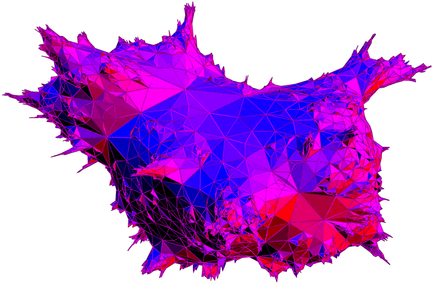
Bipolar orientations of planar maps

Mireille Bousquet-Mélou, CNRS, LaBRI, Université de Bordeaux

based on work with
Nicolas Bonichon & Éric Fusy (2010)
Éric Fusy & Kilian Raschel (2016)



Outline

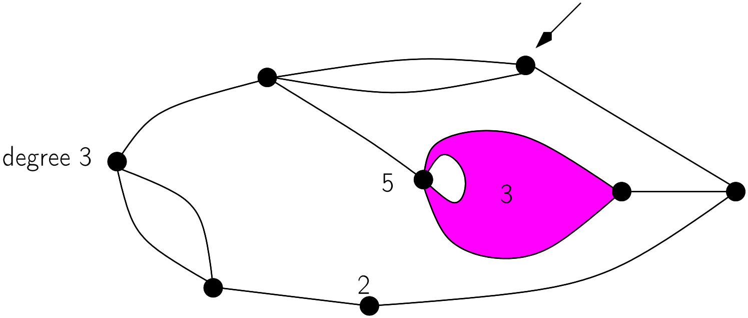
I. Introduction: planar maps with/without a structure

II. Bipolar maps [Bonichon, mbm, Fusy 10]

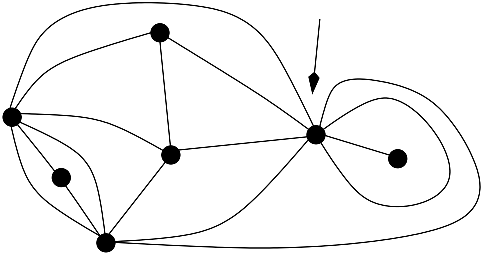
III. Bipolar triangulations, quadrangulations, etc. [mbm, Fusy, Raschel 16]

IV. Some experiments

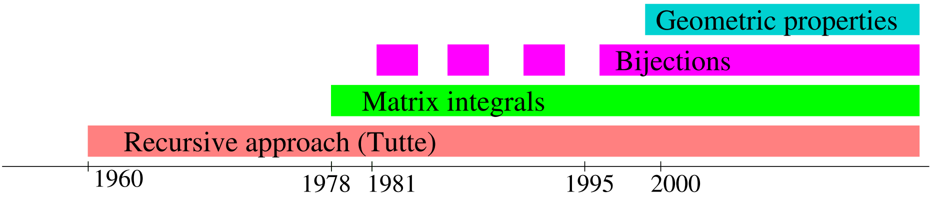
Rooted planar maps



With degree constraints: rooted triangulations



A chronology of planar maps



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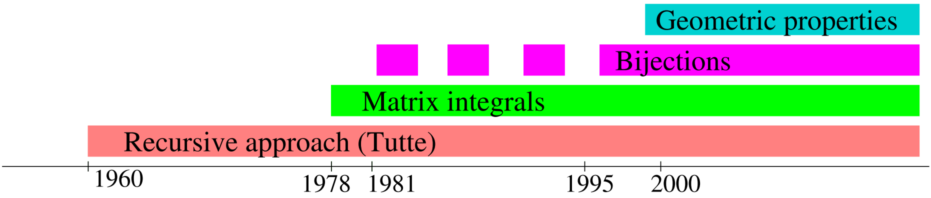

- Recursive approach: Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Bernardi, mbm...

- Matrix integrals: Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Zinn-Justin, Boulatov, Kazakov, Mehta, Duplantier, Bouttier, Di Francesco, Guitter, Eynard...

- Bijections: Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, mbm, Chapuy, Bettinelli...

- Geometric properties: Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Le Gall, Miermont, Curien, Addario-Berry, Albenque, Budd, Abraham...

Illustration: general planar maps and triangulations

	general maps (n edges)	triangulations (n vertices)
Recursive enumeration (algebraic series)	$a(n) = \frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$ [Tutte 63]	$\frac{2\cdot 4^{n-2}\,(3n-6)!!}{n!(n-2)!!}$ [Mullin, Nemeth, Schellenberg 70]
Bijective enumeration (connection with trees)	[Cori-Vauquelin 81, Schaeffer 98]	[Bouttier, Di Francesco, Guitter 02]
Limit behaviour (diameter $\simeq n^{1/4}$)	[Bettinelli, Jacob, Miermont 14]	[Le Gall 13]

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An algebraic series satisfies a polynomial equation. For instance, if $A(t) = \sum_n a(n)t^n$, then

$$27t^2A(t)^2 + (1-18t)A(t) + 16t - 1 = 0.$$

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0$$

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- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

$\Leftrightarrow$ linear recurrence for $a(n)$:

$$\sum_{k=0}^{e} a(n-k) p_k(n) = 0$$

Maps equipped with an additional structure

- Spanning trees [Mullin 67, Bernardi]
- Spanning forests [Bouttier et al., Sportiello et al., mbm-Courtiel]
- Proper colourings [Tutte 68-84]
- Self-avoiding walks [Duplantier-Kostov]
- Hard particles [Bouttier et al., mbm, Schaeffer, Jehanne]
- The q-state Potts model (equivalent to the Tutte polynomial) [Eynard-Bonnet 99, Baxter, Bermardi-mbm, Borot et al.]
- Loop models [Borot et al., Eynard, Kristjansen, Zinn-Justin]
- Bipolar orientations [Fusy et al., Bonichon et al., Felsner et al., Kenyon et al.]

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Many of these structures are special/limit cases of the Potts model.

Maps weighted by their Tutte (or: Potts) polynomial

	general maps | triangulations
recursive enumeration	[mbm-Bernardi 15] (differentially algebraic series)
bijective enumeration	in some cases
limit behaviour (diameter $\simeq n^?$)	?

