [Kostov, Gaudin, Duplantier, Staudacher, Eynard, Kristjansen...]
- Sheffield and Werner (2012) introduced Conformal Loop Ensembles which are the conjectural scaling limits of the model at a critical point on a regular lattice. Coupling with Liouville quantum gravity for the random map version?

The $O(n)$ loop model on random maps: definition

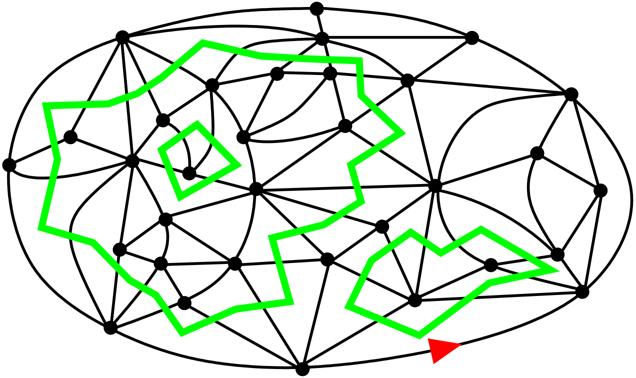
A rooted planar map is a graph embedded in the plane, considered up to continuous deformation, with a distinguished root edge incident to the outer face.



A triangulation with a boundary (each inner face has degree 3)

The $O(n)$ loop model on random maps: definition

Given a rooted map, a loop configuration is a collection of disjoint simple cycles (loops) on the dual map. By convention the outer face is not visited.



A loop configuration on a triangulation with a boundary

A configuration of the $O(n)$ loop model is a map endowed with a loop configuration.

The $O(n)$ loop model on random maps: definition

For simplicity we restrict here to the case where the underlying map is a triangulation with a boundary. To each such configuration $\mathcal{C}$ we associate a weight

$$w(\mathcal{C}) = n^{L(\mathcal{C})} g^{UF(\mathcal{C})} h^{VF(\mathcal{C})} u^{V(\mathcal{C})}$$

where $L(\mathcal{C})$, $UF(\mathcal{C})$, $VF(\mathcal{C})$ and $V(\mathcal{C})$ are respectively the numbers of loops, unvisited inner faces, visited faces and vertices of $\mathcal{C}$.

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$$F_\ell \equiv F_\ell(n, g, h, u) = \sum_{\mathcal{C} \text{ of outer degree } \ell} w(\mathcal{C}).$$

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If $n, g, h, u \geq 0$ are such that $F_\ell < \infty$, the model is said well-defined and the normalized weight yields a probability distribution over configurations. The model is then said critical if it ceases to be well-defined whenever u is increased, and subcritical otherwise. (Those properties do not depend on ℓ.)

Critical exponents

Suppose n, g, h are such that the model is at a critical point for $u = 1$.

- Volume exponent γ_{str} (string susceptibility):

$$[u^V]F_\ell \sim V^{\gamma_{\mathrm{str}}-2}, \qquad V \to \infty$$

- Perimeter exponent a:

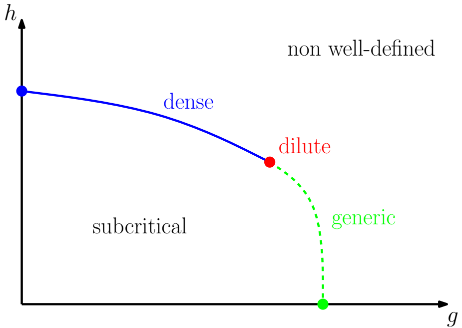
$$F_\ell \sim \frac{C^\ell}{\ell^a}, \qquad \ell \to \infty$$

(also defined for subcritical points !).

For a fixed value of $n \in (0,2)$, each of these exponents can only take a small finite number of possible values, which are rational functions of $b = \frac{1}{\pi} \arccos\left(\frac{n}{2}\right) \in (0, 1/2)$ [see e.g. BBG'12 and BBD'16].

	generic	dilute	dense	subcritical
γ_{str}	$-1/2$	$-b$	$-b/(1-b)$	
a	$5/2$	$2+b$	$2-b$	$3/2$

Phase diagram for fixed $n = 2\cos(\pi b) \in (0, 2)$



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For $n > 2$ we only have generic critical points (same exponents as for maps without loops).

