Carleman for Stokes-Ventcel

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Presentation

Introduction

In this conference, we will be concerned with the control of Stokes System with "Ventcel boundary conditions"

$$P_S := \begin{cases} u_t(t,x) & - & \Delta u + \nabla p & = v1_\omega & in & Q \\ & & \nabla . u & = 0 & in & Q \\ & & u_\tau & = 0 & in & \Sigma \\ & & u.n & = \varphi & in & \Sigma \\ \varphi_t(t,x) & + & (\frac{\partial u}{\partial n}.n - p) - \Delta_\tau \varphi & \equiv 0 & in & \Sigma \\ u(0,x) & = & u_0(x) & & on & \Omega \\ \varphi(0,x) & = & \varphi_0(x) & & on & \Gamma \end{cases}$$

Here $Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^N$ $(N = 2 \, or \, 3)$ is a bounded open set non empty and connexe, whose $\partial\Omega$ is smooth
$\omega \Subset \Omega$ is a non empty open subset , $\Sigma = \partial\Omega \times (0, T)$,
Δ_τ is Laplace-Beltrami operator on Γ,
v stands for the control function wich acts over the set ω
$u_\tau =$ tangential component of u (vector)
$u.n$=normal component of u (scalar)
The problem is to look for a control v such that
$u(T) = 0$ in Ω

The existence of v is established via a (several) Carleman inequalities.
Before discussing this problem, let's first look at a simpler system which is the heat equation with the same type of boundary conditions .
The first Carleman inequality, concerns the system :

$$P_H := \begin{cases} L\varphi := \varphi_t(t,x) - \Delta\varphi = f & in \quad Q \\ \varphi = \psi & on \quad \Sigma \\ \psi_t(t,x) + \frac{\partial\varphi}{\partial n} - \Delta_\tau\psi = g & in \quad \Sigma \\ \varphi(0,x) = \varphi_0(x) & on \quad \Omega \\ \psi(0,x) = \psi_0(x) & on \quad \Gamma \end{cases}$$

Remark :The boundary conditions imposed comes from the fact that in reality the domain Ω is covered by a thin layer, which is considered of zero thickness (boundary layer), but this is not the purpose of the presentation to justify that condition.

The presentation breaks down as follows :

1. We show that the heat system is well posed in suitable spaces and the solution is smooth, then do the same for the Stokes system.
2. Look (briefly) into control.
3. Carleman inequality is written for the heat equation
4. We then consider the Stokes system

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Presentation

The heat equation

Let

$$V_H = \{\Phi = (\phi_1, \phi_2) \in H^1(\Omega) \times H^1(\Gamma); \quad \phi_1|_\Gamma = \phi_2\}$$

and so $(\Phi = (\phi, \phi|_\Gamma)$. Let

$$a(\Phi, \Psi) := \int_\Omega \nabla\varphi \nabla\psi dx + \int_\Gamma \nabla_\tau \varphi \nabla_\tau \psi d\sigma$$

Weak formulation of the problem P_H is :

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Weak formulation of the problem P_H is :

Given $\Phi_0 := (\phi_{01}, \phi_{02}) \in H := L^2(\Omega) \times L^2(\Gamma)$, and $F := (f, g) \in L^2((0, T); H)$

find $\Phi = (\varphi_1, \varphi_2) \in L^2((0, T); V) \bigcap \mathcal{C}([0, T]; H)$ such that :

$$\frac{d}{dt}(\Phi, \Psi)_H + a(\Phi, \Psi) = (F, V)_H \quad \forall V \in V_H$$

$$\Phi(0) = \Phi_0$$

Strong formulation can be write as :

$$\frac{d}{dt}U + AU = F$$

where :

$$A = \begin{pmatrix} -\Delta & 0 \\ \frac{\partial}{\partial n} & -\Delta_\tau \end{pmatrix}$$

Smoothness : In fact, we can prove that
$\Phi \in L^2((0,T);(H^2(\Omega) \times H^2(\Gamma)) \bigcap V_H)$

we have a chain :

Data+ first equation
$\Rightarrow \frac{\partial \varphi}{\partial n} \in H^{-1/2}(\Gamma) \Rightarrow \varphi_{|\Gamma} \in H^{3/2}(\Gamma) \Rightarrow \varphi \in H^2(\Omega) \Rightarrow$
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Presentation