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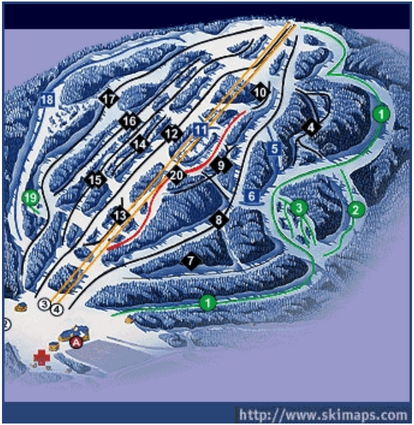
limit behaviour (diameter $\simeq n^?$)	? (1)

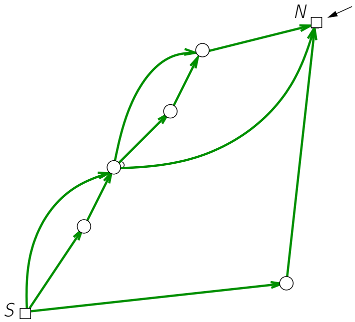
(1) recent results on the convergence in the peanosphere sense [Gwynne, Kassel, Kenyon, Miller, Sheffield, Wilson 15–]

II. Bipolar orientations: a structure with very rich underlying combinatorics



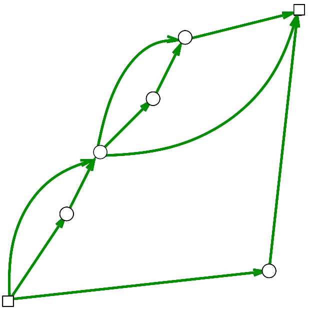
Bipolar maps: definition

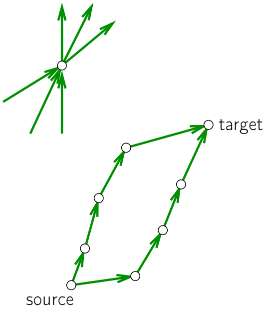
- a rooted planar map, with root vertex N
- two marked vertices S and N (the poles) in the outer face
- an acyclic orientation
- S is the only source and N the only target



Bipolar maps: basic facts

- simple orientations around a vertex/face
- dual orientation





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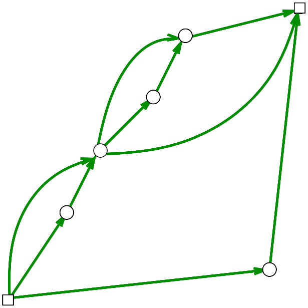
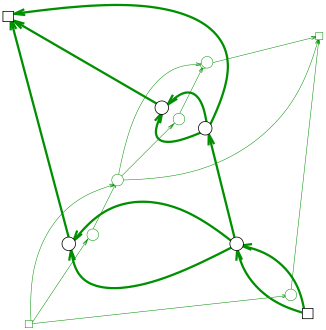
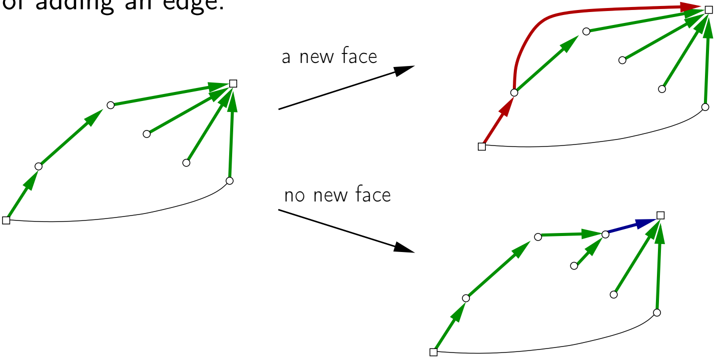




How to grow a bipolar map? a simple recursive structure

Two ways of adding an edge:



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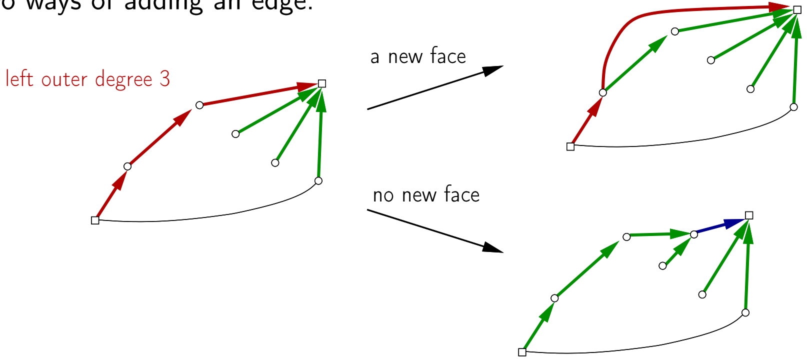

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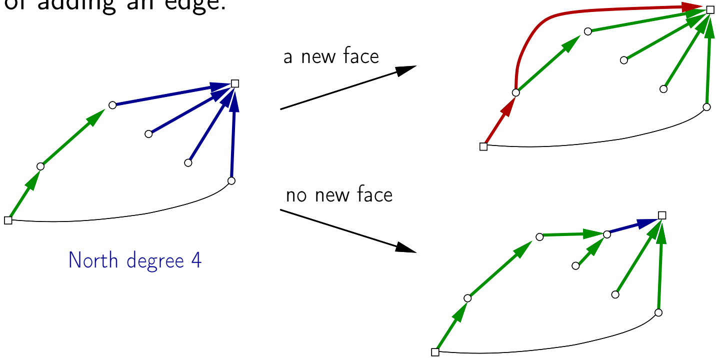

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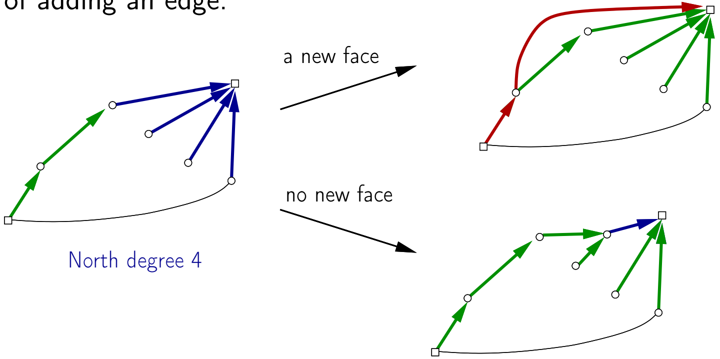

