

Inverse problems for linear PDEs via variational formulations: Robust numerical approximations

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General context and purpose

Given a suitable observation $y_{obs}(= B(y))$ of y , unique solution of a linear well-posed PDE

$$\left\{ \begin{array}{l} PDE(y, \nabla y, \dots) = f, \quad \Omega \times (0, T), \\ + \text{boundary and initial conditions} \end{array} \right\},$$

find a convergent (numerical) approximation of the following **linear inverse problem** :

reconstruct the solution y and the source f such that $B(y) = y_{obs}$.

The main aim is to highlight that space-time **variational approach** of first and second order leads to robust approximation.

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Hyperbolic situation

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ ($N \geq 1$) - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \mathbf{H}$, $f \in X$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & \Omega. \end{cases} \quad (1)$$

► Inverse Problem 1: Distributed observation on $q_T = \omega \times (0, T)$, $\omega \subset \Omega$

$$\begin{cases} H = L^2 \times H^{-1}, X = L^2(H^{-1}), \\ \text{Given } (y_{obs}, f) \in L^2(q_T) \times X, \text{ find } y \text{ s.t. } \{(1) \text{ and } y - y_{obs} = 0 \text{ on } q_T\} \end{cases}$$

► Inverse Problem 2: Boundary observation on $\Gamma_T \subset \partial\Omega \times (0, T)$

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Inverse problem 1

$$Z := \left\{ y : y \in C([0, T], L^2(\Omega)) \cap C^1([0, T], H^{-1}(\Omega)), Ly \in X, y|_{\Sigma_T} = 0 \right\}.$$

Introducing the operator $P : Z \rightarrow X \times L^2(q_T)$

$$Py := (Ly, y|_{q_T}),$$

Inverse Problem 1 is reformulated as :

$$\text{find } y \in Z \text{ solution of } Py = (f, y_{obs}). \quad (IP)$$

If a unique continuation property holds for (1) and if y_{obs} is a restriction to q_T of a solution of (1), then (IP) is well-posed: the state y corresponding to the pair (y_{obs}, f) is unique.

Most natural approach: Relaxation via Least-squares method

The most natural (and widely used in practice) approach consists in introducing a **least-squares type technic**, i.e. consider the extremal problem

$$(LS) \quad \begin{cases} \text{minimize} & J(y_0, y_1) := \frac{1}{2} \|y - y_{obs}\|_{L^2(q_T)}^2 \\ \text{subject to} & (y_0, y_1) \in \mathbf{H} \\ \text{where } y & \text{solves (1)} \end{cases}$$

A minimizing sequence $(y_{0k}, y_{1k})_{(k>0)} \in \mathbf{H}$ is defined in term of an adjoint problem.

Drawback : it is difficult to minimize over a finite dimensional subspace of the set of constraints

The minimization procedure **first** requires the **discretization of J** and of the system (1);

This raises the issue of **uniform coercivity property** of the discrete functional w.r.t. the approximation parameter.

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A not so different approach : Luenberger observers type approach

[Auroux-Blum 2005],[Chapelle,Cindea,Moireau,2012], [Ramdani-Haine 2011], etc...

Define a dynamic

$$\begin{aligned} L\bar{y} &= G(y_{obs}, q_T) \\ \bar{y}(\cdot, 0) &\text{ fixed} \end{aligned}$$

such that

$$\|\bar{y}(\cdot, t) - y(\cdot, t)\|_{N(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

$N(\Omega)$ - appropriate norm

The **reversibility** of the eq. then allows to recover y for any time.

But again, on a numerically point of view, this method requires to prove uniform discrete observability properties.

Keeping y as the main variable ...

