Some results on null controllability of the heat equation in presence of singularities

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Some references on the singularity of the solution of elliptic problems in singular domains

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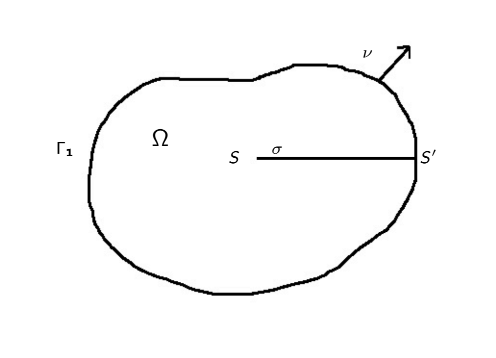
Presentation of the Problem

- We suppose that Ω contains one straight crack σ emerging at a point S' of the boundary $\partial\Omega$. We will designate by S its tip and by Γ_1 the part $\Gamma \setminus \sigma$, Γ_1 is supposed regular.



We consider

$$\begin{cases} \partial_t u - \Delta u = v 1_{\mathcal{O}} & \text{in} \quad Q_T = \Omega \times (0, T), \\ u = 0 & \text{on} \quad \Sigma_T = \Gamma \times (0, T), \\ u(.,0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

Question:

For $u_0 \in L^2(\Omega)$ and $\mathcal{O} \subset\subset \Omega$, does there exists $v \in L^2(Q_T)$ such that $u(T) = 0$?

Known results: $\Omega \in C^2$

$$\boxed{\Omega \in C^2}$$

A. Fursikov, O. Imanuvilov: For a general parabolic equation

G. Lebeau, L. Robbiano: For heat equation

- The result in [Fursikov-Imanuvilov] was obtained by proving Carleman estimates that imply an observability inequality equivalent to the null controllability of the parabolic equation.
- In [Lebeau- Robbiano], the result was obtained through a spectral inequality and this inequality was proved by proving a local Carleman estimate.

Fursikov-Imanuvilov

For $\Omega \in C^2$, and $\mathcal{O} \subset \Omega$, A. Fursikov and O .Imanuvilov constructed a function

$$\boxed{\beta \in C^2(\Omega)}$$

satisfying:

$$\beta > 0 \text{ in } \Omega,\ \beta = 0 \text{ on } \Gamma,\ |\nabla \beta| \geq C > 0 \text{ in } \overline{\Omega \backslash \mathcal{O}}.$$

Carleman Inequalities

$$\alpha(t,x) = \frac{e^{2\lambda m\|\beta\|_\infty} - e^{\lambda\left(m\|\beta\|_\infty + \beta(x)\right)}}{t(T-t)}$$

$$\xi(t,x) = \frac{e^{\lambda\left(m\|\beta\|_\infty + \beta(x)\right)}}{t(T-t)}$$

s, λ sufficiently large

Carleman Estimate

$$s^{-1}\int_{Q_T} e^{-2s\alpha}\xi^{-1}\left(|q_t|^2+|\Delta q|^2\right)dxdt+s\lambda^2\int_{Q_T} e^{-2s\alpha}\xi\,|\nabla q|^2\,dxdt$$
$$+s^3\lambda^4\int_{Q_T} e^{-2s\alpha}\xi^3\,|q|^2\,dxdt\leq C_1(s^3\lambda^4\int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3\,|q|^2\,dxdt)$$

$$\begin{cases} q_t+\Delta q = 0 & \text{sur } Q_T \\ q = 0 & \text{sur } \Sigma_T \\ q(T) = q_T & \text{sur } \Omega \end{cases}$$

$$q\in C^0\left([0,T],L^2(\Omega)\right)\cap C^0\left([0,T[,D(-\Delta)\right)\cap C^1\left([0,T[,L^2(\Omega)\right)$$

$$D(-\Delta)=H^2(\Omega)\cap H_0^1(\Omega)$$

Remark

- Note that to prove the Carleman inequalities, Fursikov and Imanuvilov used the regularity of the solution to justify some integrations by parts