- Every bipolar map is obtained exactly once

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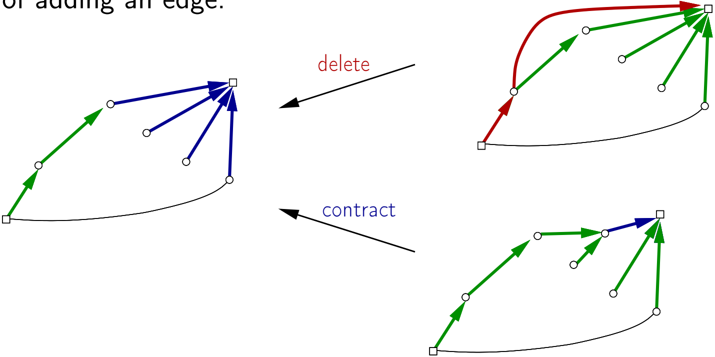

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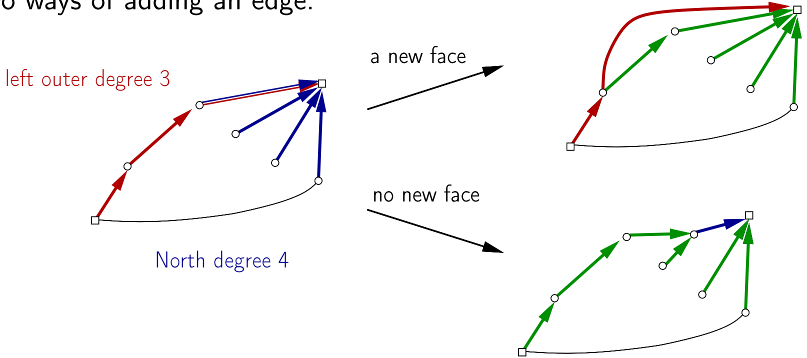

- Every bipolar map is obtained exactly once
- The number of descendants of a map with left outer degree i and North degree j is $i+j$, and their respective left outer degree/North degree are

$$(i,j) \rightsquigarrow \begin{cases} (1,j+1), & (2,j+1), & \dots & (i,j+1), \\ (i+1,j), & \dots & (i+1,2), & (i+1,1). \end{cases}$$

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Functional equation: Let $B(x,y) \equiv B(t;x,y)$ be the generating function of bipolar orientations counted by edges, left outer degree and North degree:

$$B(x,y) = \sum_O t^{e(0)} x^{lod(O)} y^{nd(O)}.$$

Then

$$B(x,y) = txy + txy\,\frac{B(x,y)-B(1,y)}{x-1} + txy\,\frac{B(x,y)-B(x,1)}{y-1}$$

A linear equation with two divided differences (or discrete derivatives).

Recursive enumeration of bipolar maps

$$B(t;x,y) \equiv B(x,y) = txy + txy\,\frac{B(x,y)-B(1,y)}{x-1} + txy\,\frac{B(x,y)-B(x,1)}{y-1}$$

Proposition [Baxter 01, mbm 11]

The number of bipolar maps with n edges is

$$b(n) = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

The associated series is D-finite:

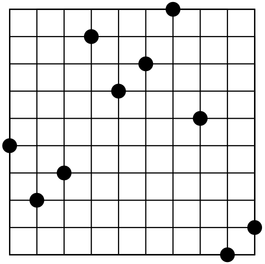
$$(n+6)(n+5)b(n+2) = (7n^2+49n+82)b(n+1) + 8(n+2)(n+1)b(n)$$

(+ Refinements with left outer degree and North degree).

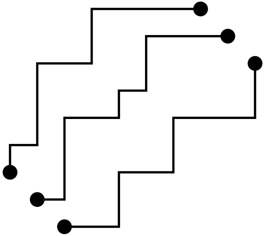
Towards bijections

Several other objects can be recursively described by isomorphic constructions:

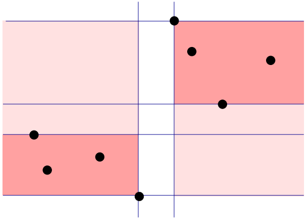
- Baxter permutations
- watermelons with three lines



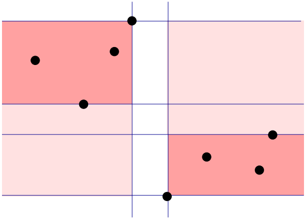
$\sigma = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$



Baxter permutations [Baxter 64]

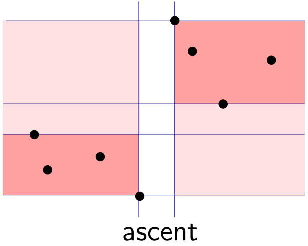


ascent

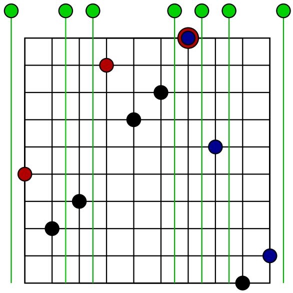


descent

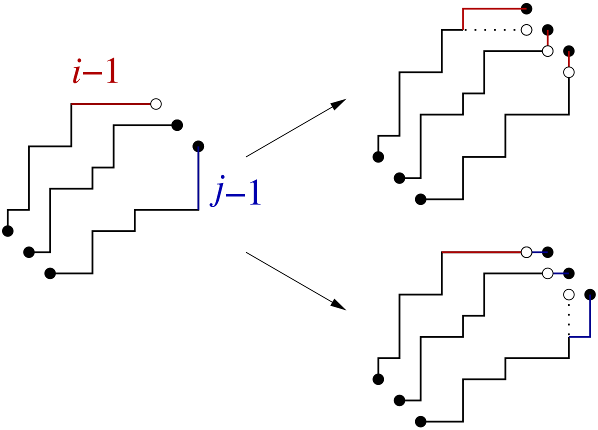
Baxter permutations [Baxter 64]



Recursive construction: insert $n+1$ in the permutation
i: number of left-to-right maxima
j: number of right-to left maxima

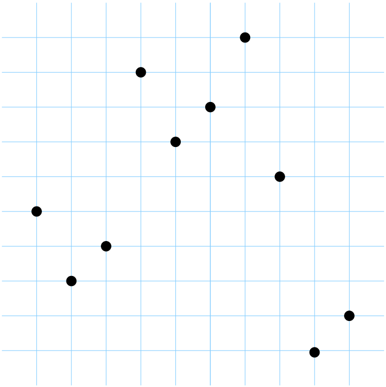


Watermelons



- Recursive construction: insert a North step in each path, or an East step
- Easy to count using the Lindström-Gessel-Viennot determinant.