Remark: connection with the $O(n)$ model on regular lattice

On a 2D regular lattice, one also observes dilute and dense critical points for $n \in (0, 2)$, and the critical exponents are still rational functions of b, which is closely related to the so-called Coulomb gas coupling constant

$$\mathfrak{g} = \begin{cases} 1 + b & \text{(dilute)}, \\ 1 - b & \text{(dense)}. \end{cases}$$

In particular the central charge reads [see e.g. Duplantier'04]

$$\mathfrak{c} = 1 - 6(\mathfrak{g} - 1)^2/\mathfrak{g}$$

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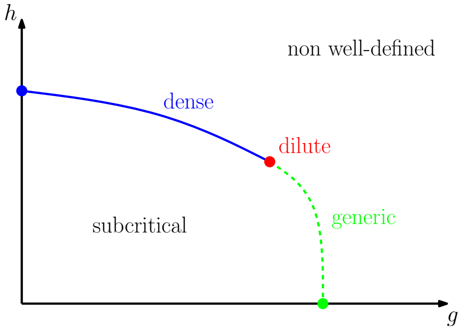
There is perfect agreement with the Knizhnik-Polyakov-Zamolodchikov formulas predicting relations between critical exponents on a 2D regular lattice and on random maps.

Phase diagram for fixed $n = 2\cos(\pi b) \in (0, 2)$



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$\mathfrak{c}$	0	$1-6b^2/(1+b)$	$1-6b^2/(1-b)$	
κ	$8/3$	$4/(1+b)$	$4/(1-b)$	

Gasket exponents

Beyond these global critical exponents, little is known about the geometrical properties of $O(n)$ configurations at a critical point. In particular, if we condition the maps to have a fixed large volume V, we expect the typical distance between vertices to be of order V^{1/d_H} for some $d_H > 0$, but there is not even a consensus on what d_H should be.

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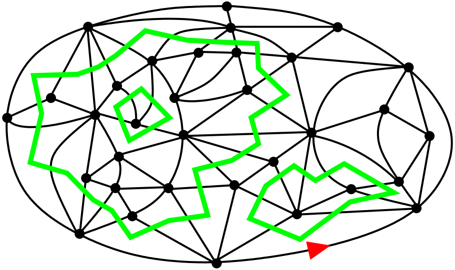
In BBG'12, we obtained information about the geometry of the gasket of $O(n)$ configurations, namely the submap formed by the edges which are exterior to all the loops.



A configuration

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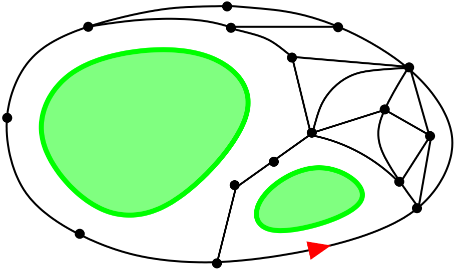

Its gasket

Gasket exponents

We showed that, at a critical point, the gasket is distributed as a Boltzmann map with large faces, as studied by Le Gall and Miermont (2011). Consequently the Hausdorff dimension for the scaling limit(s) of the gasket reads

$$d_H^{\mathrm{gasket}} = \begin{cases} 3 + 2b & \text{(dilute)}, \\ 3 - 2b & \text{(dense)}. \end{cases}$$

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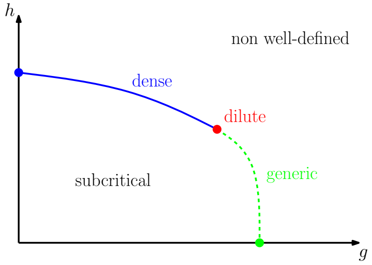
In BBD'16 we also studied the probability that, in an $O(n)$ configuration of fixed volume V, a uniformly chosen vertex belongs to the gasket. At a non generic critical point this probability decays as $V^{-\nu}$ for $V \to \infty$ with

$$\nu = \begin{cases} \frac{1-2b}{2} & \text{(dilute)}, \\ \frac{1-2b}{2(1-b)} & \text{(dense)}. \end{cases}$$

Phase diagram for fixed $n = 2\cos(\pi b) \in (0, 2)$



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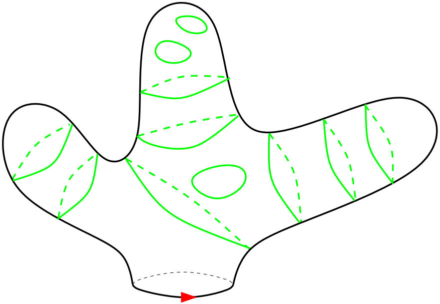
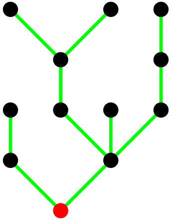
d_H	4	?	?	2
d_H^{gasket}	4	$3+2b$	$3-2b$	2
ν	0	$(1-2b)/2$	$(1-2b)/(2-2b)$	0

Nesting tree

Rather than considering distances (which we do not understand yet), we may study the structure of nestings between loops. In the planar case they are coded by the nesting tree.





