Stabilization of viscoelastic wave equations with time delay

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The problem

We analyze the stability properties of viscoelastic models for second–order evolution equations. In particular, we consider a model combining memory damping and a "small" time delay feedback and a model combining memory damping and on–off time delay feedback, namely the time delay feedback is intermittently present.

It is well–known (see e.g. Giorgi, Muñoz Rivera and Pata 2001, Alabau–Boussouira, Cannarsa and Sforza 2008) that, under appropriate assumptions on the memory kernel, wave–type equations with viscoelastic damping are exponentially stable, i.e. the energy of all solutions is exponentially decaying to zero.

On the other hand, time delay effects appear in many applications and practical problems and it is by now well–known that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly exponentially stable in absence of delay.

The problem

For some examples of this destabilizing effect of time delays we refer to [Datko 1988, Datko, Lagnese and Polis 1986, Nicaise and P. 2006, Xu, Yung and Li 2006] .

We want to show that under suitable conditions involving the delay feedback coefficient and the memory kernel, the system is asymptotically stable or exponentially stable, in spite of the presence of the time delay term. We can obtain two complementary results. The first stability result is obtained under a smallness condition on L^∞-norm of the coefficient of the delay damping, while the second one is guaranteed if the coefficient belongs to $L^1(0, +\infty)$ and the time intervals where the delay feedback is off are sufficiently large.

The general model

Let H be a real Hilbert space and let $A : \mathcal{D}(A) \to H$ be a positive self–adjoint operator with a compact inverse in H. Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$.

Let us consider the problem

$$
\begin{aligned}
& u_{tt}(x,t) + Au(x,t) - \int_0^\infty \mu(s) Au(x,t-s)ds \\
& \qquad\qquad\qquad\qquad + b(t)u_t(x,t-\tau) = 0 \quad t>0, \\
& u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\
& u(x,t) = u_0(x,t) \quad \text{in} \quad \Omega \times (-\infty, 0];
\end{aligned}
\qquad (\mathbf{P})
$$

where the initial datum u_0 belongs to a suitable space, the constant $\tau > 0$ is the time delay, and the memory kernel $\mu : [0,+\infty) \to [0,+\infty)$ satisfies

i)$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;

ii) $\mu(0) = \mu_0 > 0$;

iii) $\int_0^{+\infty} \mu(t)dt = \tilde{\mu} < 1$;

iv) $\mu'(t) \le -\delta\mu(t), \quad$ for some $\delta > 0$.

First model

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$. Let us consider the initial boundary value problem

$$
\begin{array}{ll}
u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty \mu(s)\Delta u(x,t-s)ds & \\
\qquad\qquad\qquad\qquad + k u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) & \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) & (\mathbf{P1}) \\
u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty, 0] &
\end{array}
$$

This problem enters into our previous framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathcal{D}(A) \to H \; : \; u \to -\Delta u,$$

where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.
The operator A is a self–adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$.
Moreover, for simplicity we assume $b(t) = k \;\; \forall \; t > 0$.

The models

We know that the above problem is exponentially stable for $k = 0$ (see e.g. [Giorgi, Munõz Rivera & Pata, 2001]).
Since the viscoelastic damping is a stabilizing one, it is natural to investigate if it prevails with respect the time delay term.
We will show, by using a perturbative approach introduced in [P. 2012], that even if a time delay generates instability, an exponential stability result still holds if the delay parameter k satisfies a suitable *smallness* condition.

We first prove the exponential stability of an auxiliary problem having a decreasing energy and then, regarding the original problem as a perturbation of that one, we extend the exponential decay estimate to it. Our analysis allows to determine an explicit upper bound on $|k|$.

Well-posedness

As in [Dafermos, 1970], let us introduce the new variable

$$\eta^t(x,s) := u(x,t) - u(x,t-s).$$

Moreover, as in [Nicaise & P.,2006], we introduce the function

$$z(x,\rho,t) := u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Using η and z we can rewrite (P1) as

$$\begin{array}{l}
u_{tt}(x,t) = (1-\tilde{\mu})\Delta u(x,t) + \int_0^\infty \mu(s)\Delta\eta^t(x,s)ds \\
\qquad\qquad\qquad\qquad\qquad\qquad -kz(x,1,t) \quad \text{in } \Omega \times (0,+\infty) \\
\eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\
\eta^t(x,s) = 0 \quad \text{in } \partial\Omega \times (0,+\infty),\ t \geq 0, \\
z(x,0,t) = u_t(x,t) \quad \text{in } \Omega \times (0,+\infty), \\
u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
\eta^0(x,s) = \eta_0(x,s) \quad \text{in } \Omega \times (0,+\infty), \\
z(x,\rho,0) = z^0(x,-\tau\rho) \quad x \in \Omega,\ \rho \in (0,1),
\end{array}$$