From Baxter permutations to bipolar maps



[Bonichon, mbm & Fusy 10]

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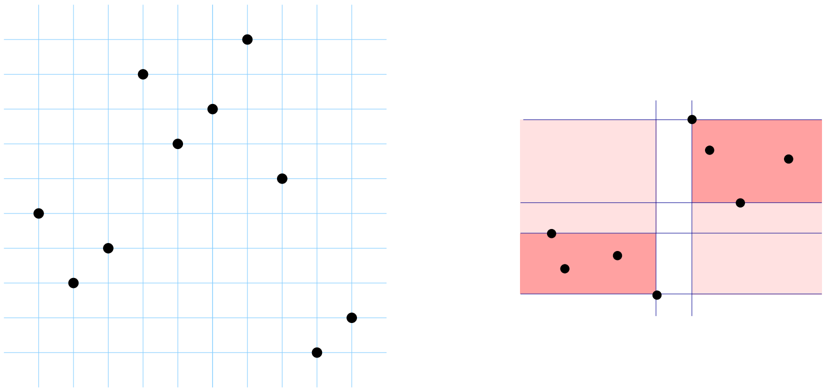

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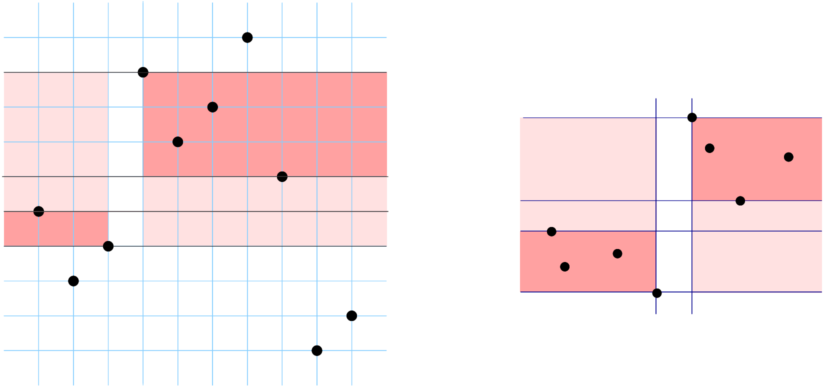

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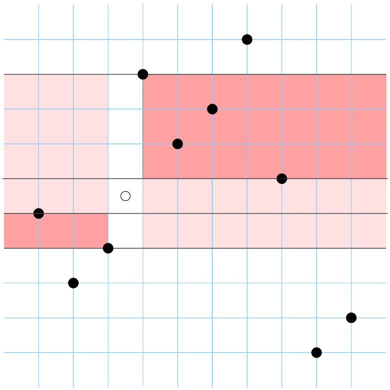

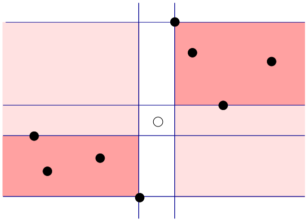

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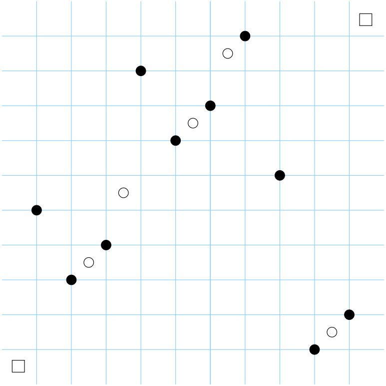

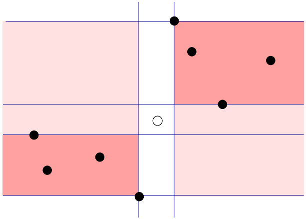

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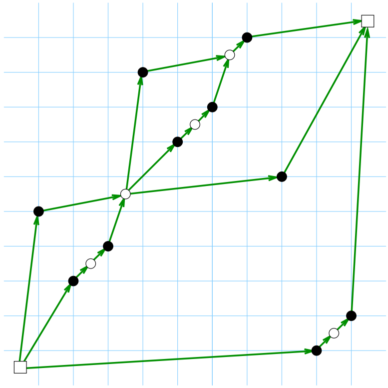

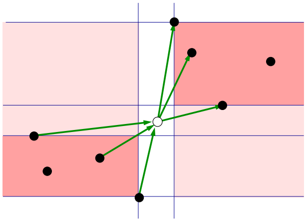

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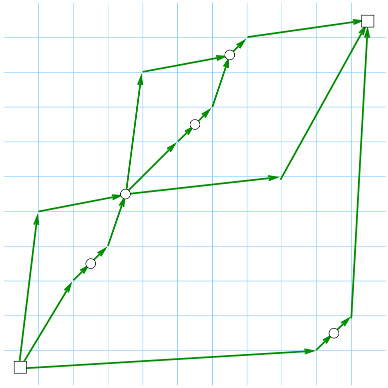

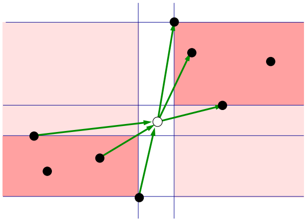

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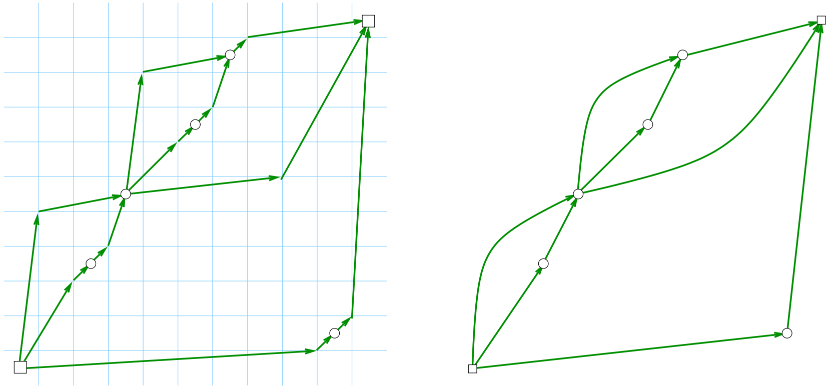

