Vanishing corrections for the position of a linear KPP front.

With: E. Brunet, S. Harris & M. Roberts.

KPP and all that

→ Introduced in 1937 by Fisher / Kolmogorov Petrovski Piskunov

$$\boxed{x \in \mathbb{R},\ t \geq 0 \quad \partial_t u = \underbrace{\partial^2_{xx} u}_{\text{Diffusion}} + \underbrace{u(1-u)}_{\text{reaction}}}$$

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(More generally $x \in \mathbb{R}^d$ $\partial_t u = \Delta u + u(1-u)$)

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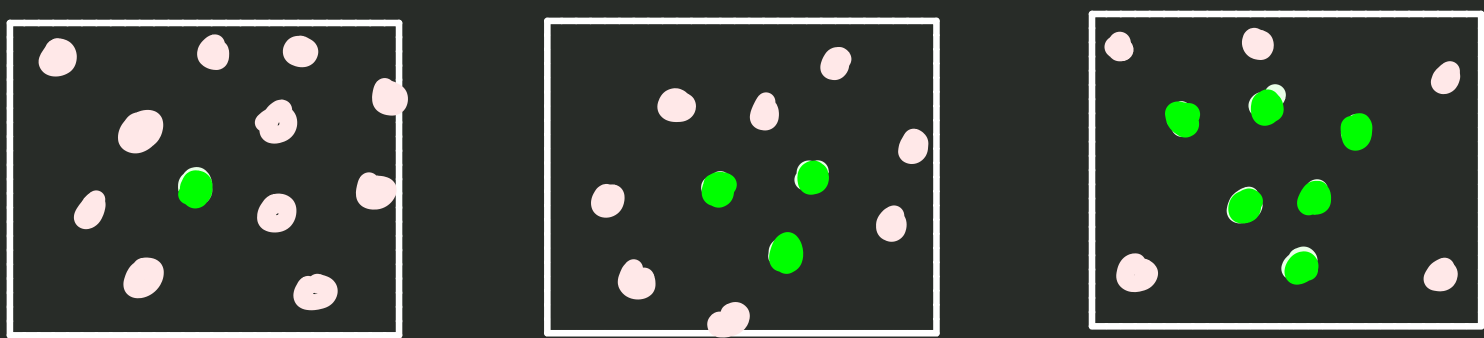
Ex: $A + B \to 2A$ in the limit of infinite densities.



KPP and all that

KPP '37 $\boxed{x \in \mathbb{R},\ t \geq 0 \quad \partial_t u = \partial^2_{xx} u + u(1-u)}$

0 : $u(x,t) \equiv 0$ and $u(x,t) \equiv 1$ are (trivial) solutions.
unstable stable.

KPP and all that

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① : A Travelling Wave solution is $\boxed{u(x,t) = \underbrace{w}_{\text{shape}}(x - \underbrace{c}_{\text{speed}}t)}$

$$\boxed{w \text{ solves } 0 = w'' + cw' + w(1-w)}$$

Q: For which c $\exists w$ s.t. w $1 \to 0$ with speed c ?

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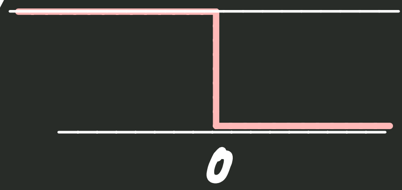
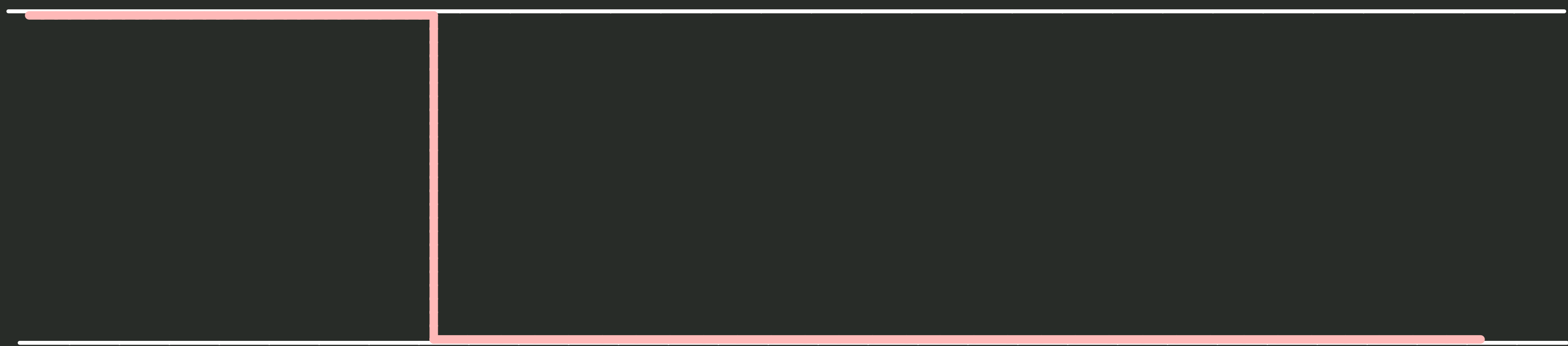
Q: For which c $\exists w$ s.t w $1 \searrow 0$ with speed c ?

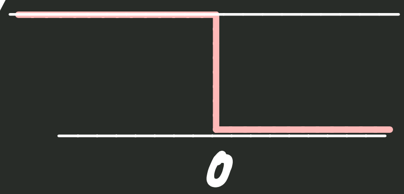
A : $\exists w$, TW of speed c, from $1 \searrow 0 \iff c \geqslant 2$
The TW of speed c is unique up to horizontal shift
w^* = critical TW of speed 2 .

KPP and all that

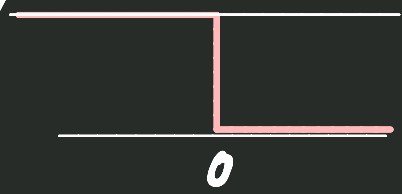
② what if $u(0,x) =$ () ?