Each node of the nesting tree corresponds to a map without loops but arbitrarily large faces. In particular the root of the tree corresponds to the gasket. What is the structure of the nesting tree at a critical point?

Nesting tree

Using the combinatorial decomposition of BBG'12 it can be seen that the nesting tree is a multitype branching process with infinitely many types (keeping track of loop perimeters). This does not seem very easy to analyze... (but see Chen-Curien-Maillard's approach in the next talk)

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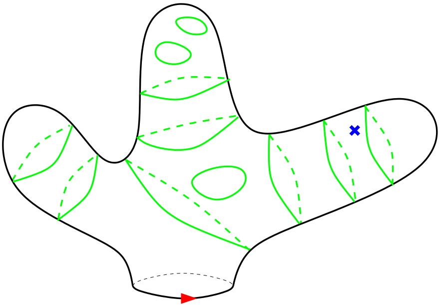
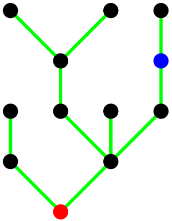
Here we consider the following simpler question: given an $O(n)$ configuration with fixed volume V, what is the distribution of the depth of a uniformly chosen vertex ? (i.e. the number of loops separating it from the outer face, or the height of the corresponding node in the nesting tree)





Main results

The depth is of order $\ln V$ and more precisely:

Theorem 1 (central limit theorem) [Borot-B.-Duplantier 2016]

Let $P_{V,\ell}$ be the depth of a uniformly chosen vertex of an $O(n)$ configuration of fixed volume V and perimeter ℓ. Then, at a non generic critical point, we have

$$\frac{P_{V,\ell} - \frac{c p_{\mathrm{opt}}}{\pi} \ln V}{\sqrt{\ln V}} \xrightarrow{(d)} \mathcal{N}(0, \sigma^2)$$

where

$$c = \begin{cases} 1 & \text{(dilute)} \\ \frac{1}{1-b} & \text{(dense)} \end{cases}, \qquad p_{\mathrm{opt}} = \frac{n}{\sqrt{4-n^2}}, \qquad \sigma^2 = \frac{4nc}{\pi(4-n^2)^{3/2}}.$$

Main results

Theorem 2 (large deviation principle) [Borot-B.-Duplantier 2016]

Let $P_{V,\ell}$ be the depth of a uniformly chosen vertex of an $O(n)$ configuration of fixed volume V and perimeter ℓ. Then, at a non generic critical point, we have

$$\mathbb{P}\left(P_{V,\ell} = \frac{c \ln V}{\pi} p\right) \sim C (\ln V)^{-1/2} V^{-\frac{c}{\pi} J(p)}, \qquad V \to \infty$$

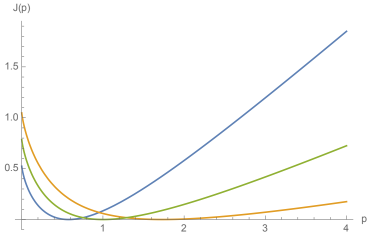
where $c = 1$ (dilute) or $c = 1/(1-b)$ (dense) and

$$J(p) = \ln\left(\frac{2}{n} \frac{p}{\sqrt{1+p^2}}\right) + \operatorname{arccot}(p) - \arccos(n/2).$$

Main results

Remarks:

- We have a similar statement when the perimeter ℓ is taken to be "large" (of order $V^{c/2}$), just replace J by $J/2$.
- Instead of marking a vertex, we may mark an inner face and obtain similar results.
- The large deviation function $J(p)$ is nonnegative, vanishes at $p = p_{\mathrm{opt}}$, satisfies $J''(p) = \frac{1}{p(p^2+1)}$, $J(p) \sim p \ln(2/n)$ for $p \to \infty$ and $J(0) = \arcsin(n/2) = \pi(1/2 - b)$ (consistently with the value of ν given before).