Well-posedness

where

$$\begin{aligned}
&u_0(x) = u_0(x,0), \quad x \in \Omega,\\
&u_1(x) = \frac{\partial u_0}{\partial t}(x,t)|_{t=0}, \quad x \in \Omega,\\
&\eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad x \in \Omega,\ s \in (0,+\infty),\\
&z^0(x,s) = \frac{\partial u_0}{\partial t}(x,s), \quad x \in \Omega,\ s \in (-\tau,0).
\end{aligned}$$

Let us introduce the vectorial unknown $\mathcal{U} := (u, u_t, \eta^t, z)^T$; then the problem can be written as

$$\left\{\begin{array}{l} \mathcal{U}' = \mathcal{A}\mathcal{U},\\ \mathcal{U}(0) = (u_0, u_1, \eta_0, z^0)^T, \end{array}\right. \tag{PA}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} := \begin{pmatrix} v \\ (1-\tilde{\mu})\Delta u + \int_0^\infty \mu(s)\Delta w(s)ds - kz(\cdot,1) \\ -w_s + v \\ -\tau^{-1} z_\rho \end{pmatrix},$$

Well-posedness

with domain

$$\mathcal{D}(\mathcal{A}) := \Big\{ \ (u, v, \eta, z)^T \in \\ H_0^1(\Omega) \times H_0^1(\Omega) \times L^2_\mu((0,+\infty); H_0^1(\Omega)) \times H^1((0,1); L^2(\Omega)) \ : \\ v = z(\cdot, 0), \ (1-\tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \\ \eta_s \in L^2_\mu((0,+\infty); H_0^1(\Omega)) \Big\} \,,$$

where $L^2_\mu((0,\infty); H_0^1(\Omega))$ is the Hilbert space of H_0^1- valued functions on $(0,+\infty)$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_\mu((0,\infty);H_0^1(\Omega))} = \int_\Omega \left(\int_0^\infty \mu(s) \nabla\varphi(x,s) \nabla\psi(x,s) ds \right) dx \,,$$

in the Hilbert space

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times L^2_\mu((0,\infty); H_0^1(\Omega)) \times L^2((0,1); L^2(\Omega)),$$

Well-posedness

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1-\tilde{\mu}) \int_\Omega \nabla u \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx$$
$$+ \int_\Omega \int_0^\infty \mu(s) \nabla w \nabla \tilde{w} ds dx + \int_0^1 \int_\Omega z(x,\rho) \tilde{z}(x,\rho)\, dx d\rho.$$

Combining some ideas from [Pruss, 1993] with the ones from [Nicaise & P., 2006], we can prove the next existence result.

Well-posedness

PROPOSITION

For any initial datum $\mathcal{U}_0 \in \mathcal{H}$ *there exists a unique solution* $\mathcal{U} \in C([0,+\infty),\mathcal{H})$ *of problem* (PA). *Moreover, if* $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$, *then*

$$\mathcal{U} \in C([0,+\infty),\mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty),\mathcal{H}).$$

Let us define the energy F of problem $(P1)$ as

$$F(t) = F(u,t) := \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{1-\tilde{\mu}}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_0^{+\infty}\int_\Omega \mu(s)|\nabla \eta^t(s)|^2 ds dx + \frac{\theta|k|e^\tau}{2}\int_{t-\tau}^t e^{-(t-s)}\int_\Omega u_t^2(x,s)dsdx,$$

where θ is any real constant satisfying $\theta > 1.$

Stability result

The following exponential stability result holds.

THEOREM 1 [Alabau-Boussouira, Nicaise & P., 2014]

For any $\theta > 1$ *in the definition of the energy* F *there exists a positive constant* k_0 *such that, for* k *satisfying* $|k| < k_0$, *there is* $\sigma > 0$ *such that*

$$F(t) \leq F(0)e^{1-\sigma t}, \quad t \geq 0;$$

for every solution of problem $(P1)$.
The constant k_0 *depends only on the kernel* $\mu(\cdot)$ *of the memory term, on the time delay* τ *and on the domain* Ω.

