Some results on Timoshenko systems

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Outline

Beam theories

- *Beam theories* are extensively used to analyze the **structural behavior** of *slender bodies*, such as columns, arches, blades, aircraft wing, and bridges.

Cloister at San Zeno Maggiore, Verona

The Gateway Arch in Saint Louis, Missouri

Aircraft wing

Blades

Tacoma Narrows Bridge

Beam theories

- The main advantage of beam models is that they *reduce the 3D problem* to a set of variables that only depends on the beam-axis coordinate.
- The 1D structural elements obtained are *simpler* and *computationally more efficient* than 2D (plate/shell) and 3D (solid) elements. This feature makes beam theories very attractive for the static and dynamic analysis of structures.
- The classical, most frequently employed theories are those by Euler-Bernoulli and Timoshenko.

Euler-Bernoulli beam theory

- The Euler-Bernoulli beam theory was established around 1750 with contributions from Leonard Euler and Daniel Bernoulli. Bernoulli provided an expression for the strain energy in beam bending, from which Euler derived and solved the differential equation. That work built on earlier developments by Jacob Bernoulli.

Jacob Bernoulli

1654–1705

Leonhard Euler

1707–1783

Daniel Bernoulli

1700–1782

Euler-Bernoulli beam theory

- However, the beam problem had been addressed even earlier. Galileo Galilei attempted one formulation that aimed at determining the capacity of beams in bending, but misplaced the neutral axis.

  Earlier, *Leonardo da Vinci also seems to have addressed the problem of beam bending*.

Jacob Bernoulli

1654–1705

Leonhard Euler

1707–1783

Daniel Bernoulli

1700–1782

Da Vinci-Euler-Bernoulli beam theory

Leonardo Da Vinci

1452–1519

Jacob Bernoulli

1654–1705

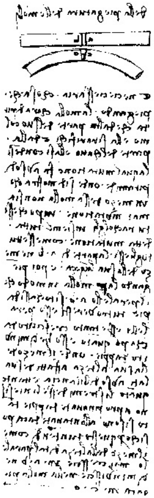
Leonhard Euler

1707–1783

Daniel Bernoulli

1700–1782

Da Vinci-Euler-Bernoulli beam theory



Folio 84 of Codex Madrid I

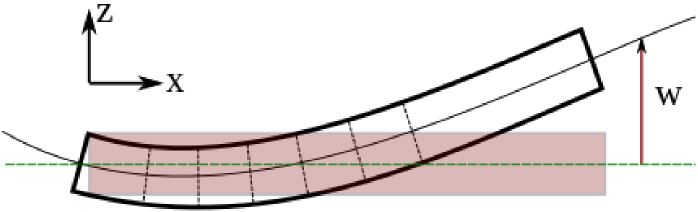
- Da Vinci made a *fundamental contribution* to what is commonly referred to as Euler-Bernoulli (engineering) beam theory 200 years before Euler and Bernoulli.
- Historians of mechanics did not cheat Leonardo; they simply were not aware that he made the fundamental hypothesis upon which Euler-Bernoulli beam theory rests in Codex Madrid I, one of two remarkable notebooks that were discovered in 1967 in the National Library of Spain (Madrid), after being misplaced for nearly 500 years.

[R. Ballarini, *Da Vinci-Euler-Bernoulli Beam Theory?*, ASME Mechanical Engineering Magazine Online, 4/18/03.]

Da Vinci-Euler-Bernoulli beam theory

The **two key assumptions** in the Euler-Bernoulli beam theory are:

- The material is linear elastic according to Hooke's law ("stress is directly proportional to the strain").
- **Plane sections** remain plane and **perpendicular to the neutral axis**, *neglecting shear deformations*.