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Theorem 3 [MWW+BBD'16]

Let $\mathcal{N}_\delta$ be the number of loops surrounding a small ball of quantum area δ in a CLE$_\kappa$ coupled to Liouville quantum gravity (for suitable γ) on the Riemann sphere. Then we have

$$\mathbb{P}\left(\mathcal{N}_\delta = \frac{cp}{\pi}\ln(1/\delta)\right) \sim C\delta^{\frac{c}{\pi}J(p)}, \qquad \delta \to 0$$

with c, $J(p)$ as in Theorem 2.

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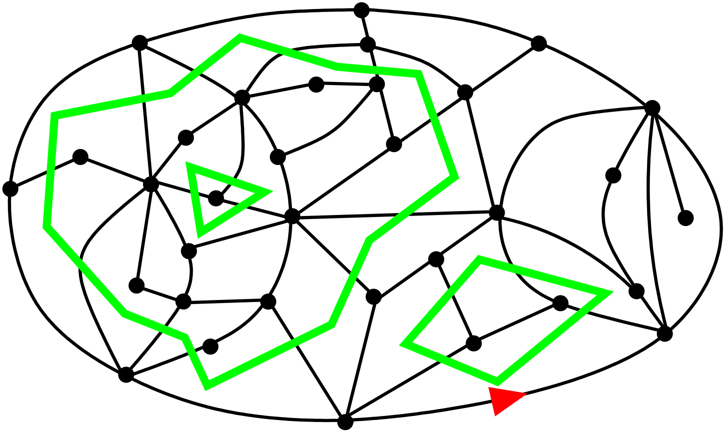
with c, $J(p)$ as in Theorem 2.

This supports the conjecture that the scaling limit of the critical $O(n)$ model on random maps is described by a CLE coupled to LQG.

Ingredients for the proof of Theorems 1 and 2

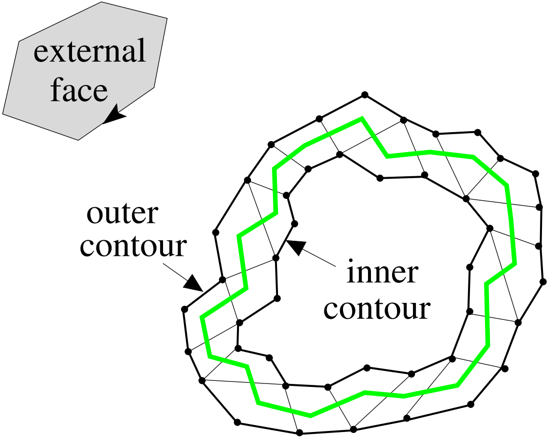
- the gasket decomposition [Borot-B.-Guitter 2012],
- a refinement allowing to control the depth of a marked vertex,
- a functional equation for the corresponding generating function that turns out to be exactly solvable (using an elliptic parametrization),
- asymptotics!

The gasket decomposition



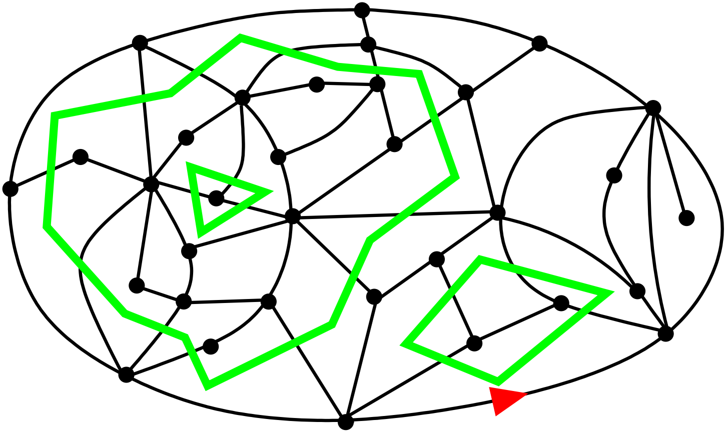
Start with a configuration of the O(n) loop model.

The gasket decomposition



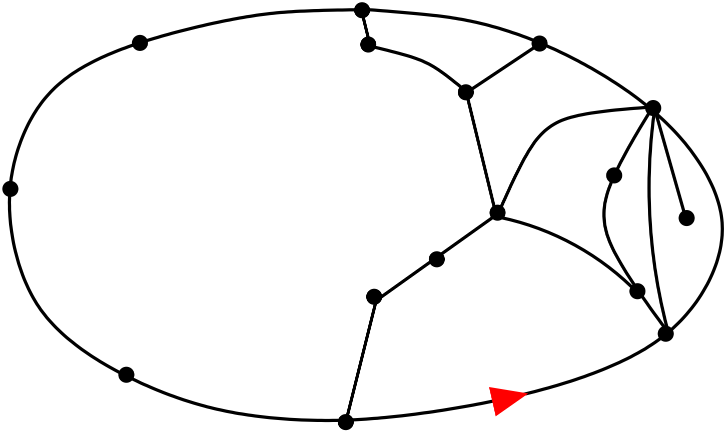
The faces visited by a loop forms a necklace.