Without loss of generality, $f \equiv 0$.

$$Z := \{y : y \in C([0, T], L^2(\Omega)) \cap C^1([0, T], H^{-1}(\Omega)), Ly \in X, y|_{\Sigma_T} = 0\}.$$

Hypothesis (Generalized Observability Inequality)

Assume that $\exists C_{obs} = C(\omega, T, \|c\|_{C^1(\bar{\Omega})}, \|d\|_{L^\infty(\Omega)})$ s.t. :

$$(H) \quad \|y(\cdot, 0), y_t(\cdot, 0)\|_H^2 \leq C_{obs} \left(\|y\|_{L^2(Q_T)}^2 + \|Ly\|_X^2 \right), \quad \forall y \in Z. \quad (2)$$

- in 1-D, (2) if $T \geq T^*(c, d)$ [Fernandez-Cara, Cindea, Münch, COCV 2013],
- in N-D, for $c = 1$ and $d = 0$, (2) if (Ω, ω, T) satisfies geometric optic condition [Bardos, Lebeau, Rauch, 1992]

$$\|y\|_{L^2(Q_T)}^2 \leq C_{\Omega, T} \left(C_{obs} \|y\|_{L^2(Q_T)}^2 + (1 + C_{obs}) \|Ly\|_X^2 \right) \quad \forall y \in Z. \quad (3)$$

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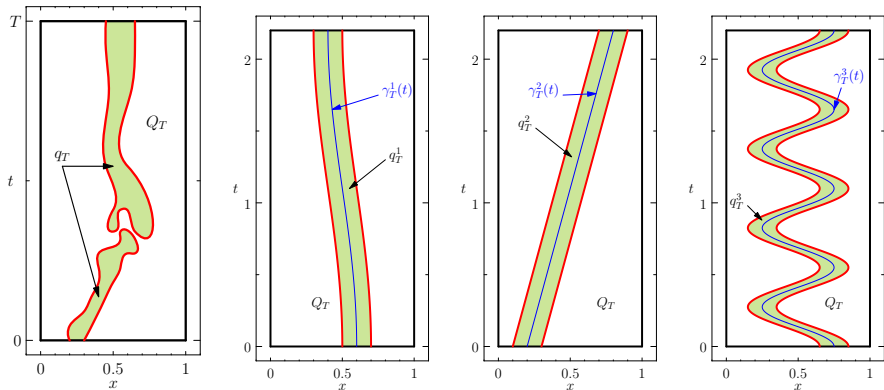
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Non cylindrical situation in 1D

[Castro-Cindea-Münch, SICON 2014],

In 1D with $c \equiv 1$ and $d \equiv 0$, the observability ineq. also holds for non cylindrical domains.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

Equivalent formulation of IP

Within this hypothesis, for **any** $\eta > 0$, we define on Z the bilinear form

$$\langle y, \bar{y} \rangle_Z := \iint_{q_T} y \bar{y} \, dx dt + \eta \int_0^T \langle Ly, L\bar{y} \rangle_{H^{-1}(\Omega)} \, dt \quad \forall y, \bar{y} \in Z. \quad (4)$$

$(Z, \|\cdot\|)$ is a Hilbert space.

Then, we consider the extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \frac{1}{2} \|y - y_{obs}\|_{L^2(q_T)}^2 + \frac{\eta}{2} \|Ly\|_{X'}^2, & \eta \geq 0 \\ \text{subject to } y \in W := \{y \in Z; Ly = 0 \text{ in } X\} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W , strictly convex and $J(y) \rightarrow +\infty$ as $\|y\|_W \rightarrow \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η .

From (2), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot, 0), y_t(\cdot, 0)) \in \mathbf{H}$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in \mathbf{H}$.