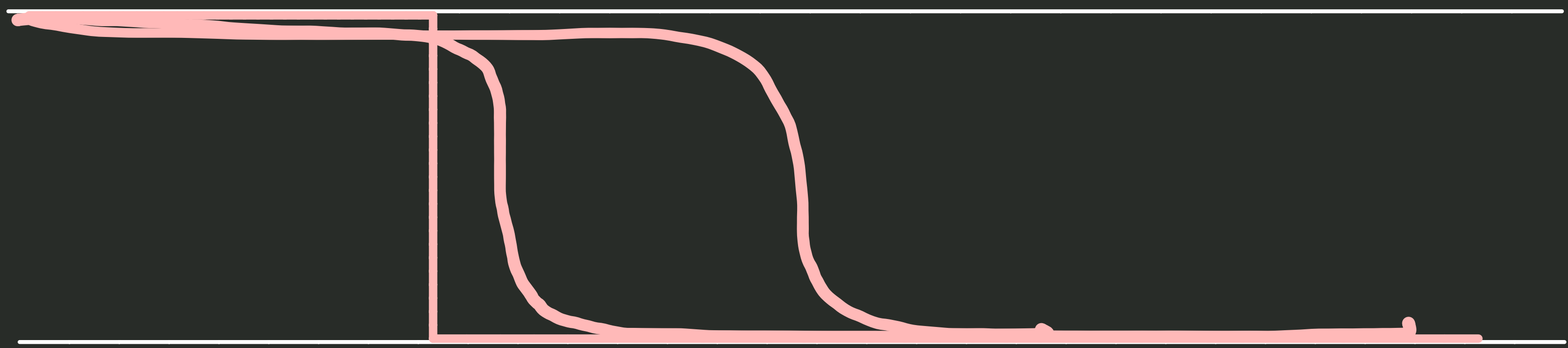


KPP and all that

② what if $u(0,x) =$ () ?

KPP and all that

② what if $u(0,x) = \Bigg($  $\Bigg)$?



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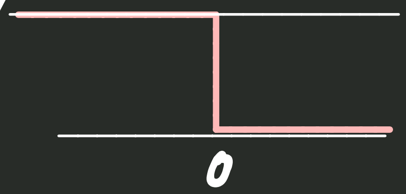
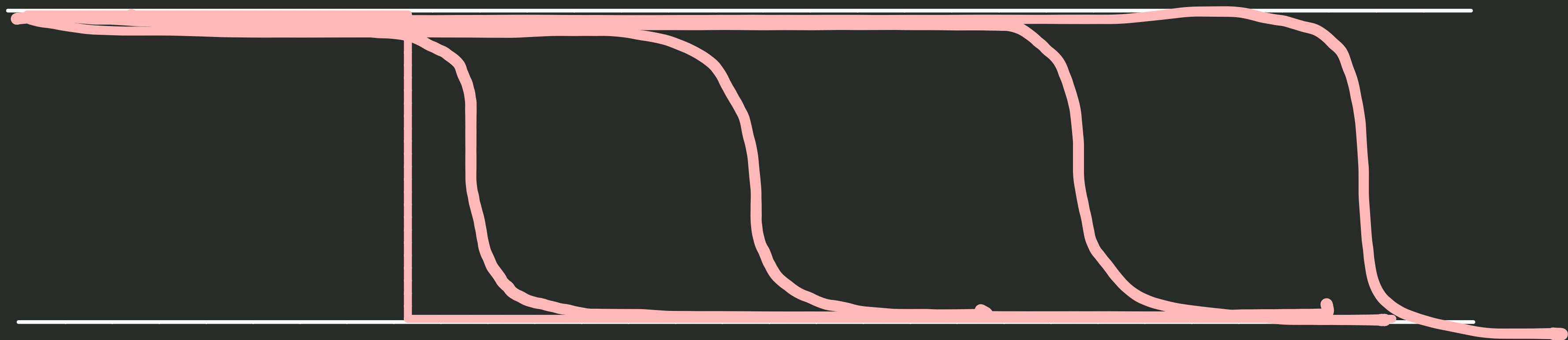
② what if $u(0,x) =$ () ?



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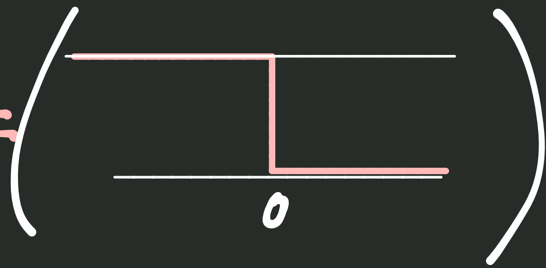
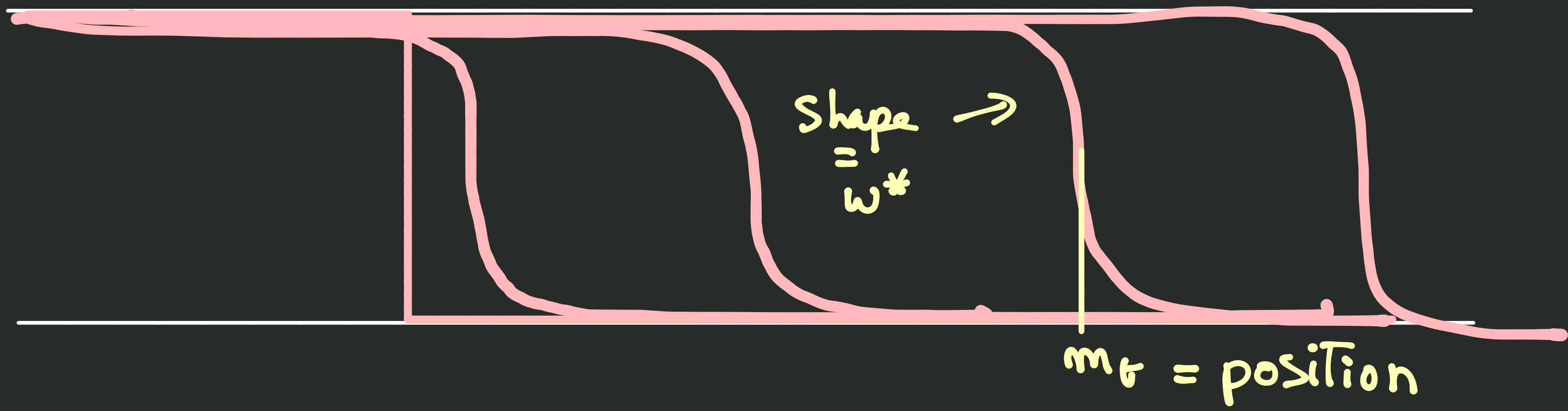
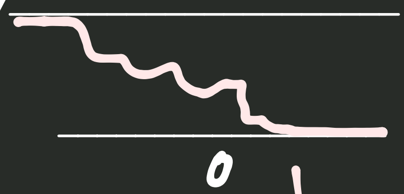
② what if $u(0,x) =$  ?