Stability

To prove our stability results we will make use of the following classical result

LEMMA [Komornik, 1994]

Let $V(\cdot)$ *be a non negative decreasing function defined on* $[0, +\infty)$. *If*

$$\int_S^{+\infty} V(t)dt \leq CV(S) \quad \forall S > 0 \,,$$

for some constants $C > 0$, *then*

$$V(t) \leq V(0) \exp\left(1 - \frac{t}{C}\right), \quad \forall \, t \geq 0 \,.$$

Stability

Let us also recall the following classical perturbation result

THEOREM [Pazy (1983), Th. 1.1 of Ch. 3]

Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X, satisfying

$$\|T(t)\| \leq Me^{\omega t}.$$

If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t}.$$

The auxiliary problem

In order to study the stability properties of problem (P1), we look at an auxiliary problem (cf. [P. 2012]) which is *near* to this one and easier to deal with. Namely,

$$\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty \mu(s)\Delta u(x,t-s)ds \\
&\qquad\qquad + \theta|k|e^\tau u_t(x,t) + k u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \\
&u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \qquad\qquad (\tilde{P}1) \\
&u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty,0].
\end{aligned}$$

The energy of every solution of the auxiliary problem is not increasing:

$$\begin{aligned}
F'(t) \le \frac{1}{2}\int_0^\infty \int_\Omega \mu'(s)|\nabla\eta^t(x,s)|^2 dxds \\
- \frac{|k|(\theta e^\tau - 1)}{2}\int_\Omega u_t^2(x,t)dx - \frac{|k|(\theta-1)}{2}\int_\Omega u_t^2(x,t-\tau)dx \\
- \frac{\theta|k|e^\tau}{2}\int_{t-\tau}^t e^{-(t-s)}\int_\Omega u_t^2(x,s)dxds\,.
\end{aligned}$$

Remark Note that the energy $F(\cdot)$ of solutions of the original problem $(P1)$ is not in general decreasing.

Stability of the auxiliary problem

THEOREM 2 [Alabau-Boussouira, Nicaise & P., 2014]

For any $\theta > 1$ in the definition of the energy F there exist positive constants C and $\overline{k}$, depending on μ, Ω and τ, such that if $|k| < \overline{k}$ then for any solution of problem $(\tilde{P}1)$ the following estimate holds

$$\int_S^{+\infty} F(t)dt \le CF(S) \quad \forall S > 0\,.$$

Our proof is based on multiplier arguments and it relies in many points on [Alabau-Boussouira, Cannarsa & Sforza, 2008]. In order to extend the exponential estimate related to the perturbed problem $(\tilde{P}1)$ to the original problem (P1), we need to determine carefully all involved constants.

We note that we could not apply the same arguments directly to our original problem since the energy is not decreasing.

Coming back to the original problem

From Theorem 2 and Lemma [Komornik], it follows that for any solution of the auxiliary problem $(\tilde{P}1)$ if $|k| < \overline{k}$, we have

$$F(t) \leq F(0)e^{1-\tilde{\sigma}t}, \quad t \geq 0,$$

with

$$\tilde{\sigma} := \frac{1}{C},$$

where C is the constant in Theorem 2.

From this and the perturbation theorem of Pazy, we deduce that Theorem 1 holds, with $\sigma := \tilde{\sigma} - e\theta|k|e^{\tau}$, if

$$-\tilde{\sigma} + e\theta|k|e^{\tau} < 0,$$

that is if the delay parameter k satisfies

$$|k| < g(|k|) := \frac{1}{Ce\theta e^{\tau}} \qquad (\star),$$

with $C := C(|k|)$ as before.

Coming back to the original problem

Now observe that $(\star)$ is satisfied for $k = 0$ because $g(0) > 0$. Moreover, by using the explicit (even if a bit involved) definition of the constant C, we can prove that $g : [0, \infty) \to (0, \infty)$ is a continuous decreasing function satisfying

$$g(|k|) \to 0 \quad \text{for} \quad |k| \to \infty.$$

Thus, there exists a unique constant $\hat{k} > 0$ such that $\hat{k} = g(\hat{k})$. We can then conclude that for any θ in the definition of the energy $F(\cdot)$, inequality $(\star)$ is satisfied for every k with

$$|k| < k_0 = \min\{\hat{k}, \overline{k}\}. \qquad \square$$

The model with on–off delay

Let H be a real Hilbert space and let $A : \mathcal{D}(A) \to H$ be a positive self–adjoint operator with a compact inverse in H. Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$.