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Optimality of $(\mathcal{P})$

In order to solve $(\mathcal{P})$, we have to deal with the constraint eq. which appears in W . We introduce a **Lagrange multiplier** $\lambda \in X'$ and the following mixed formulation: find $(y, \lambda) \in Z \times X'$ solution of

$$\begin{cases} a_r(y, \bar{y}) + b(\bar{y}, \lambda) &= l(\bar{y}), & \forall \bar{y} \in Z \\ b(y, \bar{\lambda}) &= 0, & \forall \bar{\lambda} \in \Lambda, \end{cases} \quad (5)$$

where

$$a_r : Z \times Z \rightarrow \mathbb{R}, \quad a_r(y, \bar{y}) := \iint_{q_T} y \bar{y} \, dx dt + r \int_0^T \langle Ly, L\bar{y} \rangle_{H^{-1}(\Omega)} \, dt,$$

$$b : Z \times X' \rightarrow \mathbb{R}, \quad b(y, \lambda) := \int_0^T \langle \lambda, Ly \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \, dt,$$

$$l : Z \rightarrow \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs} y \, dx dt.$$

System (20) is the **optimality system** corresponding to the extremal problem $(\mathcal{P})$.

Well-posedness of the mixed formulation

Theorem

Under the hypothesis $(\mathcal{H})$, for any $r \geq 0$,

1. The mixed formulation (20) is well-posed.
2. The unique solution $(y, \lambda) \in Z \times X'$ is the unique *saddle-point* of the Lagrangian $\mathcal{L} : Z \times X' \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. We have the estimate

$$\|y\|_Y = \|y\|_{L^2(q_T)} \leq \|y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{X'} \leq 2\sqrt{C_{\Omega,T} + \eta} \|y_{obs}\|_{L^2(q_T)}. \quad (6)$$

Well-posedness

The kernel $\mathcal{N}(b) = \{y \in Z; b(y, \lambda) = 0 \quad \forall \lambda \in X'\}$ coincides with W : we easily get

$$a_r(y, y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the **inf-sup constant property** : $\exists \delta > 0$ such that

$$\inf_{\lambda \in X'} \sup_{y \in Z} \frac{b(y, \lambda)}{\|y\|_Z \|\lambda\|_{X'}} \geq \delta. \quad (7)$$

For any fixed $\lambda \in X'$, we define $y^0 \in Z$ as the unique solution of

$$Ly^0 = -\Delta \lambda \text{ in } Q_T, \quad (y^0(\cdot, 0), y_t^0(\cdot, 0)) = (0, 0) \text{ on } \Omega, \quad y^0 = 0 \text{ on } \Sigma_T.$$

We get $b(y^0, \lambda) = \|\lambda\|_{X'}^2$, and $\|y^0\|_Z^2 = \|y^0\|_{L^2(Q_T)}^2 + \eta \|\lambda\|_{X'}^2$.

The estimate $\|y^0\|_{L^2(Q_T)} \leq \sqrt{C_{\Omega, T}} \|\lambda\|_{X'}$ implies that

$$\sup_{y \in Z} \frac{b(y, \lambda)}{\|y\|_Y \|\lambda\|_{X'}} \geq \frac{b(y^0, \lambda)}{\|y^0\|_Y \|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega, T} + \eta}} > 0$$

leading to (7) with $\delta = (C_{\Omega, T} + \eta)^{-1/2}$.

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Remark 1

From the first eq.,

$$\iint_{q_T} (y - y_{obs}) \bar{y} dt dx + \int_0^T \langle \lambda, L\bar{y} \rangle_{H_0^1, H^{-1}} dt = 0, \quad \forall \bar{y} \in Z$$

the multiplier $\lambda \in X'$ solves in the sense of transposition

$$\begin{cases} L\lambda = -(y - y_{obs}) \mathbf{1}_{q_T}, & \lambda = 0 \quad \text{in } \Sigma_T, \\ \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0 & \text{in } \Omega \end{cases} \quad (8)$$

If y_{obs} is the restriction to q_T of a solution of (1), then λ vanishes almost everywhere.