The gasket decomposition



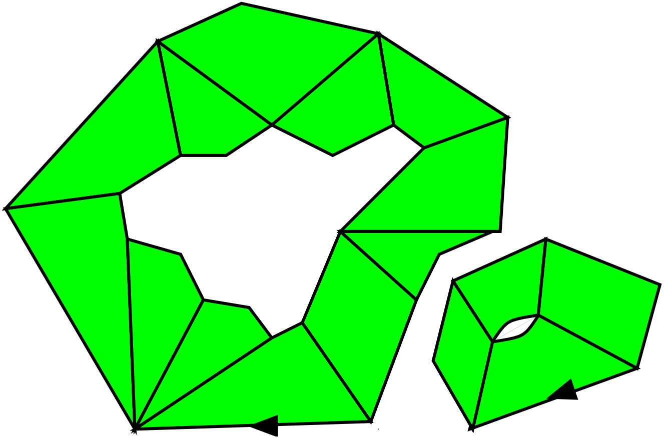
Cut along the outer and inner contours of each outermost loop.

The gasket decomposition



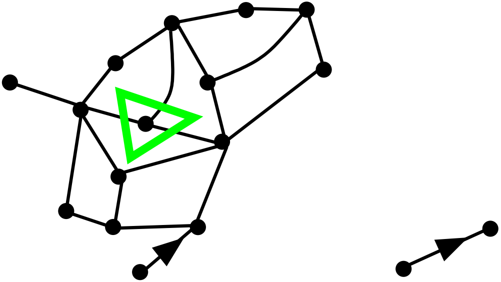
The outer component forms the gasket. It is a map without loops, with the same outer degree as the original map.

The gasket decomposition



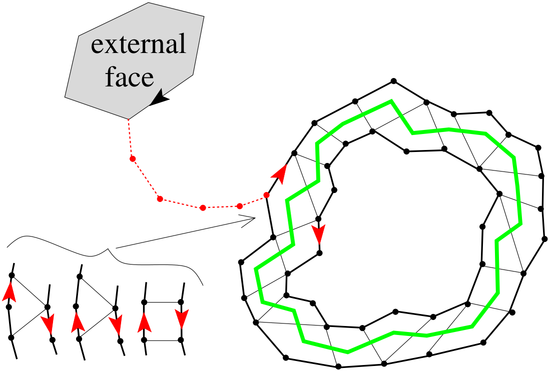
Each outermost loop forms a necklace (cyclic sequence of polygons glued side-by-side).

The gasket decomposition



Each outermost loop contains an internal configuration (of the same nature as our original object).

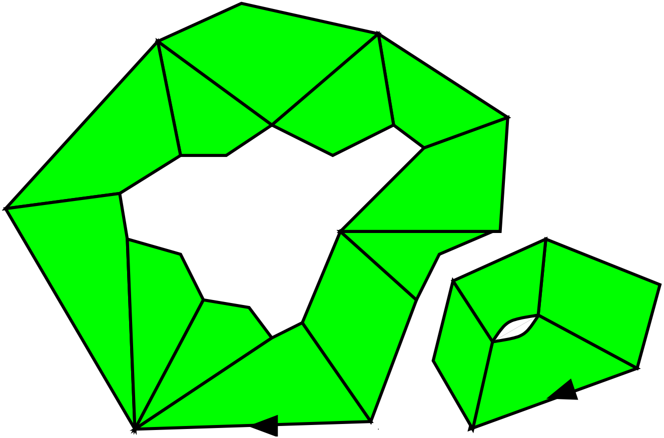
The gasket decomposition



There exists a well-defined rooting procedure:

- necklaces have a distinguished edge on the outer contour,
- internal configurations are rooted.