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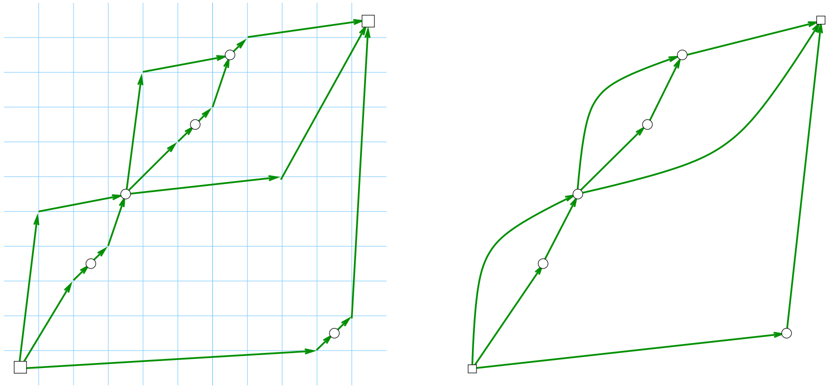

Properties:

- behaves well with respect to symmetries (including duality)
- the longest increasing subsequence of the permutation is the longest meridian (SN-path)

[Bonichon, mbm & Fusy 10]

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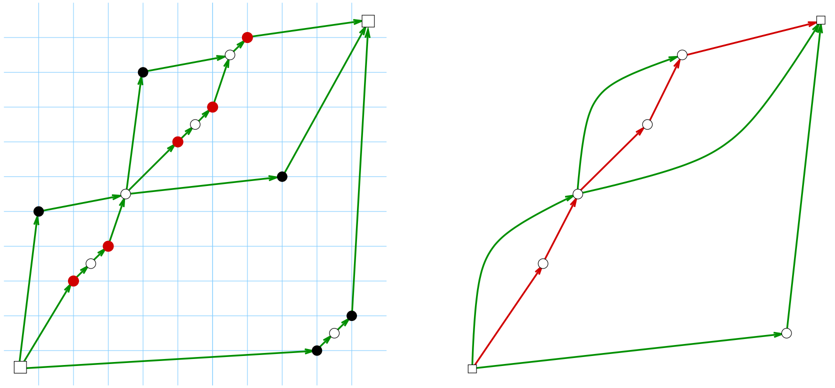

Properties:

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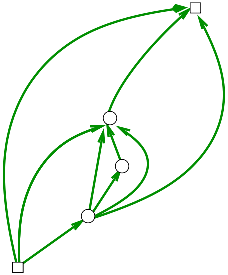
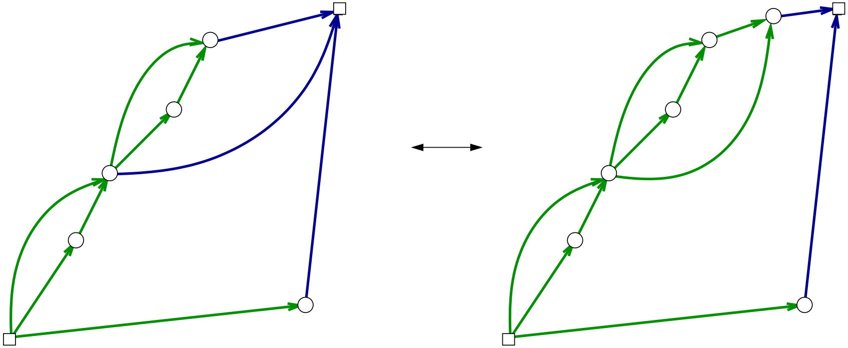
III. Bipolar triangulations, quadrangulations, etc.



Prescribing face degrees

- Due to edge contractions, the above recursive construction behaves badly (apart from triangulations [mbm 11])
- A new construction [Kenyon, Miller, Sheffield, Wilson, 15(a)]

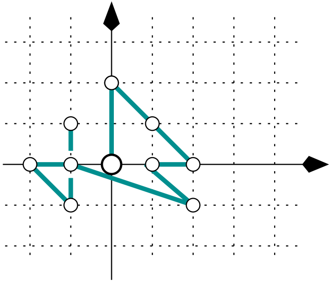


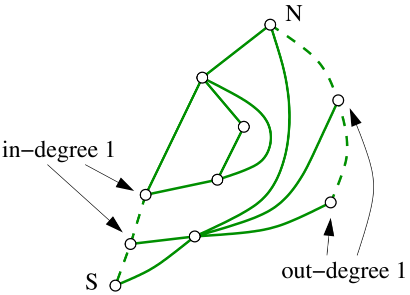
The KMSW construction

Take a lattice walk with two kinds of steps:

- SE steps $(1, -1)$
- NW steps $(-i, j)$ with $i, j \geq 0$

The construction starts from a walk and a bipolar map reduced to an edge, and yields an incomplete bipolar map.

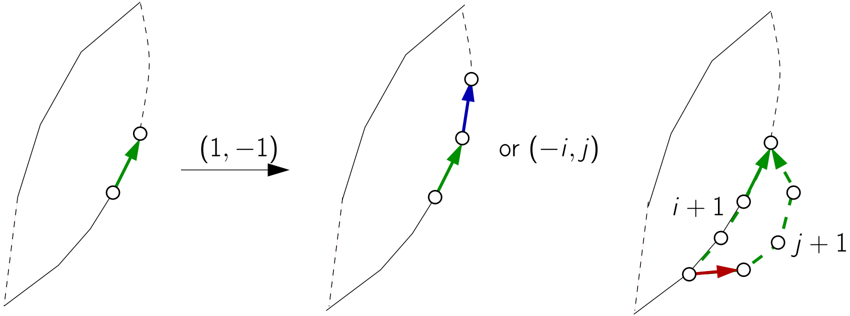




The KMSW construction

The construction starts from a walk and a bipolar map reduced to an edge, and yields an incomplete bipolar map.

- every SE step $(1, -1)$ creates an edge.
- every NW step $(-i, j)$ creates a face of degree $i + j + 2$ and an edge.



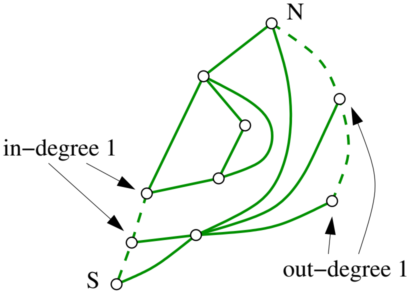
Example: walk

$(0, 2)(1, -1)(1, -1)(-1, 0)(1, -1)(-3, 1)(-1, 0)(1, -1)(0, 1)(0, 1)$

The KMSW construction

Proposition [Kenyon et al. 15(a)]

This construction is a bijection from lattice paths to incomplete bipolar maps.