In that case, $\sup_{\lambda \in \Lambda} \inf_{y \in Y} \mathcal{L}_r(y, \lambda) = \inf_{y \in Y} \mathcal{L}_r(y, 0)$

The mixed formulation is reduced to : find $y \in Z$ such that

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$$a_r(y, \bar{y}) = \iint_{q_T} y \bar{y} dx dt + r \int_0^T \langle Ly, L\bar{y} \rangle_{H^{-1}, H^{-1}(\Omega)} dt = l(\bar{y}), \quad \forall \bar{y} \in Z.$$

Remark 2

In the general case, the mixed formulation can be rewritten as follows: find $(z, \lambda) \in Z \times X'$ solution of

$$\begin{cases} \langle P_r y, P_r \bar{y} \rangle_{X \times L^2(q_T)} + \langle L \bar{y}, \lambda \rangle_{X, X'} = \langle (0, y_{obs}), P_r \bar{y} \rangle_{X \times L^2(q_T)}, & \forall \bar{y} \in Z, \\ \langle L y, \bar{\lambda} \rangle_{X, X'} = 0, & \forall \bar{\lambda} \in X' \end{cases}$$

with $P_r y := (\sqrt{r} L y, y|_{q_T})$.

Analogy with the [quasi-reversibility method](#) [Klibanov-Beilina 08, Bourgeois-Darde 10]:
for any $\varepsilon > 0$, find $y_\varepsilon \in Z$ such that

$$\langle P y_\varepsilon, P \bar{y} \rangle_{X \times L^2(q_T)} + \varepsilon \langle y_\varepsilon, \bar{y} \rangle_Z = \langle (f, y_{obs}), P \bar{y} \rangle_{X \times L^2(q_T), X \times L^2(q_T)}, \quad \forall \bar{y} \in Z, \quad (QR)$$

equivalent to the minimization over Z of

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Remark 3: Stabilized mixed formulation

$$\Lambda := \{\lambda \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), L\lambda \in L^2(Q_T), \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0\}.$$

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For $\alpha \geq 0$, find $(y, \lambda) \in Z \times \Lambda$ such that

$$\begin{cases} a_{r,\alpha}(y, \bar{y}) + b_\alpha(\bar{y}, \lambda) &= l_{1,\alpha}(\bar{y}), & \forall \bar{y} \in Y \\ b_\alpha(y, \bar{\lambda}) - c_\alpha(\lambda, \bar{\lambda}) &= l_{2,\alpha}(\bar{\lambda}), & \forall \bar{\lambda} \in \tilde{\Lambda}, \end{cases} \quad (10)$$

$$a_{r,\alpha} : Z \times Z \rightarrow \mathbb{R}, \quad a_{r,\alpha}(y, \bar{y}) := (1 - \alpha) \iint_{Q_T} y \bar{y} \, dx dt + r \int_0^T (Ly, L\bar{y})_{H^{-1}(\Omega)} \, dt,$$

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Remark 3: Stabilized mixed formulation

Proposition

Under the hypothesis $(\mathcal{H})$, for any $\alpha \in (0, 1)$, the corresponding mixed formulation is well-posed. The unique pair (y, λ) in $Z \times \Lambda$ satisfies

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_\Lambda^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|_{L^2(q_T)}^2. \quad (11)$$

with $\theta_1 := \min\left(1 - \alpha, r \eta^{-1}\right)$, $\theta_2 := \frac{1}{2} \min\left(\alpha, C_{\Omega, T}^{-1}\right)$.

Proposition

If the solution $(y, \lambda) \in Z \times X'$ of (20) enjoys the property $\lambda \in \Lambda$, then the solutions of (20) and (10) coincide.