In our case

Theorem

For $q_T \in L^2(\Omega)$ there exists a unique solution q of the adjoint system such that:

$$q \in C^0\left([0,T], L^2(\Omega)\right) \cap C^0\left([0,T[, D(-\Delta)\right) \cap C^1\left([0,T[, L^2(\Omega)\right)$$

$$D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \oplus span\{r^{\frac{1}{2}} \sin \frac{\theta}{2}\}$$

$$\boxed{q(t,r,\theta) = q_R + c(t)\eta(r) r^{\frac{1}{2}} \sin\theta}$$

- $q_R \in H^2(\Omega) \cap H_0^1(\Omega)$
- (r,θ) are the local polar coordinates at the tip S
- η is a cut-off function

$q \notin H^{\frac{3}{2}}(\Omega) \Rightarrow$

$$\boxed{\frac{\partial q}{\partial \nu} \quad is \quad not \ in \quad L^2(\Gamma)}$$

- Summarizing: two dificulties
  1. construction of a weight function β adapted to the singularity of the domain
  2. treatment of singularities of the solution near the tip S

Construction of the function β

Proposition

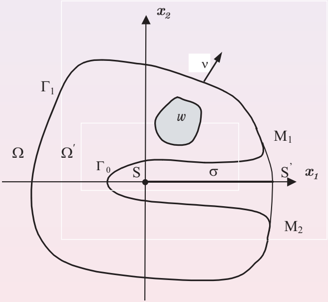
Let $\mathcal{O} \subset\subset \Omega$, there exists a function $\beta \in C^1\left(\bar{\Omega}\right) \cap W^{2,\infty}\left(\Omega\right)$ such that

$$\beta > 0 \text{ in } \Omega, \ |\nabla\beta| > 0 \text{ in } \overline{\Omega\backslash\mathcal{O}}, \ \frac{\partial\beta}{\partial\nu} < 0 \text{ on } \Gamma_1 \text{ and } \frac{\partial\beta}{\partial\nu_\pm} = 0 \text{ on } \sigma\backslash\{S\},$$

where ν_+ and ν_- denote the (opposite) outward unit normal of the crack σ.

Idea of proof

1. we divide the domain Ω in two parts in order to isolate the crack. We denote Ω_S a neighborhood of the crack and Ω' the complement in Ω.
2. we construct $\beta|_{\Omega_S} := \beta_S$ in Ω_S
3. we extend β_S in the regular part Ω' by a function β' ;



$\beta_S(x_1, x_2) = -x_1 + c$

$$\beta_S \in C^2(\overline{\Omega}),\ \beta_S > 0 \text{ in } \Omega,\ \frac{\partial \beta_S}{\partial \nu_\pm} = 0 \text{ on } \sigma \backslash \{S\} \text{ and } \frac{\partial \beta_S}{\partial \nu} < 0 \text{ on } \Gamma'' \backslash \{S'\}.$$

$$|\nabla \beta_S| > 0, \quad \overline{\Omega_s}$$

- We set $g_0 = \beta_S$ on Γ_0 and $h_0 = \dfrac{\partial \beta_S}{\partial \nu}$ on Γ_0.
- We extend g_0 (resp.h_0) to $\partial\Omega'$ by a function g (resp.h) such that $g > 0$ and $h < 0$ on $\partial\Omega'$.

$\widetilde{g}$ be a C^2 lifting of g to Ω' such that $\dfrac{\partial \widetilde{g}}{\partial \nu} = h < 0$ on $\partial\Omega'$ and $\widetilde{g} > 0$ in Ω'

Main result

- We set

$$I(u,\xi,\alpha,Q_T)=\int_{Q_T} e^{-2s\alpha}\Big((s\xi)^{-1}\big(|\partial_t u|^2+|\triangle u|^2\big)+s\lambda^2\xi|\nabla u|^2+s^3\lambda^4\xi^3|u|^2\Big)dx\,dt.$$