Euler-Bernoulli beam theory

- The motion of a beam can be described by the Euler-Beroulli beam equation *when the cross-sectional dimensions are small in comparison with the length of the beam*.
- If the cross-sectional dimensions are not negligible, the *effect of the rotatory inertia* should be considered and the motion is better described by the Rayleigh beam equation.
- If the *deflection due to shear* is also taken into account in addition to the rotatory inertia, we arrive at a still more accurate model, which is called the Timoshenko beam.

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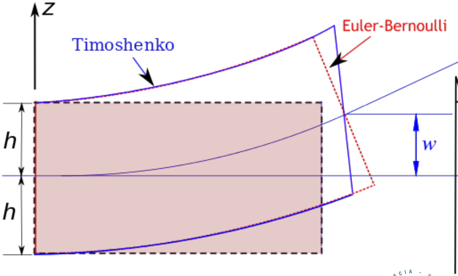
Timoshenko beam theory

Stephen Timoshenko

1878–1972

- Timoshenko beam theory **relaxes** the assumption that the sections remain perpendicular to the neutral axis, thus including shear deformation.



An example

For instance, a *cantilever beam* can be represented by

- Timoshenko model

$$w_{tt}(x,t) - [w_x(x,t) - \phi(x,t)]_x = 0, \qquad \text{in } (0,\ell)\times(0,+\infty),$$

$$\frac{1}{\alpha}\phi_{tt}(x,t) - \frac{1}{\beta}\phi_{xx}(x,t) + w_x(x,t) - \phi(x,t) = 0, \quad \text{in } (0,\ell)\times(0,+\infty),$$

with boundary conditions

$$w(0,t) = 0, \quad \phi(0,t) = 0, \qquad \text{in } (0,+\infty),$$

$$w_x(\ell,t) - \phi(\ell,t) = \phi_x(\ell,t) = 0 \quad \text{in } (0,+\infty).$$

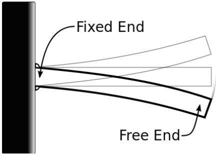
- Euler-Bernoulli model

$$w_{tt}(x,t) + \frac{1}{\beta}w_{xxxx}(x,t) = 0, \quad \text{in } (0,\ell)\times(0,+\infty),$$

with boundary conditions

$$w(0,t) = 0, \quad w_x(0,t) = 0 \quad \text{in } (0,+\infty),$$

$$w_{xx}(\ell,t) = w_{xxx}(\ell,t) = 0, \qquad \text{in } (0,+\infty).$$



Our research topics

With respect to Timoshenko systems our research activity is focusing on three main topics:

- Contact between beams and obstacles: modeling and longtime behavior of the associated energy.
- Stabilization of beams by a boundary dissipation.
- Long-term dynamics of coupled suspension bridge systems with localized dissipation.

joint work with

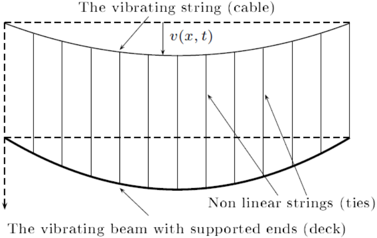
Alessia Berti

Ivana Bochicchio

Jaime E. Muñoz Rivera

- A. Berti, J. E. Muñoz Rivera, M.G. N., *A contact problem for a thermoelastic Timoshenko beam*, Z. Angew. Math. Phys., 66 (2015), no. 4, 1969-1986.
- I. Bochiocchio, J. E. Muñoz Rivera, M. G. N., *Long-term dynamics of the coupled suspension bridge system with localized Kelvin-Voigt dissipation*, work in progress.
- J. E. Muñoz Rivera, M. G. N., *Exponential stability to Timoshenko system with one boundary dissipation*, work in progress.