Stokes System

In our Stokes System, the variational space is :

$$V_S = \{U = (u, \varphi) \in H^1(\Omega)^N \times H^1(\Gamma);\ \nabla.u = 0;\ u_\tau|_\Gamma = 0;\ u.n = \varphi\}$$

and the associated bilinear form is :

$$a(U, V) := \int_\Omega \nabla u \nabla v dx + \int_\Gamma \nabla_\tau \varphi \nabla_\tau \psi d\sigma$$

Note $H := L^2(\Omega)^N \times L^2(\Gamma)$, $\mathcal{H} := L^2(\Omega)^N \times L^2(\Gamma)/\mathbb{R}$

Then, weak formulation of problem is writen :

Given $U_0 := (u_0, \varphi_0) \in H$, and
$F := (f, g) \in L^2((0,T); \mathcal{H})$

find $U \in L^2((0,T); V) \bigcap \mathcal{C}([0,T]; H)$ such that :

$$\frac{d}{dt}(U, V) + a(U, V) = (F, V) = (f, v)_{L^2(\Omega)^N} + (g, \psi)_{L^2(\Gamma)}$$
$$U(0) = U_0$$

Remark : we work with $L^2(\Gamma))/\mathbb{R}$ because $\int_\Gamma \varphi d\Gamma = 0$ and so $(g + c(t), v.n) = (g, v.n)$.

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So, De Rham insure the existence of a function p (pression) -up to additonnal constant in x- such that :

$$u_t(t,x) - \Delta u + \nabla p = f \quad \text{in} \quad Q$$

after that, we take in weak formulation $V \in V_S$, which gives by integration par parts :

$$\varphi_t(t,x) + (\frac{\partial u}{\partial n}.n - p) - \Delta_\tau \varphi \equiv g \quad \text{in} \quad L^2((0,T); L^2(\Gamma)/\mathbb{R})$$

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As for the heat equation, and using regularity for Stokes problem with Dirichlet condition, (see, eg, R. Temam (book)), we can prove regularity result :

if (f, g) as in $L^2((0,T);\mathcal{H})$, then (u, φ, p) is in $L^2((0,T); H^2(\Omega)^N \times H^2(\Gamma) \times H^1(\Omega))$

Presentation

Observability inequality

From Russel's principe, it is known that the system is null controllable if and only if there is one inequality of observability.
In the case of heat equation, this inequality can be writen :

$\exists\, C > 0$ such that :

$$|| \varphi_1^0 ||^2_{L^2(\Omega)} + || \varphi_2^0 ||^2_{L^2(\Gamma)} \leq C \int\int_{\omega\times(0,T)} | \varphi_1 |^2 \, dxdt$$

$$\forall (\varphi_1^0, \varphi_2^0) \in L^2(\Omega) \times L^2(\Gamma_V)$$

where $\varphi = (\varphi_1, \varphi_2)$ is the solution (of the adjoint problem) with $(\varphi_1, \varphi_2)(T) = (\varphi_1^0, \varphi_2^0)$

The argument that leads this result is now classical (see C.Fabre, J.P. Puel and E. Zuazua 1995) and we give only the procedure :

Let $\Phi^0 = (\varphi_1^0, \varphi_2^0) \in H$, and $\Phi = (\varphi, \psi)$ solution to :

$$P_H^* := \begin{cases} -\varphi_t(t,x) & - & \Delta\varphi & = 0 & in & Q \\ & & \varphi & = \psi & in & \Sigma \\ -\psi_t(t,x) & + & (\frac{\partial\varphi}{\partial n}) - \Delta_\tau\psi & = 0 & in & \Sigma \\ \Phi(T,x) & = & \Phi^0(x) & & & \end{cases}$$

One way to derive the null controllability from this result is to consider the functional :