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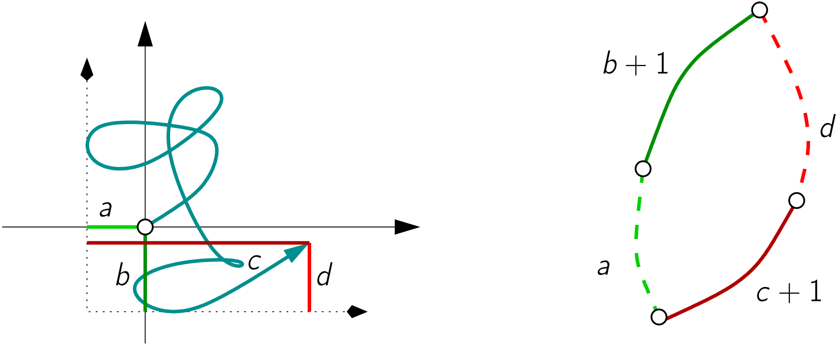
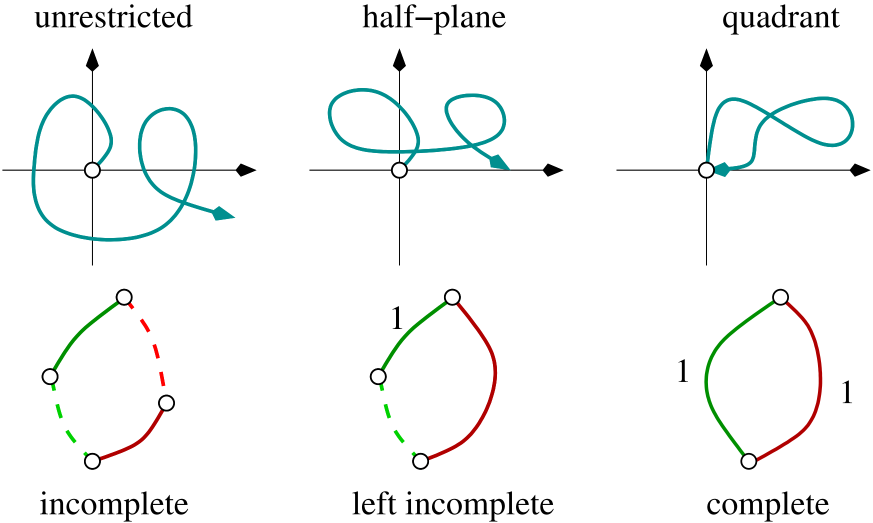
This construction is a bijection from lattice paths to incomplete bipolar maps.

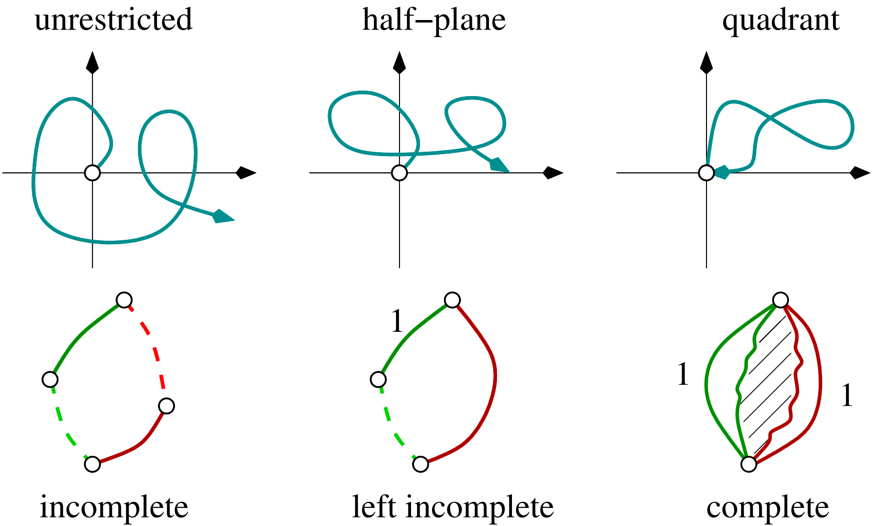
- steps ⇔ edges in the orientation (minus 1)
- steps $(-i, j)$ ⇔ faces of oriented degree $(i+1, j+1)$
- coordinates of the endpoints ⇔ left and right boundaries of the map.



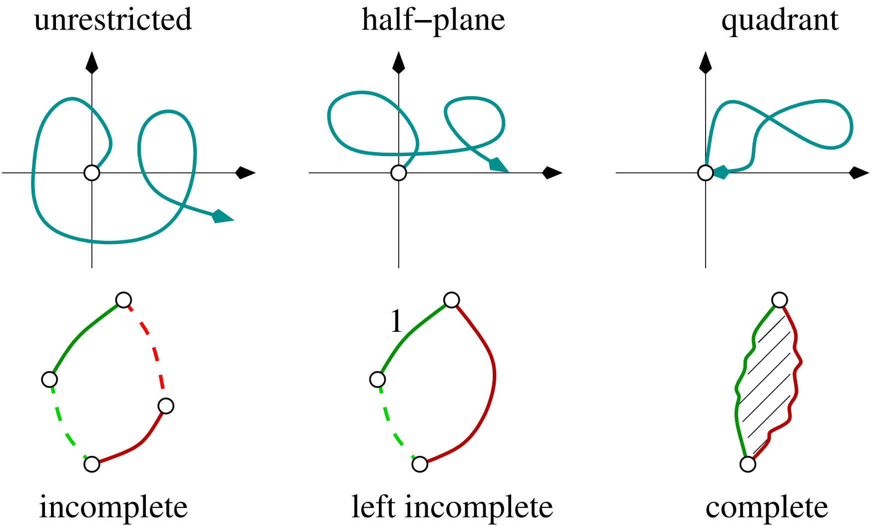
The KMSW construction: Some specializations



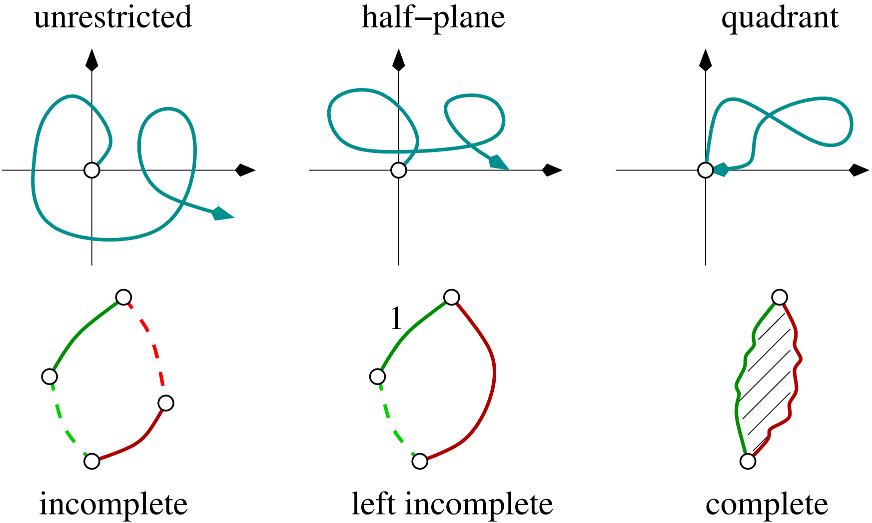
The KMSW construction: Some specializations



The KMSW construction: Some specializations



The KMSW construction: Some specializations



Enumeration?