Coupled suspension bridge system with localized dissipation

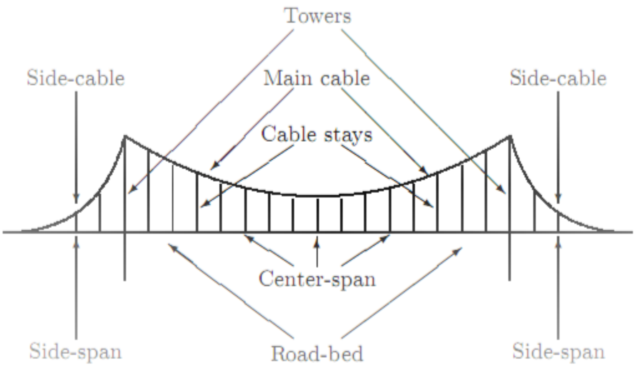


- At the begin of our study, we addressed our attention to the pioneer Glover-Lazer-McKenna approach in

  J. Glover, A.C. Lazer, P.J. McKenna, Existence and stability of large scale nonlinear oscillations in suspension bridges, Z. Angew. Math. Phys. 40 (1989), no. 2, 172–200.

  which firstly described the vertical vibrations of a non linear dynamical system modeling a suspension bridge.
- The **non linear aspect** is caused by the presence of supporting cable stays, which restrain the movement of the center span of the bridge in a downward direction, but have non influence on its behavior in the opposite direction.

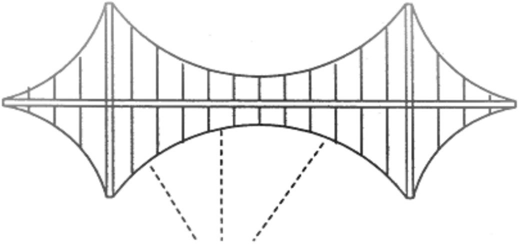
Coupled suspension bridge system with localized dissipation



- We focus on the transmission problem of a suitable dynamical system which models the motion of the deck **coupled** with the motion of the main cable holding the suspending cables.
- The **deck** can be modeled as a vibrating one-dimensional beam.
- The **main cable** can be modeled as a vibrating string.

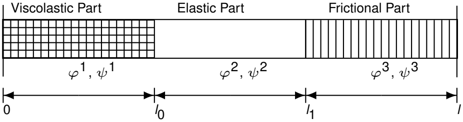
Coupled suspension bridge system with localized dissipation

- *In our model*, in describing the vibrations of a coupled suspension bridge, we consider a linear problem since
  - on one hand we neglect the effect of elongation of the road-bed,
  - on other the main cable, modeled by an elastic string, is connected to the road-bed by a *distributed system of only linear springs*.
- Precisely, we let the road-bed be supported by a symmetrical system of one-sided elastic ties (cable stays), each of which fastened on two symmetrically placed main (suspension) cables, one above and one below the road bed .

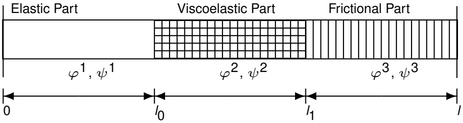


Coupled suspension bridge system with localized dissipation
Different kinds of localized dissipation

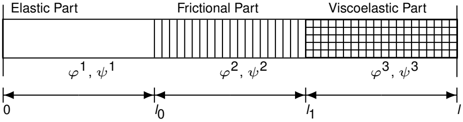
VEF Model



EVF Model

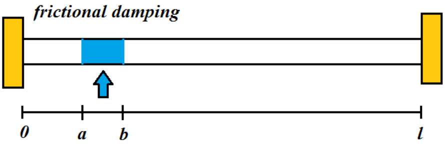


EFV Model

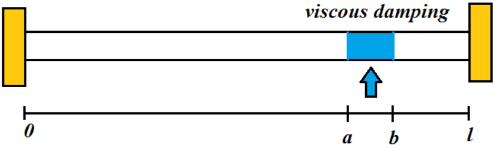


Some references

- M.L. Santos, D.S. Almeida Júnior, J.H. Rodrigues, F.A. Falcão Nascimento, *Decay rates for Timoshenko system with nonlinear arbitrary localized damping*, Differential Integral Equations 27 (2014), no. 1–2, 1–26.