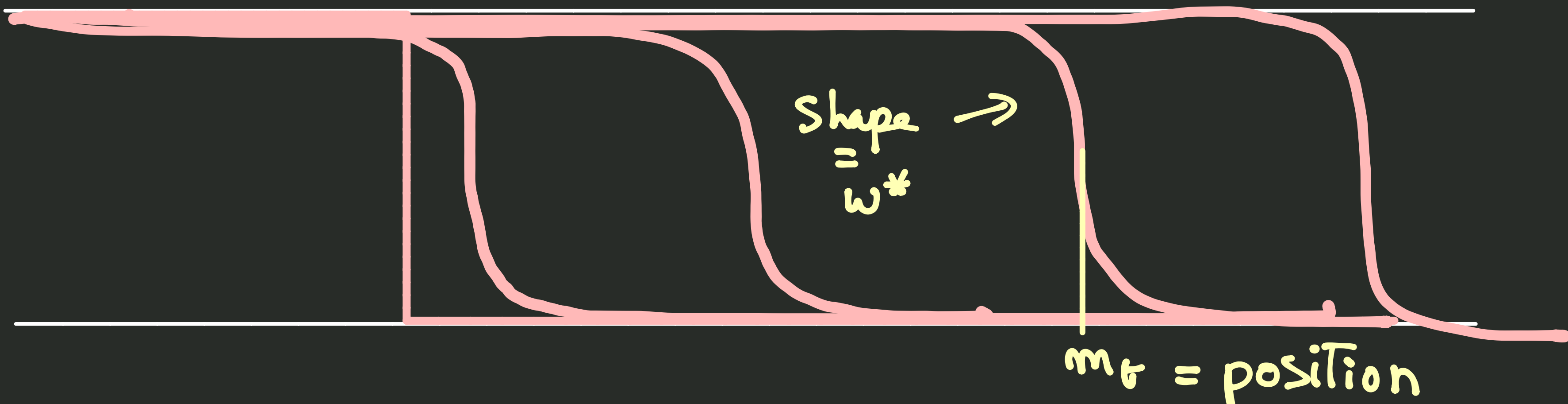
KPP '37 $\exists\, m_t$ such that

$$\boxed{u(x+m_t, t) \longrightarrow w^*(x) \text{ unif in } x. \quad \text{and} \quad m_t = 2t + o(t)}$$

KPP and Bramson

② what if $u(0,x) = \Big($  $\Big)$?

$\hookrightarrow$ decrease fast enough



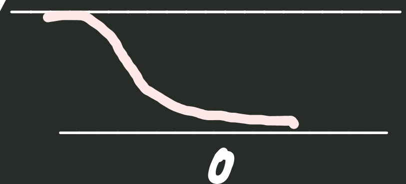
Bramson '83 $\forall$ $m_t = 2t - \frac{3}{2}\log t + c + o(1)$

$u(x+m_t,t) \longrightarrow w^*(x)$ unif in x.

$c \Leftrightarrow$ the horizontal shift.

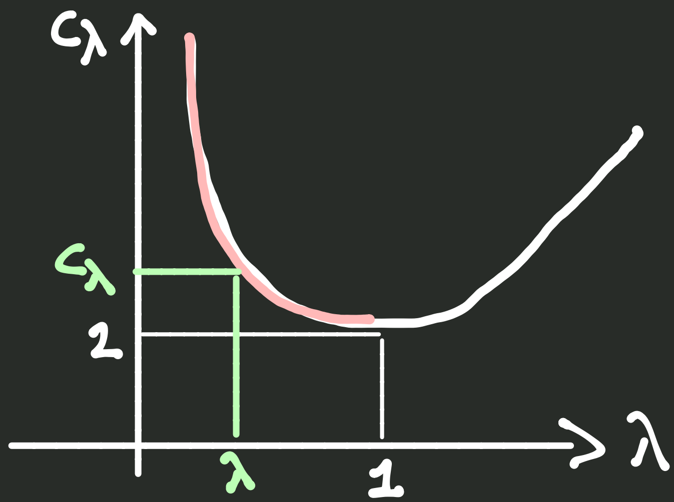
KPP and Bramson

In fact Speed selection ~ asymptotic behavior of $u(0,x)$

if $u(0,x) = \Bigg($  $\Bigg) \Longrightarrow u(x+m_t^{(\lambda)},t) \to w_\lambda(x)$

$\sim e^{-\lambda x}$

where $m_t^{(\lambda)} = c_\lambda t + o(t)$

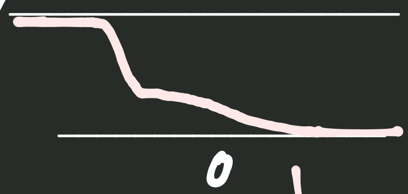


$\left(\text{ex: } c_\lambda = \lambda + \frac{1}{\lambda}\right)$

$\Longrightarrow$ TW with speed c_λ

$\sim e^{-\lambda x}$ at ∞

Ebert - Van Saarloos conjecture.

② what if $u(0,x) =$ () ?

$\hookrightarrow$ decreases fast enough

Define front position:

$m_\alpha(t) : u(m_\alpha(t), t) = \alpha$

quantile

median

$m_t = -\int x \, \partial_x u(x,t) \, dx$

Expectation

$\partial_x^2 u(m_t, x) = 0.$

inflection.