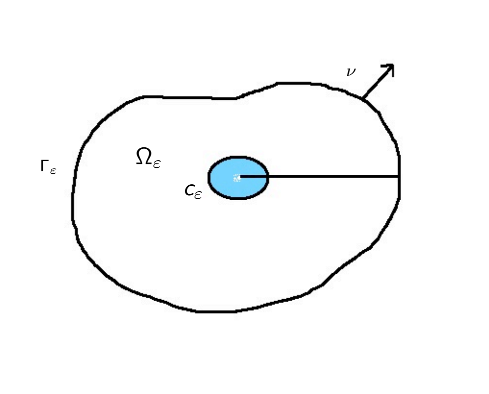
- We state our result,

Theorem

Given $q_T \in L^2(\Omega)$. There exists s_0, $\lambda_0 \in \mathbb{R}$ and $C = C(\Omega,\mathcal{O})$ such that for any $s > s_0$, $\lambda > \lambda_0$ the solution of the adjoint system satisfies

$$I(q,\xi\alpha,Q_T)\le C\int_{\mathcal{O}\times(0,T)} s^3\lambda^4\xi^3e^{-2s\alpha}|q|^2dx\,dt.$$

Step 1. Treatment of the singularity of the solution :



Step 3. Derivation of the Carleman estimate

- We set $\psi = e^{-s\alpha} q$ and for $P = \partial_t + \Delta$, we define

$$L\psi = e^{-s\alpha} P(e^{s\alpha}\psi)$$

- Our aim is a lower estimate of $||L\psi||^2_{L^2(Q_{\varepsilon,T})}$.
- We obtain $L\psi = L_1\psi + L_2\psi = F$, where

$$\begin{cases} L_1\psi = 2s\lambda^2\xi|\nabla\beta|^2\psi + 2s\lambda\xi\nabla\beta.\nabla\psi + \partial_t\psi, \\ L_2\psi = -s^2\lambda^2\left|\nabla\beta\right|^2\xi^2\psi - \Delta\psi + s\partial_t\alpha\psi, \\ F = -s\lambda\xi\Delta\beta\psi + s\lambda^2\xi\left|\nabla\beta\right|^2\psi, \end{cases}$$

Derivation of the Carleman estimate

- After integrations by parts, we get the following estimate :

$$I(\psi,\xi,\alpha,Q_{\varepsilon,T}) + J(\psi,\xi,\alpha,\Sigma_{\varepsilon,T}) \leq C\left(s^3\lambda^4 \int\limits_{\mathcal{O}\times(0,T)} \xi^3|\psi|^2 dx\, dt\right),$$

$$J(\psi,\xi,\alpha,\Sigma_{\varepsilon,T}) = -4s\lambda \int\limits_{\Sigma_{\varepsilon,T}} \xi\,(\nabla\beta.\nabla\psi)\,\frac{\partial\psi}{\partial\nu}\,d\sigma\, dt + 2s\lambda \int\limits_{\Sigma_{\varepsilon,T}} \xi\,|\nabla\psi|^2\,(\nabla\beta.\nu)\,d\sigma\, dt$$

$$J(\cdot,\cdot,\cdot,\Sigma_{\varepsilon,T}) = J(\cdot,\cdot,\cdot,\Gamma_{\varepsilon,T}) + J(\cdot,\cdot,\cdot,C_{\varepsilon,T}).$$

Step 4. Treatment of boundary terms near the crack

- **Terms on $\Gamma_{\varepsilon,T}$:**
  1. $\frac{\partial \beta}{\partial \nu} = 0$ on σ_ε then $J(\cdot,\cdot,\cdot,\sigma_{\varepsilon,T}) = 0$
  2. $\frac{\partial \beta}{\partial \nu} < 0$ on Γ_1 then $J(\cdot,\cdot,\cdot,\Gamma_{1,T}) \geq 0$

$$J(\cdot,\cdot,\cdot,\Gamma_{\varepsilon,T}) = J(\cdot,\cdot,\cdot,\Gamma_{1,T}) + J(\cdot,\cdot,\cdot,\sigma_{\varepsilon,T})$$