Let us consider the problem

$$
\begin{aligned}
& u_{tt}(x,t) + Au(x,t) - \int_0^\infty \mu(s) Au(x,t-s)ds \\
& \qquad\qquad\qquad\qquad + b(t)u_t(x,t-\tau) = 0 \quad t > 0, \\
& u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\
& u(x,t) = u_0(x,t) \quad \text{in} \quad \Omega \times (-\infty, 0];
\end{aligned}
\qquad (\mathbf{P})
$$

where the initial datum u_0 belongs to a suitable space, the constant $\tau > 0$ is the time delay, and the memory kernel $\mu : [0,+\infty) \to [0,+\infty)$ satisfies previous assumptions.

The model with on–off delay

Moreover, the function $b(\cdot) \in L^{\infty}_{loc}(0, +\infty)$ is a function which is zero intermittently. That is, we assume that for all $n \in \mathbb{N}$, there exists $t_n > 0$ with $t_n < t_{n+1}$ and such that

$$b(t) = 0 \ \forall \ t \in I_{2n} = [t_{2n}, t_{2n+1}),$$
$$|b(t)| < b_{2n+1} \neq 0 \ \forall \ t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}).$$

Also, denoting by T_n the length of the interval I_n, that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}$$

we assume that $\tau \leq T_{2n}$ for all $n \in \mathbb{N}$.

Previous literature

Models with viscoelastic damping and time delay feedback have been studied by recent papers (e.g. [Alabau–Boussouira, Nicaise and P., 2014] for infinite memory and [Dai and Yang, 2014] in the case of finite memory).

In these papers the authors prove exponential stability results if the (constant) coefficient of the delay damping is sufficiently small. These stability results can be easily extended to a variable coefficient $b(\cdot) \in L^\infty(0, +\infty)$ under a suitable *smallness* assumption on the $L^\infty-$ norm of $b(\cdot)$.

Previous literature

Stability results for second–order evolution equations with intermittent damping are first studied by [Haraux, Martinez and Vancostenoble, 2005] without any time delay term. They consider a problem with intermittent on–off or with positive–negative damping, and show that, under appropriate conditions, the *good* behavior of the system in the time intervals where only the standard dissipation, i.e. the damping with the right sign, is present prevails over the *bad* behavior where the damping is no present or it is present with the wrong sign, i.e. as anti–damping. Thus, asymptotic/exponential stability results are obtained.

Recently [Nicaise and P., 2012 & 2014] considered second–order evolution equations with intermittent delay feedback. More precisely, in the studied models, when the (destabilizing) delay term is no present, a not–delayed damping acts. Under suitable assumptions, stability results are obtained (see also [Fragnelli and P., 2015] for a nonlinear extension/improvement).

Here, the good behavior in the time intervals where the delay feedback is no present is ensured by a viscoelastic damping.

Well–posedness

For the existence result let us consider the problem

$$
\begin{aligned}
&u_{tt}(x,t) + Au(x,t) - \int_0^\infty \mu(s)Au(x,t-s)ds = f(t) \quad t>0,\\
&u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\
&u(x,t) = u_0(x,t) \quad \text{in} \quad \Omega \times (-\infty, 0].
\end{aligned}
\qquad (\mathbf{P2})
$$

By using the variable $\eta^t(x,s) := u(x,t) - u(x,t-s)$, problem (P2) may be rewritten as

Well–posedness

$$
\begin{aligned}
& u_{tt}(x,t) = -(1-\tilde{\mu})Au(x,t) - \int_0^\infty \mu(s)A\eta^t(x,s)ds \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +f(t) \quad \text{in } \Omega \times (0,+\infty) \\
& \eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\
& u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\
& \eta^t(x,s) = 0 \quad \text{in } \partial\Omega \times (0,+\infty),\, t \geq 0, \\
& u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
& \eta^0(x,s) = \eta_0(x,s) \quad \text{in } \Omega \times (0,+\infty),
\end{aligned}
\qquad (\tilde{\mathbf{P}}\mathbf{2})
$$

where

$$
\begin{aligned}
& u_0(x) = u_0(x,0), \quad x \in \Omega, \\
& u_1(x) = \frac{\partial u_0}{\partial t}(x,t)|_{t=0}, \quad x \in \Omega, \\
& \eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad x \in \Omega,\ s \in (0,+\infty).
\end{aligned}
$$

Well–posedness

Set $L^2_\mu((0,\infty);V)$ the Hilbert space of $V-$ valued functions on $(0,+\infty)$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_\mu((0,\infty);V)} = \int_0^\infty \mu(s) \langle A^{1/2}\varphi(s), A^{1/2}\psi(s)\rangle_H ds\,.$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = V \times H \times L^2_\mu((0,\infty);V),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1-\tilde{\mu})\langle A^{1/2}u, A^{1/2}\tilde{u}\rangle_H + \langle v, \tilde{v}\rangle_H + \int_0^\infty \mu(s)\langle A^{1/2}w, A^{1/2}\tilde{w}\rangle_H ds$$

Well–posedness

Let us recall the following well–posedness result (see [Giorgi, Muñoz Rivera and Pata, 2001]).