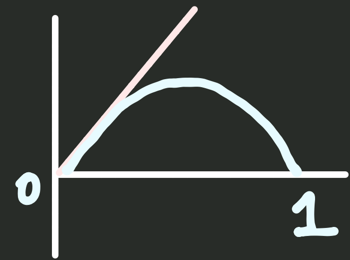
Ebert-Van Saarloos $\quad m_\alpha(t) = 2t - \frac{3}{2}\log t + c - 3\sqrt{\frac{\pi}{t}} + \cdots$

$u(x + m_t, t) \longrightarrow w^*(x)$ unif in x.

$c \Leftrightarrow$ the horizontal shift.

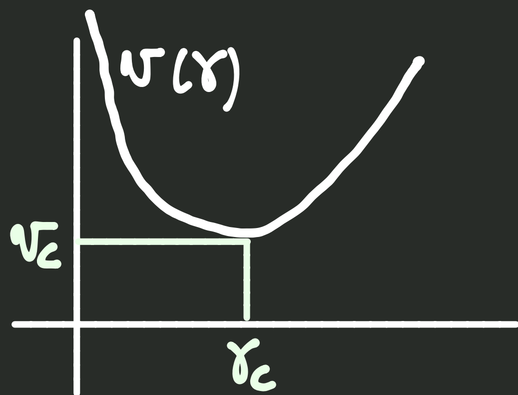
KPP and Bramson

<u>Universality</u> $\partial_t u = \partial^2_{xx} u + f(u)$ f:



$$m_\alpha(t) = 2t - \frac{3}{2}\log t + c - 3\sqrt{\frac{\pi}{t}} + \cdots$$

More generally: for "pulled" fronts



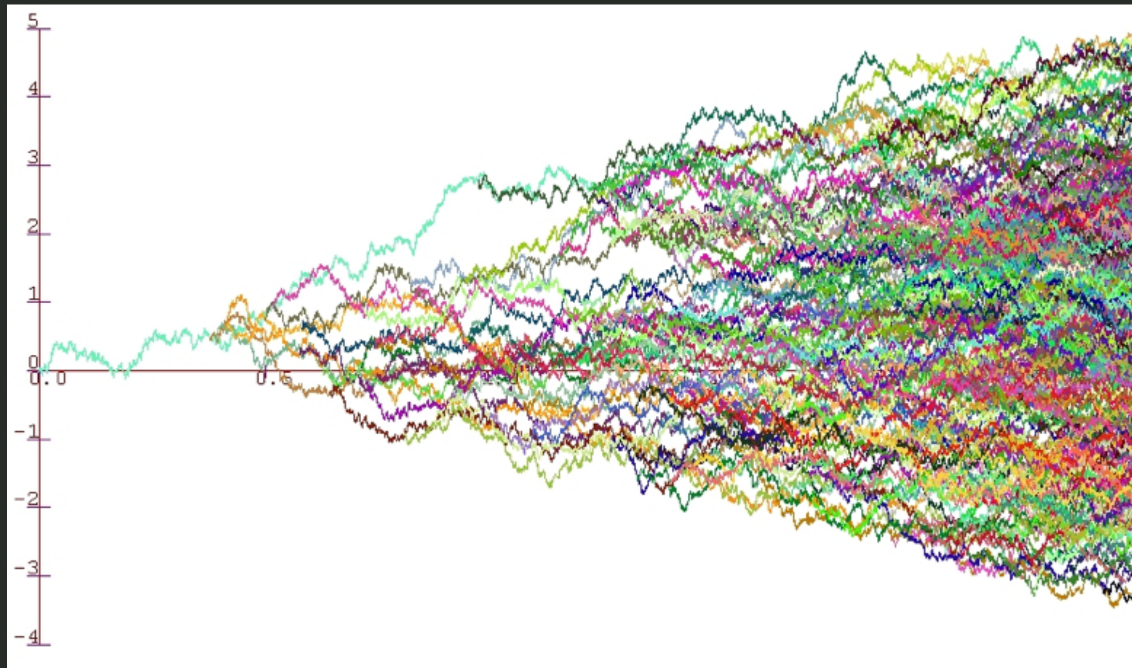
and

$$m_\alpha(t) = v_c t - \frac{3}{2\gamma_c}\log t + c - 3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-1/2} + \cdots$$

KPP & BBM

BBM = branching Brownian motion

- ptc $\sim \sqrt{2}\,B_t$ (so that generator is Δ, not $\frac{1}{2}\Delta$)
- split into two at rate 1
- ptc are independent.



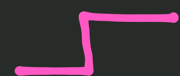
$X_i(t)$ = position of ptc at time t

Let $\varphi(x): \mathbb{R} \Rightarrow [0,1]$

McKean '75

$$u(x,t) = \mathbb{E}_0\Big[\prod_{i=1}^{N(t)} \varphi(x - X_i(t))\Big]$$

Solves $\begin{cases} \partial_t u = \partial^2_{xx} u - u(1-u) \\ u(x,0) = \varphi(x) \end{cases}$ KPP

ex $\varphi(x) =$ 

$u(x,t) = \mathbb{P}_0(\max X_i(t) \le x)$

$\Rightarrow$ cdf of $\max X_i(t)$!

Bramson for probabilists

M_t = position of rightmost ptc
$= \max_i \{ X_i(t) \}$

$u(t,x) = 1 - \mathbb{P}(M_t < x)$
solves F-KPP

Take $m_t = 2t - \frac{3}{2}\log t + c$

$$\boxed{u(x+m_t, t) = \mathbb{P}(M_t - m_t \leq x) \to w^*(x)}$$

$\Rightarrow$ $(M_t - m_t)$ converges in dist. to W

and $\mathbb{P}(W \leq x)$ is the critical travelling wave.

A linear problem

$\partial_t u = \partial^2_{xx} u + u - \cancel{u^2}$ $\Rightarrow$ unbounded growth.

$u(x,t)$ = Expected density of ptc near x.
in BBM.

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$$\begin{cases} \partial_t u = \partial^2_{xx} u + u, & x > m_t \\ u(m_t, t) = 0. \end{cases}$$

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Q: Can we chose m_t so that $u(m_t + x, t) \to \phi(x)$?

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Q: • Can we chose m_t so that $u(m_t + x, t) \to \phi(x)$?