Then

$$J(\cdot,\cdot,\cdot,\Gamma_{\varepsilon,T}) \geq 0$$

Step 4. Treatment of boundary terms near the tip

- **Terms on** C_ε **:**

  We use the density of $D(-\Delta) \cap C^1(\overline{\Omega}) \oplus span\{r^{\frac{1}{2}} sin\frac{\theta}{2}\}$ in $D(-\Delta)$

  $$\psi = \psi_R + c_S \psi_S,$$

  with $\psi_R(.,t) \in C^1(\overline{\Omega})$ for all $t \in (0, T)$.
  This implies

  $$\psi(.,t) = O(\sqrt{\varepsilon}) \quad \text{and} \quad |\nabla \psi(.,t)| = O(\frac{1}{\sqrt{\varepsilon}}).$$

$$J(\psi,\xi,\alpha,C_{\varepsilon,T}) = \int_{C_\varepsilon}\Big(\xi\frac{\partial\beta}{\partial\nu}((\frac{\partial\psi}{\partial\tau})^2-(\frac{\partial\psi}{\partial\nu})^2)-2\xi\frac{\partial\beta}{\partial\tau}\frac{\partial\psi}{\partial\nu}\Big)$$

- Let $L(\psi,\psi) := J(\psi,\xi,\alpha,C_{\varepsilon,T})$
- With $\psi=\psi_R+c_S\psi_S$, one has

$$L_\varepsilon(\psi,\psi)=L_\varepsilon(\psi_R,\psi_R)+2cL_\varepsilon(\psi_R,\psi_S)+c^2 L_\varepsilon(\psi_S,\psi_S).$$

- we verify that

$$\lim_{\varepsilon\to 0} L_\varepsilon(\psi_R,\psi_R)=\lim_{\varepsilon\to 0} L_\varepsilon(\psi_R,\psi_S)=0.$$

Lemma

$$\lim_{\varepsilon \to 0} L_{\varepsilon}(\psi, \psi) \geq 0$$

Using the expression of q_s and β in neighborhood of the tip S we verify that

$$\lim_{\varepsilon\to 0} L_\varepsilon(\psi_S,\psi_S)$$
$$= \lim_{\varepsilon\to 0}\int_{C_\varepsilon} e^{-2s\alpha}\xi\left[(\nabla\beta.\nu)\left(\left(\frac{\partial q_S}{\partial\tau}\right)^2-\left(\frac{\partial q_S}{\partial\nu}\right)^2\right)-2\nabla\beta.\tau\frac{\partial q_S}{\partial\tau}\frac{\partial q_S}{\partial\nu}\right]d\sigma$$
$$= \lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{e^{-2s\alpha}\xi}{4}d\theta \geq 0.$$

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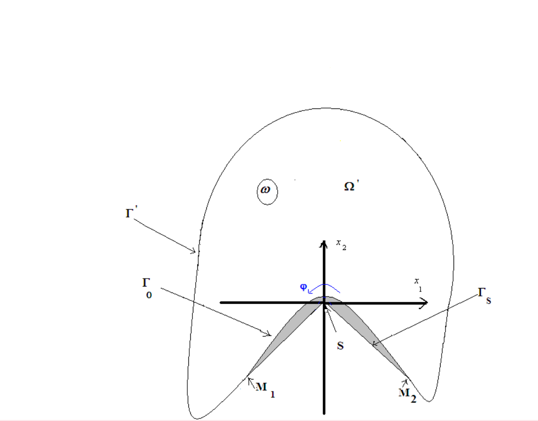
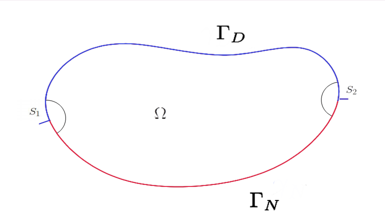
Carleman estimate for domains with corners



$u = u_r + cr^{\frac{\pi}{\omega}} \sin(\frac{\pi}{\omega}\theta)$

$$u \in H^s(\Omega), \quad s > \frac{3}{2}$$

Carleman estimate for the heat equation with mixed boundary conditions



$$u = u_r + c_1\eta_1 r^{\frac{1}{2}} \sin(\frac{1}{2}\theta) + c_2\eta_2 r^{\frac{1}{2}} \sin(\frac{1}{2}\theta)$$

Open problems

- Extension to general elliptic operators
- Mixed boundary conditions for polygonal domains
- Neumann boundary conditions for polygonal domains

Thank you for your attention