Walk enumeration

Parameters and variables:

- steps/edges: variable t
- steps $(-i, j)$ (faces): variable z_{i+j} (degree selection)
- coordinates of the endpoint: variables x, y

Example: $w = (0,2)(1,-1)(1,-1)(-1,0)(1,-1)$

$$\Rightarrow \mathrm{weight}(w) = t^5 z_2 z_1 x^2 \bar{y}$$

where $\bar{y} := y^{-1}$.

The step polynomial (generating function of the steps)

$$S(x,y) := tx\bar{y} + t \sum_{i,j \geq 0} z_{i+j} \bar{x}^i y^j$$

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Unrestricted walks: a rational series

$$U(x,y) = \sum_{w} \mathrm{weight}(w) = \frac{1}{1 - S(x,y)}$$

Walk enumeration

The step polynomial:

$$S(x,y) := tx\bar{y} + t\sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j,$$

Unrestricted walks: a rational series

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Unrestricted walks: a rational series

$$U(x,y) = \frac{1}{1-S(x,y)}$$

Half-space walks: an algebraic series

$$H(x) = \frac{Y_0(x)}{tx},$$

where Y_0 is the unique series in t satisfying $1 = S\left(x, Y_0(x)\right)$.

$$Y_0(x) = tx + t^2 x\sum_{i\geq 0} z_i\bar{x}^i + O(t^3).$$

Walk enumeration

The step polynomial:

$$S(x,y) := tx\bar{y} + t\sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j,$$

Unrestricted walks: a rational series

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Half-space walks: an algebraic series

$$H(x) = \frac{Y_0(x)}{tx},$$

where Y_0 is the unique series in t satisfying $1 = S\left(x, Y_0(x)\right)$.

Quadrant walks: a D-finite series

$$Q = [x^0]\frac{Y_0(x)}{tx}\left(1 - \frac{1}{tx^2} + \sum_{i\geq 0}(i+1)z_i\bar{x}^{i+2}\right).$$

Walk enumeration: a proof!

Half-space walks: an algebraic series

$$H(x) = \frac{Y_0(x)}{tx},$$

where Y_0 is the unique series in t satisfying $1 = S(x, Y_0(x))$.

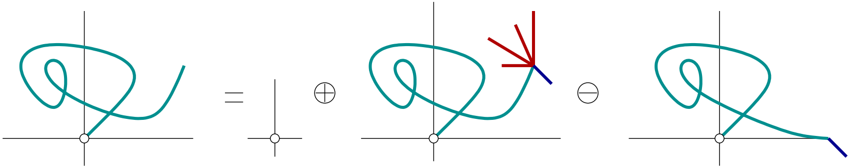
Proof. Functional equation for walks in the upper half-plane:

$$H(x,y) = 1 + H(x,y)S(x,y) - tx\bar{y}H(x,0),$$

where $S(x,y)$ is the step generating function. Equivalently,

$$(1 - S(x,y))\,H(x,y) = 1 - tx\bar{y}H(x,0).$$

Let $y = Y_0(x)$ cancel the l.h.s. Then $H(x,0) = Y_0/(tx)$. □



Walk enumeration: the quadrant case

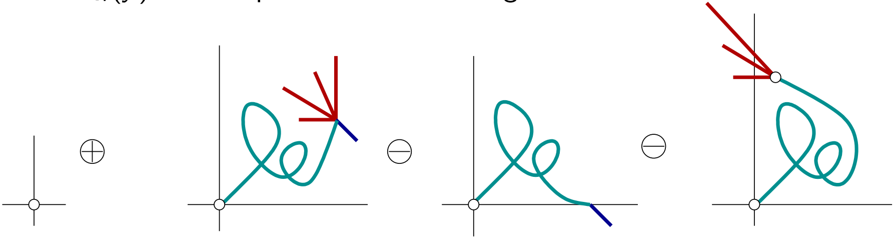
Quadrant walks: a D-finite series

$$Q = [x^0]\frac{Y_0(x)}{tx}\left(1-\frac{1}{tx^2}+\sum_i (i+1)z_i\bar{x}^{i+2}\right).$$

Functional equation:

$$Q(x,y) = 1 + Q(x,y)S(x,y) - tx\bar{y}Q(x,0) - t\sum_{i>0,j\geq 0} z_{i+j}\bar{x}^i y^j \left(Q_0(y) + xQ_1(y) + \cdots + x^{i-1}Q_{i-1}(y)\right)$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i.

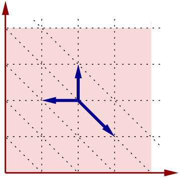


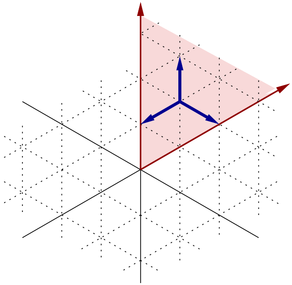
Walk enumeration: the quadrant case

Quadrant walks: a D-finite series

$$Q = [x^0]\frac{Y_0(x)}{tx}\left(1 - \frac{1}{tx^2} + \sum_i (i+1) z_i \bar{x}^{i+2}\right).$$

A simple case: triangulations. Take $z_1 = 1$ and $z_i = 0$ if $i \neq 1$





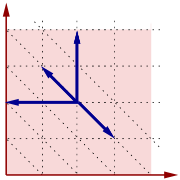
Walks confined to a Weyl chamber, solvable using the reflection principle [Gessel-Zeilberger 92]

Walk enumeration: the quadrant case

Quadrant walks: a D-finite series

$$Q = [x^0] \frac{Y_0(x)}{tx} \left(1 - \frac{1}{tx^2} + \sum_i (i+1) z_i \bar{x}^{i+2} \right).$$

Quadrangulations. Take $z_2 = 1$ and $z_i = 0$ if $i \neq 2$



Walk enumeration: the quadrant case

Quadrant walks: a D-finite series

$$Q = [x^0]\frac{Y_0(x)}{tx}\left(1 - \frac{1}{tx^2} + \sum_{i\geq 0}(i+1)z_i\bar{x}^{i+2}\right).$$

Functional equation:

$$\begin{aligned} Q(x,y) = 1 + Q(x,y)S(x,y) - tx\bar{y}Q(x,0) \\ - t\sum_{i>0,j\geq 0} z_{i+j}\bar{x}^i y^j \left(Q_0(y) + xQ_1(y) + \cdots + x^{i-1}Q_{i-1}(y)\right) \end{aligned}$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i.