• Can we chose m_t so that

. $u(m_t + 1, t) = 1 \quad \forall t$?

. $u'(m_t, t) = 1 \ \forall t$?

. $\int_0^\infty u(m_t + x, t)\, dx = 1$?

Theorem (Berestycki Brunet Harris Roberts)

Suppose $h(x) = O\left(x^{\nu} e^{-x}\right)$, $\nu < -2$

and write $m_t = 2t - \frac{3}{2}\log t + a + r(t)$

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with $\alpha = \frac{e^{-a-\Delta}}{2\sqrt{\pi}} \int h(y) y e^{-y} \psi_{\infty}(y)\, dy$

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If furthermore $r''(t) = O\left(t^{-2-\eta}\right)$ $\eta > 0$ (ex: $r(t) = t^{-\eta}$)

$$\boxed{\begin{aligned} U(x+m_t,t) = \alpha\, x e^{-x} \Big[& 1 - r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}} + O\left(t^{1+\nu/2}\right) + O\left(\frac{1}{t^{1/2+\eta}}\right) \\ & + O\left(\frac{\log t}{t}\right) + O\left(r(t)^2\right) \Big] \end{aligned}}$$

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chose $r(t) = -\frac{3\sqrt{\pi}}{\sqrt{t}} + o\left(t^{-1/2}\right)$
To kill this Term

$\implies$ yields the fastest convergence!

$$u(x+m_t,t) = \alpha\, x e^{-x}\Big[1 \boxed{- r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}}} + O\big(t^{1+\nu/2}\big) + O\Big(\frac{1}{t^{1/2+\eta}}\Big)$$
$$+ O\Big(\frac{\log t}{t}\Big) + O\big(r(t)^2\big)\Big]$$

chose $r(t) = -\frac{3\sqrt{\pi}}{\sqrt{t}} + o(t^{-1/2})$

To kill this Term

$\Longrightarrow$ yields the fastest convergence!

- For most choice of $r(t)$ $\quad |u(x+m_t,t) - \phi(x)| = O(t^{-1/2})$

- For $r(t)$ well chosen $\left(\approx -\frac{3\sqrt{\pi}}{\sqrt{t}}\right)$ if $\nu < -4$

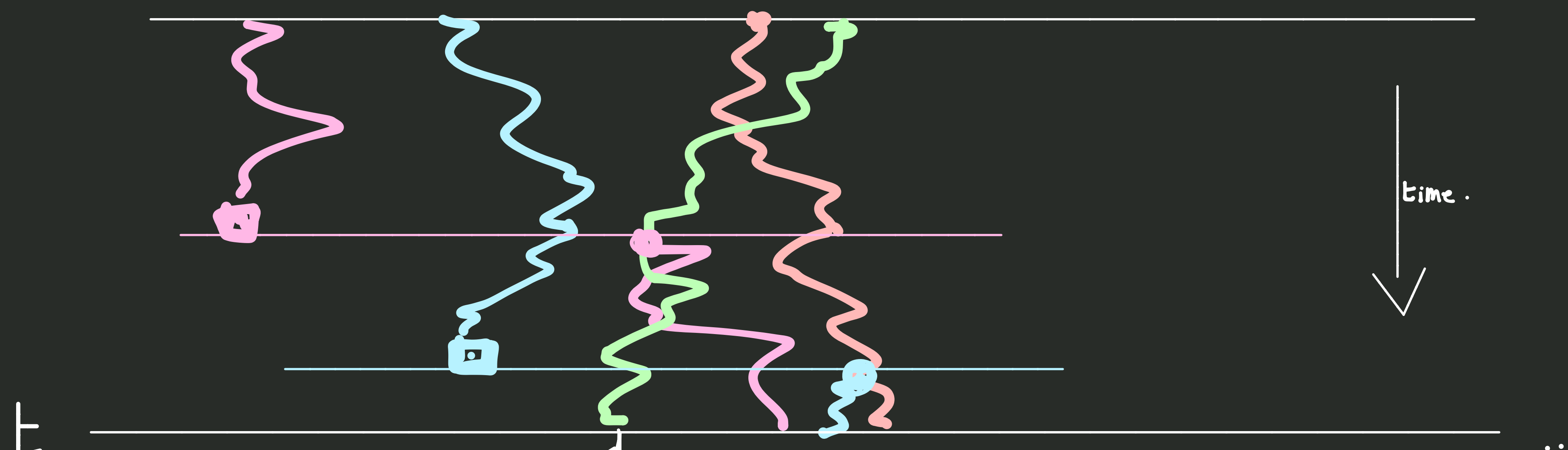
  can have $\quad |u(x+m_t,t) - \phi(x)| = O(\log t / t)$

Why is this interresting ?

N-BBM : when a particle branches kill the leftmost to keep $\# = N$.

$\rightarrow$ model of pop. under selection.

$\mu_t^N(\cdot) =$ empirical meas. (each ptc $= 1/N$)



$L_t^N =$ pos. of leftmost

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N-BBM : when a particle branches kill the leftmost to keep $\# = N$.

$\to$ model of pop. under selection.

L^N_t = pos. of leftmost $\qquad \mu^N_t(\cdot)$ = empirical meas. (each ptc = $1/N$)

<u>Conjecture</u> : On $[0T]$, $(L^N_t, \mu^N_t(dx))$ converges weakly to $(L_t, u(t,x)dx)$ solution of free boundary pb:

- $\partial_t u = \Delta u + u \quad \forall x > L_t$
- $u(L_t, t) = 0$
- $\forall t \int_{L_t}^{\infty} dy\, u(t,y)dy = 1$

(hydrodynamic limit)

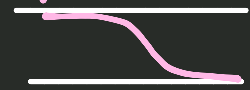
We believe $L_t = 2t - \frac{3}{2}\log t + c - \frac{3\sqrt{\pi}}{\sqrt{t}} + \cdots$

$\Rightarrow$ Durrett - Remenik prove it for a related model.