- H.L. Zhao, K.S. Liu, C.G. Zhang, *Stability for the Timoshenko beam system with local Kelvin-Voigt damping*, Acta Math. Sin. (Engl. Ser.) 21 (2005), no. 3, 655–666.



Evolutionary system

The evolutionary system consists on a *wave equation* coupled to *beam equations of Timoshenko type* and it is given by

$$\begin{cases} \rho v_{tt}(x,t) - \alpha v_{xx}(x,t) - \boxed{\beta\left[\varphi(x,t) - v(x,t)\right]} = -\gamma_0 v_t(x,t) \\ \rho_1(x)\varphi_{tt}(x,t) - S_x + \boxed{\beta\left[\varphi(x,t) - v(x,t)\right]} = -\gamma_1(x)\varphi_t(x,t) \\ \rho_2(x)\psi_{tt}(x,t) - M_x + S = -\gamma_2(x)\psi_t(x,t) \end{cases} \tag{1}$$

- $v = v(x,t) : [0,\ell]\times[0,T] \to \mathbb{R}$ vertical displacement of the main cable
- $\varphi = \varphi(x,t) : [0,\ell]\times[0,T] \to \mathbb{R}$ vertical deflection of the beam's cross section
- $\psi = \psi(x,t) : [0,\ell]\times[0,T] \to \mathbb{R}$ the angle of rotation of a cross section (that is supposed to remain plane).

Evolutionary system

Denoting by $\mathfrak{I}^i = (\ell_{i-1}, \ell_i)$, $i = 1, 2, 3$, where $0 = \ell_0 < \ell_1 < \ell_2 < \ell_3 = \ell$, we have

$$\begin{cases} \rho v_{tt}(x,t) - \alpha v_{xx}(x,t) - \beta\,[\varphi(x,t) - v(x,t)] = -\gamma_0 v_t(x,t) \\ \rho_1(x)\varphi_{tt}(x,t) - S_x + \beta\,[\varphi(x,t) - v(x,t)] = -\gamma_1(x)\varphi_t(x,t) \\ \rho_2(x)\psi_{tt}(x,t) - M_x + S = -\gamma_2(x)\psi_t(x,t) \end{cases}$$

- M stands for the bending moment and S for the shear force:

$$M = b(x)\psi_x(x,t) + \boxed{b_0(x)\psi_{xt}(x,t)},$$

$$S = \kappa(x)\,[\varphi_x(x,t) + \psi(x,t)] + \boxed{\kappa_0(x)\,[\varphi_{xt}(x,t) + \psi_t(x,t)]}.$$

- $\kappa(x) = \kappa_i \in \mathbb{R}^+$, per $x \in \mathfrak{I}^i$, is related to the *shear modulus of elasticity*,
- $b(x) = b_i \in \mathbb{R}^+$, per $x \in \mathfrak{I}^i$, is related to *rigidity coefficients of cross section of the beam*,
- $\kappa_0(x) = \begin{cases} \kappa_0, & \text{if } x \in \mathfrak{I}^i_V, \\ 0, & \text{otherwise}. \end{cases}$ and $b_0(x) = \begin{cases} b_0, & \text{if } x \in \mathfrak{I}^i_V, \\ 0, & \text{otherwise}. \end{cases}$, with $\kappa_0, b_0 \in \mathbb{R}^+$ account for the **viscoelastic** component

Evolutionary system

Moreover,

$$\begin{cases} \rho v_{tt}(x,t) - \alpha v_{xx}(x,t) - \beta\left[\varphi(x,t) - v(x,t)\right] = -\gamma_0 v_t(x,t) \\ \rho_1(x)\, \varphi_{tt}(x,t) - S_x + \beta\left[\varphi(x,t) - v(x,t)\right] = -\, \gamma_1(x)\, \varphi_t(x,t) \\ \rho_2(x)\, \psi_{tt}(x,t) - M_x + S = -\, \gamma_2(x)\, \psi_t(x,t) \end{cases} \tag{2}$$