Definition.

Set $I=[0,T]$, for $T>0$, and let $f\in L^1(I,H)$. A function $U:=(u,u_t,\eta)\in\mathcal{H}$ is a solution of problem $(\tilde{P}2)$ in the interval I, with initial data $U(0)=U_0=(u_0,u_1,\eta_0)\in\mathcal{H}$, provided

$$
\begin{aligned}
\langle u_{tt},\tilde{v}\rangle = -(1-\tilde{\mu})\langle A^{1/2}u, A^{1/2}\tilde{v}\rangle_H \\
-\int_0^\infty \mu(s)\langle A^{1/2}\eta(s), A^{1/2}\tilde{v}\rangle_H ds + \langle f,\tilde{v}\rangle_H\,;
\end{aligned}
$$

$$
\int_0^\infty \mu(s)\langle \eta_t(s)+\eta_s(s), A\tilde{\eta}(s)\rangle_H ds = \int_0^\infty \mu(s)\langle u_t, A\tilde{\eta}(s)\rangle_H ds\,;
$$

for all $\tilde{v}\in V$ and $\tilde{\eta}\in L^2_\mu(\mathbb{R}^+,\mathcal{D}(A))$, and a.e. $t\in I$.

Well–posedness

THEOREM [Giorgi, Munõz Rivera & Pata, 2001]

For given $T > 0$, problem $(\tilde{P}2)$ has a unique solution U in the time interval $I = [0, T]$, with initial datum U_0.

Under the above assumptions, we obtain the following result

THEOREM [P. 2015]

Under the above assumptions, for any $U_0 \in \mathcal{H}$, the system (P) has a unique solution $U \in C([0, \infty); \mathcal{H})$.

Well–posedness

Proof.

We prove the existence and uniqueness result on the interval $[0, t_2]$; then the global result follows by translation.

First, in the interval $[0, t_1]$, since $b(t) = 0 \ \forall \ t \in [0, t_1)$, we can apply the theorem of [Giorgi, Munõz Rivera & Pata, 2001] with $f \equiv 0$. Then we obtain a solution U, in the sense of previous definition, on the interval $[0, t_1]$.

The situation is more delicate in the time interval $[t_1, t_2]$ where the delay feedback is present. In this case, we decompose the interval $[t_1, t_2]$ into the successive intervals $[t_1 + j\tau, t_1 + (j+1)\tau)$, for $j = 0, \dots, N-1$, where N is such that $t_1 + (N+1)\tau \geq t_2$. The last interval is then $[t_1 + N\tau, t_2]$.

Well–posedness

Now, look at the problem on the interval $[t_1, t_1 + \tau]$. Here $u_t(t - \tau)$ can be considered as a known function. Indeed, for $t \in [t_1, t_1 + \tau]$, then $t - \tau \in [0, t_1]$, and we know the solution U on $[0, t_1]$ by the first step. Thus, problem (P) may be rewritten on $[t_1, t_1 + \tau]$ as

$$
\begin{aligned}
&u_{tt}(x,t) + Au(x,t) - \int_0^\infty \mu(s)Au(x,t-s)ds = f_1(t) \quad t \in [t_1, t_1 + \tau],\\
&u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [t_1, t_1 + \tau],\\
&u(x,t) = u_0^1(x,t) \quad \text{in} \quad \Omega \times (-\infty, t_1];
\end{aligned}
$$

where $f_1(t) = b(t)u_t(t - \tau)$ the initial datum is $u_0^1(x,t) = u_0(x,t)$ in $\Omega \times (-\infty, 0]$ and $u_0^1(x,t) = u(x,t)$ in $\Omega \times [0, t_1]$.
Then we can apply once more the theorem of [Giorgi, Munõz Rivera & Pata, 2001] obtaining a unique solution U on $[0, t_1 + \tau)$. Proceedings analogously in the successive time intervals $[t_1 + j\tau, t_1 + (j+1)\tau)$, we obtain a solution on $[0, t_2]$. $\square$