Summary

1) For F-KPP $\partial_t u = \Delta u + u(1-u)$ with $u_0(x) =$

fast decreasing



$m_\alpha(t) = \inf\{x : u(t,x) = \alpha\}$

Then $m_\alpha(t) = 2t - \frac{3}{2}\log t + c_\alpha + \frac{\gamma_\alpha}{\sqrt{t}} + O(t^{-0.99})$

→ recent work. Nolen, Roquejoffre, Ryzhik. But, private communication, in fact $\gamma_\alpha = -3\sqrt{\pi}$!

2) The linear free boundary pb. $(L_t, u(t,x))$

$$\begin{cases} \partial_t u = \Delta u + u \\ u(L_t, t) = 0 \\ + \text{extra cond.} \end{cases}$$

Soft: $u(L_t + x, t)$ converges

hard $\int_{L_t}^{\infty} u(y)\,dy = 1$: Open

V.S. correction gives the fastest convergence.

Theorem (Berestycki Brunet Harris Roberts)

Suppose $h(x) = O\left(x^{\nu} e^{-x}\right)$, $\nu < -2$

and write $m_t = 2t - \frac{3}{2}\log t + a + r(t)$

Then $r(t) \to 0 \implies U(m_t + x, t) \longrightarrow \phi(x) = \alpha x e^{-x}$

with $\alpha = \frac{e^{-a-\Delta}}{2\sqrt{\pi}} \int h(y)\, y e^{-y} \psi_\infty(y)\, dy$

If furthermore $r''(t) = O\left(t^{-2-\eta}\right)$ $\eta > 0$ (ex: $r(t) = t^{-\eta}$)

$$U(x + m_t, t) = \alpha x e^{-x}\left[1 - r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}} + O\left(t^{1+\nu/2}\right) + O\left(\frac{1}{t^{1/2+\eta}}\right) + O\left(\frac{\log t}{t}\right) + O\left(r(t)^2\right)\right]$$

<u>Key idea</u> Write $U(x,t) = \int_0^\infty h(y)\, q(t,x,y)\, dy$

where q sol started from δ_y (linear!)

$q(t,x,y)$ sol of: $\begin{cases} \partial_t q = \partial_x^2 q + q \\ q(t, m_t, y) = 0 , \quad q(0,x,y) = \delta_{x-y} \end{cases}$

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$q(t,x,y)$ sol of: $\begin{cases} \partial_t q = \partial_x^2 q + q \\ q(t, m_t, y) = 0 \end{cases}$, $q(0,x,y) = \delta_{x-y}$

$q(t,x,y) = e^t\, \mathbb{P}_y \left(B_t \in dx,\ B_s > m_s \ \forall s \in (0,t) \right)$

density of ptc near x, started from y, killing at m_s

$$U(x,t) = \int_0^\infty h(y)\, q(t,x,y)\, dy$$

$$q(t,x,y) = e^t \, \mathbb{P}_y\left(B_t \in dx,\ B_s > m_s \quad \forall s \in (0,t)\right)$$

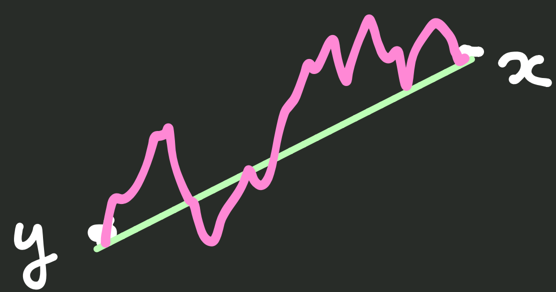
$$q(t, x+m_t, y) = e^t \, \mathbb{P}_y\left(B_t \in dx,\ B_s > 0\right) \mathbb{E}_y\Big[\underbrace{Z^{(m)}}_{\text{Girsorov Term.}}(B) \,\Big|\, B_t \in dx,\ B_s > 0\ \forall s\Big]$$

$$q(t, x+m_t, y) = \frac{1}{\sqrt{\pi t}} \sinh\frac{xy}{2t}\, e^{-\frac{x^2-y^2}{4t}}\, e^{t - \frac{1}{4}\int_0^t ds\, m'(s)^2}\, e^{\frac{m(t)}{2t}(y-x)}\, \underline{\psi_t(y,x)}$$

where

$$\psi_t(y,x) = \mathbb{E}_y\left[\exp\left(\frac{1}{2}\int_0^t m(s)'' \left(\underline{\xi_s^{t:y\to x}} - \underline{y + \frac{x-y}{t}s}\right) ds\right)\right]$$

Bessel bridge
from y To x of length t.



$$\Psi_t(y,x) = \mathbb{E}_y\left[\exp\left(I_t(x,y) \right)\right]$$

where $$I_t(x,y) = \frac{1}{2}\int_0^t \underline{m(s)''} \left(\xi_s^{t:y\to x} - y + \frac{x-y}{t}s\right) ds$$

Note: $$\left(\xi_s^{t:y\to x} - y + \frac{x-y}{t}s\right) \xrightarrow[t\to\infty]{} \left[\xi_s^{(y)} - y\right]$$

$$I(y) = \frac{1}{2}\int_0^\infty m''(s)\left[\xi_s^{(y)} - y\right] ds$$

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$$I(y) = \frac{1}{2}\int_0^\tau m''(s)\left[\xi_s^{(y)} - y\right] dy$$

Assume only $m''(s) = O(1/s^2) \Rightarrow m'(s) = v + O(1/s) \Rightarrow m(s) = vs + O(\lg s)$

Lemma $\lim_t \Psi_t(y,x) = \Psi_\infty(y) = \mathbb{E}\left[e^{I(y)}\right]$

$$\Psi_t(y,x) = \mathbb{E}_y\left[\exp\left(I_t(x,y) \right)\right]$$

where $$I_t(x,y) = \frac{1}{2}\int_0^t m(s)'' \left(\xi_s^{t:y\to x} - y + \frac{x-y}{t}s\right) ds$$