- given $Y^0 = (y_1^0, y_2^0) \in L^2(\Omega) \times L^2(\Gamma_V)$ and $\varepsilon > 0$ let :
- $$J_\varepsilon(\varphi_1^0, \varphi_2^0) = \frac{1}{2} \int\int_{\omega\times(0,T)} \mid \varphi_1 \mid^2 dxdt$$ $$+\varepsilon \mid\mid (\varphi_1^0, \varphi_2^0) \mid\mid_{L^2(\Omega)\times L^2(\Gamma)}$$ $$+((\varphi_1, \varphi_2)(0), (y_1^0, y_2^0))_{L^2(\Omega)\times L^2(\Gamma_V)}$$
- Let $\Phi_\varepsilon = (\varphi_{\epsilon 1}, \varphi_{\epsilon 1})$ the solution (of the adjoint problem) associated to unique minimum Φ_ε^0 of J_ε.
- Introduce $v_\varepsilon := \varphi_{\epsilon 1} 1_\omega,$ and prove that v_ε converges to v
- Then Y the associated solution to v, satisfies $Y(T) = 0$

Presentation

We adopt the same approch as in the heat equation, with a similar J_ε :

$$J_\varepsilon(U^0) = \frac{1}{2}\int\int_{\omega\times(0,T)} \mid u \mid^2 dxdt + \varepsilon \parallel U^0 \parallel_H + (U^0, Y^0)_H$$

$$H = L^2(\Omega)^N \times L^2(\Gamma)$$

Global Carleman estimate

The powerful tools to prove this inequality (observability), in the case of the heat equation, is global estimates

$$\int\int_{\Omega\times(0,T)} \rho \mid \varphi_1 \mid^2 dxdt + \int\int_{\Gamma\times(0,T)} \rho \mid \varphi_2 \mid^2 dxdt \leq$$

$$\int\int_{\omega\times(0,T)} \rho \mid \varphi_1 \mid^2 dxdt$$

where $\rho = \rho(x,t)$ is a continuous and strictly positive function for $t \in (0,T)$. and Φ is solution of the adjoint problem to P_H
This inequality is a direct result of Carleman inequality

Presentation

Global Carleman estimate

From A, Fursikov and O, Yu.Imaunvilov (1995), we know that :
given a nonempty $\omega \Subset \Omega$, there exist a $\mathcal{C}^2(\overline{\Omega})$ function $\eta > 0$ on Ω, such that $| \nabla_x \eta | \geq C > 0$ in $\Omega \backslash \omega$, $\quad \partial_n \eta < 0$ on $\partial\Omega$ and $\eta = 0$ on $\partial\Omega$

Let $\xi := \frac{e^{\lambda\eta}}{t(T-t)}, \quad \alpha := \frac{e^{\lambda\eta} - e^{2\lambda||\eta||_{\mathcal{C}(\overline{\Omega})}}}{t(T-t)}$

Proposition

There exist $C > 0$, such that

$$s\lambda^2\|e^{s\alpha}\xi^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(Q)} + +s^3\lambda^4\|e^{s\alpha}\xi^{\frac{3}{2}}\varphi\|^2_{L^2(Q)}$$

$$+s\lambda\|e^{s\alpha}\xi^{\frac{1}{2}}\nabla_\tau\varphi\|^2_{L^2(\Sigma)}) + s^3\lambda^3\|e^{s\alpha}\xi^{\frac{3}{2}}\varphi\|^2_{L^2(\Sigma)}$$

$$\leq C(\|e^{s\alpha}f\|^2_{L^2(Q)} + \|e^{s\alpha}g\|^2_{L^2(\Sigma)} + s^3\lambda^4\|e^{s\alpha}\xi^{\frac{3}{2}}\varphi\|^2_{L^2((0,T)\times\omega)})$$

for all λ, and s large enough.

Sketch of the proof

We set :$w := e^{s\alpha}\varphi$; $Pw := e^{s\alpha}Le^{-s\alpha}w = e^{s\alpha}f$

We develop $||Pw||_{L^2(Q)}$, (see F.I), we integrate by parts, we obtain two types of terms :

Integrals over Q consisting of positive terms and terms that can be absorbed and which are the same as those for the classic heat equation

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Boundary terms consisting of :

- $$s^3\lambda^3(-\frac{\partial\eta}{\partial n})^3\xi^3 w^2, \texttt{(good, we keep),}$$

- $$s\lambda(-\frac{\partial\eta}{\partial n})\xi(\nabla_T w)^2, \texttt{(good, we keep),}$$

- more positive terms , more absorbable terms, and a term of the form $(\frac{\partial w}{\partial n})F(w_t, \nabla_T w, w)$

We go back to the second equation of P_H (Ventcel) to transform $(\frac{\partial w}{\partial n}) = G(w_t, \Delta w, g)$, and after other manipulations (integration by parts, absorption,...), we find the result.