Stability result

By using the variable

$$z(x,\rho,t) := u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

and the variable η^t, we can rewrite problem (P) as

$$\begin{array}{l}
u_{tt}(x,t) = -(1-\tilde{\mu})Au(x,t) - \int_0^\infty \mu(s)A\eta^t(x,s)ds \\
\qquad\qquad\qquad\qquad\qquad\qquad -b(t)z(x,1,t) \quad \text{in } \Omega \times (0,+\infty) \\
\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \\
\eta^t(x,s) = 0 \quad \text{in } \partial\Omega \times (0,+\infty),\ t \geq 0, \\
z(x,0,t) = u_t(x,t) \quad \text{in } \Omega \times (0,+\infty), \\
u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
\eta^0(x,s) = \eta_0(x,s) \quad \text{in } \Omega \times (0,+\infty), \\
z(x,\rho,0) = z^0(x,-\tau\rho) \quad x \in \Omega,\ \rho \in (0,1),
\end{array}$$

where

$$
\begin{aligned}
&u_0(x) = u_0(x,0), \quad x \in \Omega, \\
&u_1(x) = \tfrac{\partial u_0}{\partial t}(x,t)|_{t=0}, \quad x \in \Omega, \\
&\eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad x \in \Omega,\ s \in (0,+\infty), \\
&z^0(x,s) = \tfrac{\partial u_0}{\partial t}(x,s), \quad x \in \Omega,\ s \in (-\tau,0).
\end{aligned}
$$

Let us now introduce the energy functional:

$$
\begin{aligned}
E(t) = E(u,t) := \frac{1}{2}\|u_t(t)\|_H^2 + \frac{1-\tilde{\mu}}{2}\|u(t)\|_V^2 \\
+\frac{1}{2}\int_0^{+\infty} \mu(s)\|A^{1/2}\eta^t(s)\|_H^2 ds + \frac{1}{2}\int_{t-\tau}^t |b(s+\tau)|\|u_t(s)\|_H^2 ds\,.
\end{aligned}
$$

Then,

$$
E(t) = E_S(t) + \frac{1}{2}\int_{t-\tau}^t |b(s+\tau)|\|u_t(s)\|_H^2 ds\,,
$$

where $E_S(\cdot)$ denotes the standard energy for wave equation with viscoelastic damping.

Stability result

Let us now recall the following result.

THEOREM [Giorgi, Munõz Rivera & Pata 2001, Pata 2009]

Assume $b \equiv 0$. Then, for every solution of problem (P), the energy $E_S(\cdot)$ is not increasing and

$$E_S'(t) \leq \frac{1}{2} \int_0^\infty \mu'(s) \|A^{1/2}\eta^t(s)\|_H^2 ds \,.$$

Moreover, there are two positive constant C, α, $C > 1, \alpha > 0$, depending only on Ω and on the memory kernel $\mu(\cdot)$, such that for every solution of problem (P) it results

$$E_S(t) \leq C e^{-\alpha t} E_S(0) \,.$$

Stability result

Now, let T_0 be the time such that

$$T_0 := \frac{1}{\alpha} \ln C \,,$$

that is the time for which $Ce^{-\alpha T} = 1$.

As an immediate application we have the following result.

PROPOSITION

Assume $T_{2n} > T_0$. Then, there exists a constant $c_n \in (0,1)$ such that

$$E_S(t_{2n+1}) \le c_n E_S(t_{2n}) \,. \qquad (\mathbf{O})$$

Stability result

On the *bad* interval we have the following estimate.

PROPOSITION

Assume $T_{2n} \geq \tau, \ \forall \ n \in \mathbb{N}$. Then,

$$E'(t) \leq b_{2n+1} \|u_t(t)\|_H^2 , \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \, \forall \, n \in \mathbb{N} \, . \qquad \textbf{(B)}$$

By using these estimates on the *bad* and *good* intervals we obtain our stability result.

Asymptotic stability

THEOREM [P., 2015]

Assume $T_{2n} \geq \tau$ and $T_{2n} > T_0$, for all $n \in \mathbb{N}$, where T_0 is the time defined before. Then, if

$$\sum_{n=0}^{\infty} \ln \left[e^{2b_{2n+1}T_{2n+1}} (c_n + T_{2n+1} b_{2n+1}) \right] = -\infty \,, \tag{$\star$}$$

the system (P) is asymptotically stable, that is any solution u of (P) satisfies $E_S(u,t) \to 0$ for $t \to +\infty$.

Asymptotic stability

Proof.