- $\rho_j(x) = \begin{cases} \rho_j^i, & \text{if } x \in \mathfrak{I}^i, \\ 0, & \text{otherwise.} \end{cases}$, for $j = 1, 2$ and with $\rho_j^i \in \mathbb{R}^+$.
- $\gamma_j(x) = \begin{cases} \gamma_j^i, & \text{if } x \in \mathfrak{I}_F^i, \\ 0, & \text{otherwise.} \end{cases}$, for $j = 1, 2$ and with $\gamma_j^i \in \mathbb{R}^+$ account for the **frictional** component.

Evolutionary system

Initial data:

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{in } (0,\ell),$$
$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) \quad \text{in } (0,\ell),$$
$$\psi(x,0) = \psi_0(x), \quad \psi(x,0) = \psi_1(x) \quad \text{in } (0,\ell).$$

Boundary conditions:

$$v(0,t) = v(\ell,t) = 0, \quad \varphi(0,t) = \psi(0,t) = 0, \quad \varphi(\ell,t) = \psi(\ell,t) = 0.$$

Trasmission conditions:

$$\varphi^i(\ell_i,t) = \varphi^{i+1}(\ell_i,t), \quad \psi^i(\ell_i,t) = \psi^{i+1}(\ell_i,t),$$
$$S^i(\ell_i,t) = S^{i+1}(\ell_i,t), \quad M^i(\ell_i,t) = M^{i+1}(\ell_i,t).$$

The semigroup approach

- Introducing the state vector $Z(t) = \Big(v(t), \tilde{v}(t), \varphi(t), \tilde{\varphi}(t), \psi(t), \tilde{\psi}(t)\Big)$, our system becomes the linear ODE in $\mathcal{H}$

$$\frac{d}{dt} Z(t) = \mathbb{A}\, Z(t).$$

- An application of the classical Lumer-Phillips theorem shows that the operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup

$$S(t) = e^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Exponential stability

- Recalling a standard and widely used technique for the investigation of the decay properties of an abstract contraction semigroup $S(t) = e^{t\mathbb{A}}$ on a Hilbert space:

Lemma

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ generated by $\mathcal{A}$. Then the semigroup is exponentially stable if and only if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq C \qquad \forall \lambda \in \mathbb{R}.$$

(see J. Prüss, *On the spectrum of C_0-semigroups*, Trans. AMS., **284** (1984), 847–857)

we can prove

Theorem

The semigroup associated to the transmission problem decays exponentially as time goes to infinity if and only if the viscous (V) component is not in the middle of the beam.

- This means that the VEF, EFV, VFE, and FEV models are exponentially stable.

Sketch of the proof

In particular, in order to prove that the resolvent operator is uniformly bounded over the imaginary axes, we apply some observability inequalities of the following type

Lemma

Any strong solution of the system

$$
\begin{aligned}
i\lambda v - V &= f_1 \quad \text{in} \quad (a,b)\\
i\lambda \varphi - \Phi &= f_2^1 \quad \text{in} \quad (a,b)\\
i\lambda \psi - \Psi &= f_2^2 \quad \text{in} \quad (a,b)\\
i\lambda \rho V - \alpha v_{xx} + \gamma_0 V - \beta(\varphi - v) &= \rho f_3 \quad \text{in} \quad (a,b)\\
i\lambda \rho_1 \Phi - \kappa(\varphi_x + \psi)_x + \beta(\varphi - v) + \gamma_1 \Phi &= \rho_1 f_4^1 \quad \text{in} \quad (a,b)\\
i\lambda \rho_2 \Psi + \kappa(\varphi_x + \psi) - b\psi_{xx} + \gamma_2 \Psi &= \rho_2 f_4^2 \quad \text{in} \quad (a,b)
\end{aligned}
$$