The gasket decomposition



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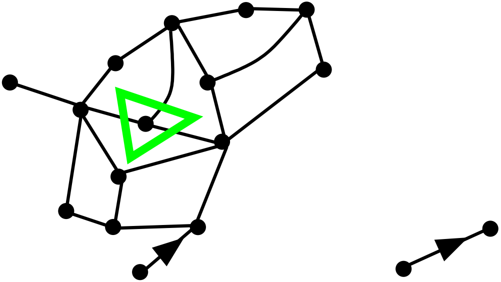

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The gasket decomposition

Bijection

$$\{\text{configurations}\} \simeq \{(\text{gasket, necklaces, internal configurations})\}$$

- A gasket is a map whose faces are either regular faces or holes.
- Each hole of degree $k \geq 1$ is associated with a necklace of outer length k.
- Each necklace of inner length $k' \geq 0$ is associated with an internal configuration of outer degree k'.

The gasket decomposition: consequences

To translate this bijection into equations, we introduce the partition function of Boltzmann maps with outer degree ℓ

$$\mathcal{F}_\ell(g_1, g_2, \ldots; u) = \sum_{\substack{\text{maps with} \\ \text{outer degree } \ell}} \prod_{k \geq 1} g_k^{\#\{\text{inner faces of degree } k\}} u^{\#\{\text{vertices}\}}$$

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and the necklace generating function

$$A(x, y) = \sum_{k \geq 1} \sum_{k' \geq 1} A_{k,k'} x^k y^{k'} := \sum_{\text{necklaces}} f(\text{necklace}) x^{\text{outer length}} y^{\text{inner length}}.$$

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In the case of the $O(n)$ loop model on triangulations, we have

$$A_{k,k'} = \binom{k + k' - 1}{k} h^{k+k'}$$

hence

$$A(x, y) = \frac{hx}{1 - h(x + y)}.$$

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The gasket decomposition: consequences

Proposition [BBG'12]

The partition function of the O(n) loop model on random maps is obtained from the generating function for Boltzmann maps via

$$F_\ell = \mathcal{F}_\ell(g_1, g_2, \ldots; u)$$

where the g_k's satisfy the fixed-point condition

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} \mathcal{F}_{k'}(g_1, g_2, \ldots; u).$$

In particular, the gasket is distributed according to the Boltzmann measure with face weights $g_1, g_2, \ldots$.

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where

$$F_\ell^\bullet = u \frac{\partial}{\partial u} F_\ell, \qquad \mathcal{F}_\ell^\bullet = u \frac{\partial}{\partial u} \mathcal{F}_\ell, \qquad \mathcal{F}_{\ell,k}^{(2)} = \frac{\partial}{\partial g_k} \mathcal{F}_\ell.$$

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Combinatorial meaning of this equation: the first term corresponds to the case where the marked vertex is within the gasket, and the second term to the case where it is surrounded by at least one loop.

Refinement

This suggests to consider the refined generating function $F_\ell^\bullet[s]$ determined by

$$F_\ell^\bullet[s] = \mathcal{F}_\ell^\bullet(g_1, g_2, \ldots; u) + n\, s \sum_{k,k' \geq 0} \mathcal{F}_{\ell,k}^{(2)}(g_1, g_2, \ldots; u) A_{k,k'} F_{k'}^\bullet[s]$$

where the parameter s counts the number of loops surrounding the marked vertex, i.e. its depth!

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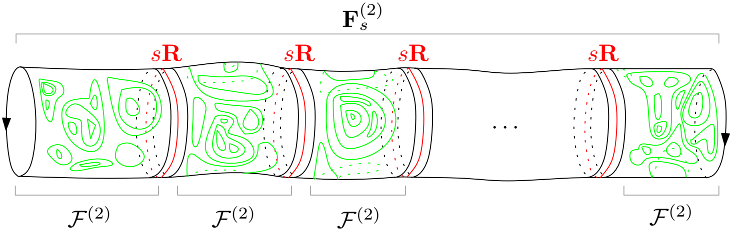
By a similar idea we may also count cylinders (two boundaries), etc.



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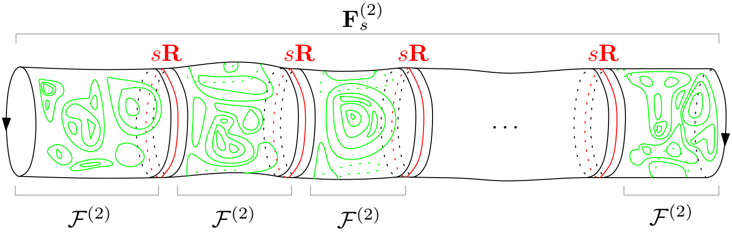

The refined functional equation can then be solved by a direct generalization of the method used for the non refined case.