Note that (B) implies

$$E'(t) \leq 2b_{2n+1}E(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),\ n \in \mathbb{N}.$$

Then we have

$$E(t_{2n+2}) \leq e^{2b_{2n+1}T_{2n+1}}E(t_{2n+1}), \quad \forall\ n \in \mathbb{N}. \qquad (\mathbf{E1})$$

From the definition of the energy E,

$$E(t_{2n+1}) = E_S(t_{2n+1}) + \frac{1}{2}\int_{t_{2n+1}-\tau}^{t_{2n+1}} |b(s+\tau)| \|u_t(s)\|_H^2 ds\,.$$

Note that, for $t \in [t_{2n+1} - \tau, t_{2n+1})$, then
$t + \tau \in [t_{2n+1}, t_{2n+1} + \tau) \subset I_{2n+1} \cup I_{2n+2}$. Now, if $t + \tau \in I_{2n+2}$, then $b(t+\tau) = 0$. Otherwise, if $t + \tau \in I_{2n+1}$, then $|b(t+\tau)| \leq b_{2n+1}$.

Asymptotic stability

Then, we deduce

$$E(t_{2n+1}) = E_S(t_{2n+1}) + \frac{1}{2} b_{2n+1} \int_{t_{2n+1}-\tau}^{\min(t_{2n+2}-\tau, t_{2n+1})} \|u_t(s)\|_H^2 ds \,,$$

since if $t_{2n+1} > t_{2n+2} - \tau$, then $b(t) = 0$ for all
$t \in [t_{2n+2}, t_{2n+1} + \tau) \subset [t_{2n+2}, t_{2n+3})$.
Then, since the energy $E_S(\cdot)$ is decreasing in the intervals I_{2n},

$$\begin{aligned} E(t_{2n+1}) &\leq E_S(t_{2n+1}) + T_{2n+1} b_{2n+1} E_S(t_{2n+1} - \tau) \\ &\leq E_S(t_{2n+1}) + T_{2n+1} b_{2n+1} E_S(t_{2n}) \qquad (\mathbf{E2})\,. \end{aligned}$$

Using (E2) in (E1), we deduce

$$E_S(t_{2n+2}) \leq E(t_{2n+2}) \leq e^{2b_{2n+1}T_{2n+1}} (c_n + T_{2n+1} b_{2n+1}) E_S(t_{2n}), \quad \forall\, n \in \mathbb{N},$$

where we have used once more the observability estimate (O).

Asymptotic stability

Iterating this procedure we arrive at

$$E_S(t_{2n+2}) \leq \Pi_{p=0}^{n} e^{2b_{2p+1}T_{2p+1}}(c_p + T_{2p+1}b_{2p+1})E_S(0), \quad \forall\, n \in \mathbb{N},$$

Then, we have asymptotic stability if

$$\Pi_{p=0}^{n} e^{2b_{2p+1}T_{2p+1}}(c_p + T_{2p+1}b_{2p+1}) \longrightarrow 0, \quad \text{for } n \to \infty,$$

or equivalently

$$\ln\left[\Pi_{p=0}^{n} e^{2b_{2p+1}T_{2p+1}}(c_p + T_{2p+1}b_{2p+1})\right] \longrightarrow -\infty, \quad \text{for } n \to \infty\,.$$

This concludes the proof. $\square$

REMARK

Clearly $(\star)$ is verified if the following conditions are satisfied:

$$\sum_{n=0}^{\infty} b_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln(c_n + b_{2n+1}T_{2n+1}) = -\infty\,,$$

and therefore if

$$\sum_{n=0}^{\infty} b_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln c_n = -\infty\,.$$

In particular, from $(\star)$ we have stability e.g. when $b \in L^1(0, +\infty)$ and when the length of the *good* intervals I_{2n} is greater then a fixed time $\bar{T}$, $\bar{T} > T_0$ and $\bar{T} \geq \tau$, namely

$$T_{2n} \geq \bar{T}, \quad \forall\, n \in \mathbb{N}\,.$$

Indeed, in this case there exists $\bar{c} \in (0,1)$ such that $0 < c_n < \bar{c}$.

Exponential stability

We now show that under additional assumptions on the coefficients T_n, b_{2n+1}, c_n an exponential stability result holds.