verifies, in any elastic interval I_E,

$$
\begin{aligned}
&|\varphi_x(a) + \psi(a)|^2 + |\psi_x(a)| + |\Phi(a)|^2 + |\Psi(a)|^2 + |\varphi_x(b) + \psi(b)|^2 + \\
&|\psi_x(b)| + |\Phi(b)|^2 + |\Psi(b)|^2 \le C \int_{I_E} \left(|\Phi|^2 + |\Psi|^2 + |\varphi_x + \psi|^2 + |\psi_x|^2 \right) dx + C\,\|Z\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} \; ;
\end{aligned}
$$

$$
\begin{aligned}
\int_{I_E} \left(|\Phi|^2 + |\Psi|^2 + |\varphi_x + \psi|^2 + |\psi_x|^2 \right) dx \le \quad & C\left(|\varphi_x(a) + \psi(a)|^2 + |\psi_x(a)| + |\Phi(a)|^2 + |\Psi(a)|^2 \right) + \\
& C\,\|Z\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\\
\int_{I_E} \left(|\Phi|^2 + |\Psi|^2 + |\varphi_x + \psi|^2 + |\psi_x|^2 \right) dx \le \quad & C\left(|\varphi_x(b) + \psi(b)|^2 + |\psi_x(b)| + |\Phi(b)|^2 + |\Psi(b)|^2 \right) + \\
& C\,\|Z\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}
\end{aligned}
$$

where $Z = (v, \varphi, \psi, V, \Phi, \Psi) = (v, \varphi, \psi, v_t, \varphi_t, \psi_t)$ and $F = (f_1, f_2^1, f_2^2, f_3, f_4^1, f_4^2) \in \mathcal{H}$.

The lack of exponential stabiliy

If elastic and frictional part are not contiguous, but divided by a viscoelastic ones, namely EVF, FVE, the model is not exponential stable.

Theorem

The system, where elastic and frictional part are not contiguous, but separated by a viscoelastic ones, is not exponential stable.

Theorem

The solution of the system in which each elastic part is not associated to a frictional ones decays polynomially as t^{-2}. Moreover the rate of decay is optimal over $\mathcal{D}(\mathcal{A})$.

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The solution of the system in which each elastic part is not associated to a frictional ones decays polynomially as t^{-2}. Moreover the rate of decay is optimal over $\mathcal{D}(\mathcal{A})$.

Polynomial decay

To show the polynomial decay and the optimality we use a result appeared in

A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Math. Ann., ***347*** *(2009), pp. 455–478.*

Lemma

Let $(S(t))_{t\geq 0}$ *be a bounded* C_0*-semigroup on a Hilbert space* $\mathcal{H}$ *with generator* $\mathcal{A}$ *such that* $i\mathbb{R} \subset \rho(\mathcal{A})$*. Then*

$$\frac{1}{|\lambda|^{\alpha}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq C \quad \forall \lambda \in \mathbb{R} \quad \Leftrightarrow \left\| S(t)\mathcal{A}^{-1} \right\|_{\mathcal{D}(\mathcal{A})} \leq \frac{C}{t^{1/\alpha}}$$

Timoshenko beam with one boundary dissipation
Modeling

- Let us consider the mechanical behaviour of a Timoshenko homogeneous beam of length ℓ.
- We denote by $\varphi = \varphi(x,t) : (0,\ell) \times (0,+\infty) \to \mathbb{R}$ the transverse displacement (vertical deflection) of the cross section at $x \in (0,\ell)$ and at time $t \in (0,+\infty)$.
- Assuming that plane cross sections remain plane, the angle of rotation of a cross section is defined by $\psi = \psi(x,t) : (0,\ell) \times (0,+\infty) \to \mathbb{R}$.