- Walks with small steps in the quadrant:

Bostan, mbm, Fayolle, Kauers, Kourkova, Koutschan, Mishna, Raschel, Zeilberger...

- Walks with large steps in the quadrant: Fayolle & Raschel 15 – Bostan, mbm & Melczer 16

Recurrence relations for $(p+2)$-angulations by edges

Quadrant walks: a D-finite series

$$Q = [x^0]\frac{Y_0(x)}{tx}\left(1 - \frac{1}{tx^2} + \sum_{i\geq 0}(i+1)z_i\bar{x}^{i+2}\right).$$

- $p = 1$ (triangulations)

$$(n+3)(n+2)a(n+1) = 3(3n+2)(3n+1)a(n)$$

- $p = 2$ (quadrangulations)

$$(n+4)(n+3)^2a(n+2) = 4(2n+3)(n+3)(n+1)a(n+1)+12(2n+3)(2n+1)(n+1)a(n)$$

- $p = 3$ (pentagulations)

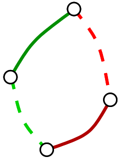
$$27(3n+8)(3n+4)(5n+3)(3n+5)^2(3n+7)^2(n+2)^2a(n+2) =$$
$$60(5n+7)(3n+5)(5n+9)(5n+6)(3n+4)(8+5n)(145n^3+532n^2+626n+233)a(n+1)$$
$$-800(5n+6)(5n+1)(5n+7)(5n+2)(5n+3)(5n+9)(5n+4)(8+5n)^2a(n)$$

Software: [Bostan, Lairez, Salvy 13]

Asymptotic results for $(p+2)$-angulations with n edges

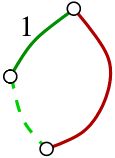
- Unconstrained walks (incomplete orientations)

$$u(n) \sim c_0\,(p+2)^n n^0$$



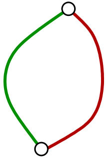
- Half-plane walks (left incomplete orientations)

$$h(n) \sim c_1\,(p+2)^n n^{-3/2}$$



- Quadrant walks (complete orientations)

$$q(n) \sim c_2\,\mu^n n^{-4}$$



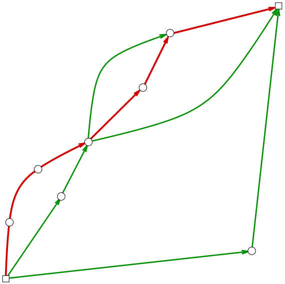
with

$$\mu = \frac{p+2}{p}\left(\frac{p(p+1)}{2}\right)^{2/(p+2)}$$

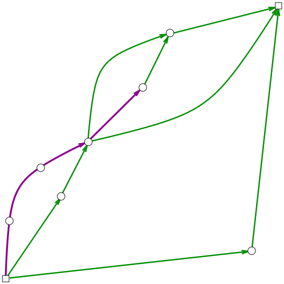
IV. Distance parameters: experiments

Distance parameters

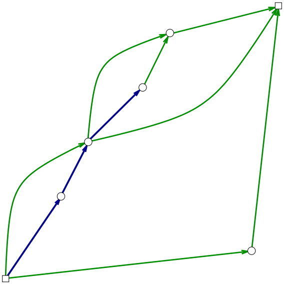
- longest meridian (SN oriented path)
- left path to the South pole
- shortest path to the South pole



longest meridian



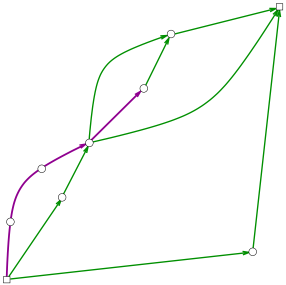
left path to the South pole



shortest path to the South pole

Distance parameters

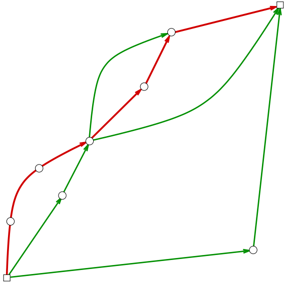
- longest meridian (SN oriented path)
- left path to the South pole $\sim n^{1/2}$
- shortest path to the South pole



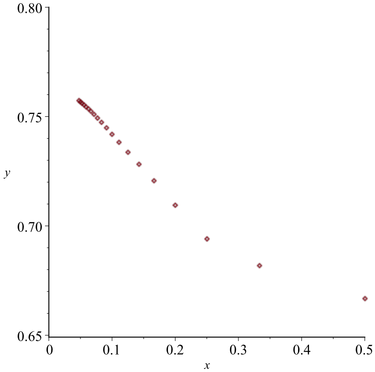
left path to the South pole

Distance parameters

- longest meridian (SN oriented path) $\sim n^{0.76\ldots}$
- left path to the South pole $\sim n^{1/2}$
- shortest path to the South pole

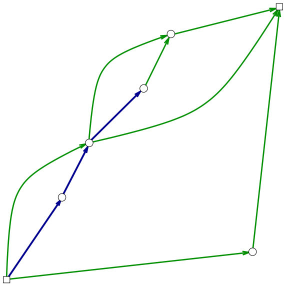


longest meridian



Distance parameters

- longest meridian (SN oriented path) $\sim n^{0.76\ldots}$
- left path to the South pole $\sim n^{1/2}$
- shortest path to the South pole $\sim n^{0.45\ldots}$



shortest path to the South pole

In conclusion

- Very rich combinatorics
- Connection with quadrant walks, with the longest increasing sequence in (Baxter) permutations...
- Enumerative results
- What about large random bipolar maps? large Baxter permutations?