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Presentation

Stokes System

Here too, the observability inequality is written in the same manner as in the case of heat equation.

$$\| u^0 \|^2_{L^2(\Omega)^N} + \| \varphi^0 \|^2_{L^2(\Gamma)} \leq C \int\int_{\omega\times(0,T)} | u |^2 \, dxdt$$

$$\forall \, (u^0, \varphi^0) \in L^2(\Omega)^N \times L^2(\Gamma)$$

Remark :
In this inequality , the pressure does not appear, and the most delicate point in the demonstration is to get rid of it.

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Here also, we use Carleman's inequality , however , it is too technical with weight less simple than those of the heat equation , therefore , I will not write it, but rather describe the process to achieve it .

$$\xi := \frac{e^{\lambda(m||\eta||_{\mathcal{C}(\overline{\Omega})}+\eta(x))}}{t^4(T-t)^4}, \quad \alpha := \frac{e^{\lambda(m||\eta||_{\mathcal{C}(\overline{\Omega})}+\eta(x))}-e^{2\lambda m||\eta||_{\mathcal{C}(\overline{\Omega})}}}{t^4(T-t)^4} \quad m > 2$$

• First, we write the system in the form :

$$u_t(t,x) - \Delta u = f - \nabla p \text{ in } \quad Q$$

which gives N equations, there are treated as in N heat equations , except that the boundary terms cannot be treated separately.

• We sum these N equations in order to obtain boundary conditions that achieve coupling and result in a term of the form

$$(\frac{\partial w}{\partial n}.n)F(w_t, \nabla_T w, w)$$

+ good terms (see heat equation)

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• Ventcel condition involves :

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To rid it, we adopt approach used by Fernandez-Cara et al. (2004), for the Stokes system with CL Dirichlet , with adequate modifications for CL "Ventcel"

- Write $\Delta p = \nabla . f \quad in \quad \Omega, \quad ae$ in t

- $p(t) \in H^1(\Omega)$ and the main result of Imanuvilov-Puel (2003) give,

$$\| e^{\tau\mu} \nabla p(t) \|^2_{L^2(\Omega)} + \| e^{\tau\mu} p \|^2_{L^2(\Gamma)} \leq$$

$$\lambda^2 \tau^2 \int_{\omega_1} \mu^2 e^{2\tau\mu} p^2 dx + \sqrt{\tau} e^{2\tau} \| p \|^2_{H^{1/2}(\Gamma)} + \text{good terms.}$$

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- $$\nabla p = H(f, u_t, \Delta u) \quad in \quad \omega_2 \Subset \omega \Subset Omega$$

- As for Stokes-Dirichlet problem, (Fernandez-Cara et al.), we choose p such that, $\int_{\omega_2} p dx = 0 \quad (\omega_2 \Subset \omega)$.
This allows the use Poincaré–Wirtinger's inequality :

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This allows the use Poincaré–Wirtinger's inequality :

$$\int_{\omega_2} |\, p \,|^2 \, dx \leq C \int_{\omega_2} |\, \nabla p \,|^2 \, dx$$

After many integration by part, decomposition of the solution and using uniqueness for Stokes problem and after few modifications in demonstration (because boundary conditions are different), we get in the end that p no longer appears in inequality.

These modifications are minor, and no major problem appears.

Remarks

1) To get rid of the pressure, we used a complicated article (I- P) establishing an inequality for elliptic operators with data is in $H^{-1}(\Omega)$.

This article uses OPD "P" that he decomposes into a tangential and a normal operators via symbols

Thus this method doesn't allow to consider nonsmooth opens, even if the solution is smooth.

If Ω is e.g. convex , the solution is smooth, but if for example there are angles , it cannot define tangential operator.

How to circumvent this problem ?

2) If Ω is smooth , and that we have mixed boundary conditions (Dirichlet - Ventcel) , we can use "P_τ and P_n", but solution is not regular ($p \notin H^1(\Omega)$) and IP demonstration stuck .

Can we use the decomposition method of singularity (see e,g P. Grisvard or O. Zair et al.) , or spectral method (as in L. Robiano et al.).

However , this does not seem easy, because in both methods, the data is given in $L^2(\Omega)$ and the solution is at least H^1 .

Thank You