The evolution of the system is given by

$$\begin{aligned} &\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 && \text{in } (0,\ell) \times (0,+\infty), \\ &\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0 && \text{in } (0,\ell) \times (0,+\infty), \end{aligned}$$

where $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = KAG$, $b = EI$. Here $S = \kappa(\varphi_x + \psi)$ stands for the shear force, $M = b\psi_x$ the bending moment, ρ denotes the density, A the cross-sectional area, I is the area moment of inertia, K the shear coefficient for measuring the stiffness of materials ($K < 1$), E and G are elastic constants.

Timoshenko beam with one boundary dissipation
Modeling

We supplement our system with initial conditions

$$\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x) \quad \text{in } (0,\ell),$$

with boundary conditions

$$\varphi(0,t)=0, \quad \varphi_x(\ell,t)=0, \quad \psi(\ell,t)=0 \quad \text{in } (0,+\infty),$$

and with a boundary dissipation acting only at $x=0$, namely

$$b\,\psi_x(0,t)=\kappa\,\psi_t(0,t) \quad \text{in } (0,+\infty).$$

Some references

J.U. Kim, Y. Renardy, Boundary control of the Timoshenko beam, SIAM J. Control Optim. 25 (1987), no. 6, 1417–1429.

- The Timoshenko beam can be uniformly stabilized by means of a boundary control.
- The boundary control corresponds to a *control mechanism* which monitors φ_t and ψ_t at $x = \ell$, and transforms them into the lateral force and moment applied ad $x = \ell$, respectively, namely

$$K\psi(\ell, t) - K\,\varphi_x(\ell, t) = \alpha\, w_t(\ell, t)$$

$$EI\psi_x(\ell, t) = -\beta\,\psi_t(\ell, t).$$

F. Ammar-Khodja, S. Kerbal, A. Soufyane, Stabilization of the nonuniform Timoshenko beam, J. Math. Anal. Appl. 327 (2007), no. 1, 525–538.

- The authors consider a Timoshenko beam with variable physical parameters.
- Exploiting the fact that the Timoshenko beam consists of two weakly coupled waves, the authors show that if the two wave speeds are the same, then the beam can achieve uniform stability with a feedback acting only on the rotation angle or under the following boundary conditions:

$$\psi_x(0, t) = c\,\psi_t(0, t), \quad \psi_x(\ell, t) = -d\,\psi_t(\ell, t), \quad c, d > 0.$$

In particular, the uniform stability under boundary dissipation has been proved up to a finite-dimensional space of initial data.

Timoshenko beam with one boundary dissipation

Main aim

- Our main aim is to prove that the exponential stability holds if and only if the two wave speeds are the same and the coefficients of the beam satisfy the following property

$$\frac{\rho_1 \kappa}{\rho_1 b + \rho_2 \kappa} \left(\frac{2\ell}{\pi} \right)^2 \neq \frac{(j_1^2 - j_2^2)^2}{(j_1^2 + j_2^2)}, \quad \text{with } j_1, j_2 \in \mathbb{Z} \setminus \{0\}.$$

Timoshenko beam with one boundary dissipation

Sketch of the proof

- We appeal to the frequency domain approach:

Theorem

Let $S(t) = e^{\mathbb{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if

(i) $i\mathbb{R} \subset \rho(\mathbb{A})$, *where* $\rho(\mathbb{A})$ *denotes the resolvent set of* $\mathbb{A}$, *and*

(ii) $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathbb{I} - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty.$

Timoshenko beam with one boundary dissipation
Sketch of the proof

- The following Lemma is fundamental to obtain the exponential decay of the related energy.

Lemma

Let us suppose that the two wave speeds are the same. Then, $i\mathbb{R} \subset \rho(\mathbb{A})$*, if and only if*

$$\frac{\rho_1 \kappa}{\rho_1 b + \rho_2 \kappa}\left(\frac{2\ell}{\pi}\right)^2 \neq \frac{(j_1^2 - j_2^2)^2}{(j_1^2 + j_2^2)}, \quad \textit{with } j_1, j_2 \in \mathbb{Z} \setminus \{0\}.$$