Note: $$\left(\xi_s^{t:y\to x} - y + \frac{x-y}{t}s\right) \xrightarrow[t\to\infty]{} \left[\xi_s^{(y)} - y\right]$$

$$I(y) = \frac{1}{2}\int_0^t m''(s)\left[\xi_s^{(y)} - y\right] dy$$

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Lemma $$\lim_t \Psi_t(y,x) = \Psi_\infty(y) = \mathbb{E}\left[e^{I(y)}\right]$$

in fact: $\Psi_t(y,x) = \Psi_t(y,0)\left(1 + x\, O\left(\frac{\log t}{t}\right)\right)$, $K_1 \leq \Psi_t(y,x) \leq K_2$

$$\lim_y \Psi_\infty(y) = e^{\Delta} \quad \text{where } \Delta = \frac{1}{4}\int_0^\infty ds\, (m'(s) - v)^2$$

write $m(s) = \sqrt{2}\, s + \delta(s)$, $\delta(s) = O(\log s)$, $\delta'(s) = O(1/s)$

then

$$q(t, m(t)+x, y) = \frac{1}{\sqrt{\pi t}} \sinh \frac{xy}{2t}\, e^{-\frac{x^2-y^2}{4t}}\, e^{t - \frac{1}{4}\int_0^t ds\, m'(s)^2}\, e^{\frac{m(t)}{2t}(y-x)}\, \psi_t(y,x)$$

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$$= \frac{1}{\sqrt{\pi t}} e^{t\left(1-\frac{v^2}{4}\right) - \frac{v}{2}\delta(t) - \Delta - \frac{v}{2}x + O(\log t/t)}$$

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so $U(x+m(t),t) = \frac{1}{\sqrt{\pi t}} e^{t\left(1-\frac{v^2}{4}\right) - \frac{v}{2}\delta(t) - \Delta - \frac{v}{2}x + O(\log t/t)} H(x,t)$

$$H(x,t) = \int_0^\infty dy\, \sinh \frac{xy}{2t} e^{\frac{v}{2}y + \frac{\delta(t)}{2t}y} \psi_t(y) e^{-y^2/4t}\, h(y)$$

$$\boxed{\begin{aligned}&\text{so } U(x+m(t),t) = \frac{t^{-3/2}}{\sqrt{4\pi}}\, e^{t\left(1-\frac{v^2}{4}\right)-\frac{v}{2}\delta(t)-\Delta-\frac{v}{2}x+O(\log t/t)}\, H(x,t)\\[1em]&H(x,t) = \int_0^\infty dy\; 2t\,\sinh\frac{xy}{2t}\; e^{\frac{v}{2}y+\frac{\delta(t)}{2t}y}\,\psi_t(y)\, e^{-y^2/4t}\, h(y)\end{aligned}}$$

Pick v and $\delta(t)$ so that has a non trivial limit

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$\underline{v = 2}$.

If $h(y) \sim A y^{\nu} e^{-y}$, $\nu < -2$

then $H(x,t) \sim x \int_0^t h(y)\, y\, e^{y}\, \psi_\infty(y)$ (bounded)

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So $e^{-\delta(t)}$ must compensate $t^{-3/2} \Rightarrow \underline{\delta(t) = -\frac{3}{2}\log t}$.

Thm : $\forall\ h(x) = u(x,0)$ as follow, if m is such that $m''_t = O\left(\frac{1}{t^2}\right)$
then ϕ non Trivial if and only if

$h(x) = u(x,0)$	m_t	$\phi(x)$
$\sim A x^{\nu} e^{-\gamma x}$ $\gamma \in (0,1)$	$c_\gamma t + \frac{\nu}{\gamma}\log t + a + o(1)$	$= \alpha\left(e^{-\gamma x} + e^{-x/\gamma}\right)$ $\alpha = A e^{-\gamma a}\left(\gamma + \frac{1}{\gamma}\right)$
$\sim A x^{\nu} e^{-x}$ $\nu > -2$	$2t - \frac{1-\nu}{2}\log t + a + o(1)$	$= \alpha\, x e^{-x}$ $\alpha = A \frac{e^{-a}}{\sqrt{\pi}} 2^{\nu}\, \Gamma\left(1 + \frac{\nu}{2}\right)$
$\sim A x^{-2} e^{-x}$	$2t - \frac{3}{2}\log t + \log\log t + a + o(1)$	$= \alpha\, x e^{-x}$ $\alpha = \frac{A e^{-a}}{4\sqrt{\pi}}$
$= O\left(x^{\nu} e^{-x}\right)$ $\nu < -2$	$2t - \frac{3}{2}\log t + a + o(1)$	$= \alpha\, x e^{-x}$ $\alpha = \frac{e^{-a-\Delta}}{2\sqrt{\pi}} \int h(y)\, y e^{-y} \psi_\infty(y)\, dy$

Thanks!

We focus on the "Bramson" case:

Thm

$h(x) = O(x^{\nu} e^{-x})$, $\nu < -2$ $\quad m_t = 2t - \frac{3}{2}\log t + a + r(t)$

$$\phi(x) = \alpha x e^{-x} \quad \text{with} \quad \alpha = \frac{e^{-a-\Delta}}{2\sqrt{\pi}} \int h(y) y e^{-y} \psi_\infty(y)\, dy$$

with $r(t) \to 0$, $r''(t) = O(t^{-2-\eta})$ $\quad \eta > 0$

$$\boxed{u(x+m_t, t) = \alpha x e^{-x}\left[1 - r(t) - \frac{3\sqrt{\pi}}{\sqrt{t}} + O(t^{1+\nu/2}) + O\left(\frac{1}{t^{1/2+\eta}}\right) + O\left(\frac{\log t}{t}\right) + O(r(t)^2)\right]}$$

$\Rightarrow$ rate of conv. $u(x+m_t,t)$ to $\alpha x e^{-x}$ is

$$\max\left(|r(t)|,\ t^{-1/2},\ t^{1+\nu/2}\right).$$

$\Rightarrow$ If $\nu < -3$ take $r(t) = -\frac{3\sqrt{\pi}}{\sqrt{t}}$, if $-3 < \nu < -2$, $r(t) = O(t^{1+\nu/2})$.