THEOREM [P., 2015]

Assume

$$T_{2n} = T^* \quad \forall\ n \in \mathbb{N},$$

with $T^* \geq \tau$ and $T^* > T_0$, where the time T_0 is as before. Assume also that

$$T_{2n+1} = \tilde{T} \quad \forall\ n \in \mathbb{N}.$$

Moreover, assume that

$$\sup_{n\in\mathbb{N}}\ e^{2b_{2n+1}\tilde{T}}(c + b_{2n+1}\tilde{T}) = d < 1,$$

where $c = c_n,\ n \in \mathbb{N}$, is as in (O). Then, there exist two positive constants γ, β such that

$$E_S(t) \leq \gamma e^{-\beta t} E_S(0), \quad t > 0,$$

for any solution of problem (P).

Finite memory

REMARK

Analogous result could be obtained for the case of finite memory, namely if system (P) is replaced by

$$
\begin{array}{l}
u_{tt}(x,t) + Au(x,t) - \int_0^t \mu(s) Au(x,t-s)ds + b(t)u_t(x,t-\tau) = 0 \quad t > 0, \\
u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\
u(x,t) = u_0(x,t), \quad u_t(x,t) = u_1(x)(\text{in} \quad \Omega;
\end{array}
$$

with memory kernel $\mu(\cdot)$ and delay coefficient $b(t)$ satisfying the same assumptions that before.
Indeed, also for such a problem it is well-known that an exponential decay estimate holds on the time intervals where the delay feedback is null (see e.g. [Alabau–Boussouira, Cannarsa and Sforza, 2008]).
Therefore, in such intervals an observability type estimate like (O) is available if the length of the intervals is sufficiently large.

Stability under the restriction $T_{2n+1} \leq \tau$

Now, assume that the length of the delay intervals is lower than the time delay, that is

$$T_{2n+1} \leq \tau, \quad \forall n \in \mathbb{N}\,.$$

This allows to work directly with the standard energy $E_S(\cdot)$.

PROPOSITION

Assume $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$. Then, for $t \in I_{2n+1}$,

$$E_S'(t) \leq b_{2n+1} E_S(t) + b_{2n+1} E_S(t_{2n})\,.$$

Proof: By differentiating $E_S(t)$ we get

$$E_S'(t) \leq b(t)\langle u_t(t), u_t(t-\tau)\rangle_H\,.$$

Hence,

$$E_S'(t) \leq \frac{b_{2n+1}}{2}\|u_t(t)\|_H^2 + \frac{b_{2n+1}}{2}\|u_t(t-\tau)\|_H^2 \leq b_{2n+1}E_S(t) + b_{2n+1}E_S(t-\tau)\,.$$

Now, to conclude it suffices to observe that since $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, then for $t \in I_{2n+1}$ it is $t-\tau \in I_{2n}$. Then, since $E_S(\cdot)$ is decreasing in I_{2n} , the estimate in the statement is proved. $\square$

Stability under the restriction $T_{2n+1} \leq \tau$

THEOREM [P., 2015]

Assume $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, $\forall\, n \in \mathbb{N}$. Then, if

$$\sum_{n=0}^{\infty} \ln\left[e^{b_{2n+1}T_{2n+1}}(c_n + 1 - e^{-b_{2n+1}T_{2n+1}})\right] = -\infty\,, \qquad (\star\star)$$

the system (P) is asymptotically stable, that is any solution u of (P) satisfies $E_S(u,t) \to 0$ for $t \to +\infty$.

Stability under the restriction $T_{2n+1} \le \tau$

• Note that, in case of *bad* intervals I_{2n+1} with length lower or equal than the time delay τ , the assumption $(\star\star)$ is a bit less restrictive than $(\star)$.

Indeed, since $b_{2n+1}T_{2n+1} > 0$, it results

$$e^{b_{2n+1}T_{2n+1}}(c_n+1-e^{-b_{2n+1}T_{2n+1}}) < e^{2b_{2n+1}T_{2n+1}}(c_n+b_{2n+1}T_{2n+1}), \quad \forall n \in \mathbb{N}\,.$$

For instance if $b_{2n+1}T_{2n+1} = 1/4$ and $c_n = e^{-1/2} - 1/4$ for every $n \in \mathbb{N}$, then

$$e^{2b_{2n+1}T_{2n+1}}(c_n + b_{2n+1}T_{2n+1}) = 1, \quad \forall\ n \in \mathbb{N}\ ,$$

and

$$e^{b_{2n+1}T_{2n+1}}(c_n + 1 - e^{-b_{2n+1}T_{2n+1}}) = \alpha \in (0,1), \quad \forall\ n \in \mathbb{N}\ .$$

Therefore $(\star)$ does not hold while $(\star\star)$ is clearly satisfied.

• Also in this case, under additional assumptions on the coefficients T_n, b_{2n+1}, c_n an exponential stability result holds.

Thank you for your attention!