Bessel Toolbox:

- $\xi_s^{(y)}$: Bessel started from y: $d\xi_s^{(y)} = dB_s + \frac{2}{\xi_s^{(y)}} ds$

- $\xi_s^{t:y\to 0} = \frac{t-s}{t} \xi^{(y)}_{\frac{st}{t-s}} \quad s \in [0,t)$

- $d\xi_s^{t:y\to 0} = dB_s + \left(\frac{2}{\xi_s^{t:y\to 0}} - \frac{\xi_s^{t:y\to 0}}{t-s} \right) ds$

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- $d\xi_s^{t:y\to x} = d\tilde{B}_{t,s} + \left(\coth\left(\frac{x\,\xi_s^{t:y\to 0}}{2(t-s)}\right) - \frac{\xi_s^{t:y\to 0}}{t-s}\right)ds$

$\underline{y \geq z \geq 0}$ $\quad \xi_s^{(z)} \leq \xi_s^{(y)} \leq \xi_s^{(z)} + (y-z) \quad \forall s$

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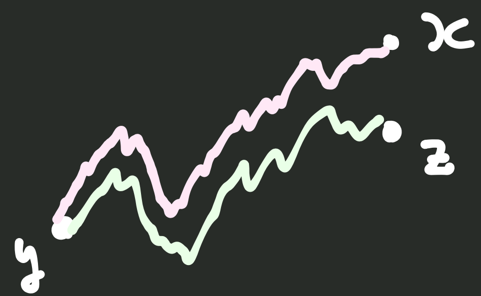

$$\xi_s^{t:y\to z} \leq \xi_s^{t:y\to x} \leq \xi_s^{t:y\to z} + \frac{(x-z)s}{t}$$

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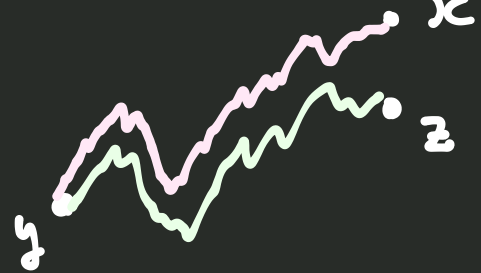


$$\xi_s^{t:y\to z} \leq \xi_s^{t:y\to x} \leq \xi_s^{t:y\to z} + \frac{(x-z)s}{t}$$

$\xi_s^{t:y\to 0} \longrightarrow \xi_s^{(y)}$ $\qquad \xi_s^{(y)} - y \longrightarrow B_s$

$\forall y_t \to \infty \quad \xi_s^{t:y_t\to 0} - y_t\,\frac{t-s}{t} \longrightarrow B_s$

$\exists G$ r.v. with Gaussian tail s.t. $\forall s, \forall y$

$$|\xi_s^{(y)} - y| \le G \max\left(s^{1/2-\varepsilon}, s^{1/2+\varepsilon}\right)$$

Assume m is C^2, $m''(s) = O(s^{-2})$

$\Rightarrow m'(s) = v + O(1/s)$, $m(s) = v(s) + O(\log s)$

$$\Delta = \frac{1}{4} \int_0^\infty (m'(s) - v)^2 \, ds$$

$$\psi_t(y, x) = \mathbb{E}\left[e^{I_t(y,x)} \right]$$

$$q(t,x,y) = e^{t} \, \mathbb{P}_y\left(B_t \in dx, \quad B_s > m_s \quad \forall s \in (0,t)\right)$$

Next: Girsanov

$$q(t, x+m_t, y) = e^{t - \frac{1}{4}\int_0^t ds \, m'(s)^2} \, \mathbb{E}_y\Big[B_t \in dx, \; B_s > 0 \; \forall s, \; \exp\Big(-\frac{1}{2}\int_0^t m'(s)\, dB_s\Big)\Big]$$

$$\text{Now, } \mathbb{E}_y\left[B_t \in dx, B_s > 0, \forall s \leq t\right] = \frac{1}{\sqrt{\pi t}} \sinh\frac{xy}{2t}\, e^{-\frac{x^2 - y^2}{4t}} \, dx$$

$$q(t, x+m_t, y) = \frac{1}{\sqrt{\pi t}} \sinh\frac{xy}{2t}\, e^{-\frac{x^2-y^2}{4t}} \, e^{t - \frac{1}{4}\int_0^t ds\, m'(s)^2} \, \mathbb{E}_y\Big[\exp\Big(-\frac{1}{2}\int_0^t m'(s)\, d\xi_s^{t:y\to x}\Big)\Big]$$

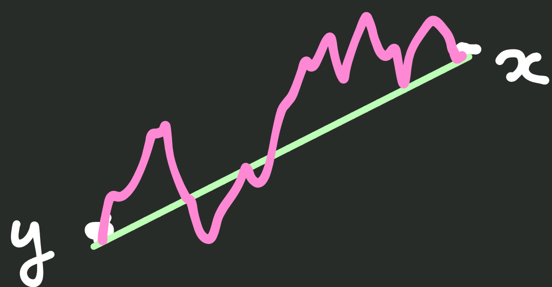
$\xi^{t:y\to x}$: Bessel bridge from y to x of length t.

$$\mathbb{E}_y\left[\exp\left(-\frac{1}{2}\int_0^t m'(s)\, d\xi_s^{t:y\to x}\right)\right]$$

$$\Psi_t(y,x) = \mathbb{E}_y\left[\exp\left(-\frac{1}{2}\int_0^t m'(s)\left(d\xi_s^{t:y\to x} - \frac{x-y}{t}ds\right)\right)\right]$$

$$= \mathbb{E}_y\left[\exp\left(-\frac{1}{2}\int_0^t m'(s)\, d\xi_s^{t:y\to x}\right)\right] e^{\frac{m(t)}{2t}(x-y)}$$

$$= \mathbb{E}_y\left[\exp\left(-\frac{1}{2}\int_0^t m(s)''\left(\xi_s^{t:y\to x} - y + \frac{x-y}{t}s\right)ds\right)\right]$$



$$q(t, x + m_t, y) = \frac{1}{\sqrt{\pi t}} \sinh \frac{xy}{2t}\, e^{-\frac{x^2 - y^2}{4t}}\, e^{t - \frac{1}{4}\int_0^t ds\, m'(s)^2}$$

$$e^{\frac{m(t)}{2t}(y - x)} \quad \psi_t(y, x)$$

$\Rightarrow$ Bessel Toolbox to compute $\psi_t(y,x)$.