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Remark 4 - Link with controllability

The mixed formulation has a structure very closed to the one we get when we address - using the same approach - the null controllability of (1): the **control of minimal $L^2(q_T)$ -norm** which drives to rest $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is given by $v = \varphi 1_{q_T}$ where $(\varphi, \lambda) \in \Phi \times L^2(0, T; H_0^1(\Omega))$ solves

$$\begin{cases} a(\varphi, \bar{\varphi}) + b(\bar{\varphi}, \lambda) &= I(\bar{\varphi}), & \forall \bar{\varphi} \in \Phi \\ b(\varphi, \bar{\lambda}) &= 0, & \forall \bar{\lambda} \in L^2(0, T; H_0^1(\Omega)), \end{cases} \quad (12)$$

where

$$a : \Phi \times \Phi \rightarrow \mathbb{R}, \quad a(\varphi, \bar{\varphi}) = \iint_{q_T} \varphi(x, t) \bar{\varphi}(x, t) \, dx \, dt$$

$$b : \Phi \times L^2(0, T; H_0^1(0, 1)) \rightarrow \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T \langle L\varphi, \lambda \rangle_{H^{-1}, H_0^1} \, dt$$

$$I : \Phi \rightarrow \mathbb{R}, \quad I(\varphi) = - \langle \varphi_t(\cdot, 0), y_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^1 \varphi(\cdot, 0) y_1 \, dx.$$

with $\Phi = \{\varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0, 1))\}$.

[Cîndea- Münch, Calcolo 2015]

Remark 5

"Reversing the order of priority" between the constraint $y - y_{obs} = 0$ in $L^2(q_T)$ and $Ly - f = 0$ in X , a possibility could be to minimize the functional

$$\begin{cases} \text{minimize} & J(y) := \|Ly - f\|_X^2 + \varepsilon \|y\|_{\mathcal{A}}^2 \\ \text{subject to} & y \in Z \quad \text{and to} \quad y - y_{obs} = 0 \quad \text{in} \quad L^2(q_T) \end{cases} \quad (13)$$

via the introduction of a Lagrange multiplier in $L^2(q_T)$.

The proof of the inf-sup property : there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(q_T)} \sup_{y \in Z} \frac{\iint_{q_T} \lambda y \, dx dt}{\|\lambda\|_{L^2(q_T)} \|y\|_Y} \geq \delta$$

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Remark 6 : Dual of the mixed problem - Minimization over λ

Theorem

$$\sup_{\lambda \in X'} \inf_{y \in Z} \mathcal{L}_r(y, \lambda) = - \inf_{\lambda \in X'} J_r^{**}(\lambda) + \mathcal{L}_r(y_0, 0)$$

where $y_0 \in Z$ solves $a_r(y_0, \bar{y}) = l(\bar{y}), \forall \bar{y} \in Y$ and

$$J_r^{**} : X' \rightarrow \mathbb{R}, \quad J_r^{**}(\lambda) = \frac{1}{2} \int_0^T \langle \mathcal{P}_r \lambda, \lambda \rangle_{X'} dt - b(y_0, \lambda).$$

Lemma

Let $\mathcal{P}_r$ be the linear operator from X' into X' defined by

$$\mathcal{P}_r \lambda := -\Delta^{-1}(Ly), \quad \forall \lambda \in X' \quad \text{where } y \in Z \text{ solves } a_r(y, \bar{y}) = b(\bar{y}, \lambda), \quad \forall \bar{y} \in Z.$$

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For any $r > 0$, the operator $\mathcal{P}_r$ is a strongly elliptic, symmetric isomorphism from X' into X' .

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Remark 7 - Boundary observation

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ - Ω of class C^2

The results apply if the distributed observation on q_T is replaced by a Neumann **boundary observation** on a sufficiently large subset Σ_T of $\partial\Omega \times (0, T)$ (i.e. assuming $\frac{\partial y}{\partial \nu} = y_{\nu, obs} \in L^2(\Sigma_T)$ is known on Σ_T).

If (Q_T, Σ_T, T) satisfy some geometric condition, then there exists a positive constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\bar{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that

$$\|y(\cdot, 0), y_t(\cdot, 0)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq C_{obs} \left(\left\| \frac{\partial y}{\partial \nu} \right\|_{L^2(\Sigma_T)}^2 + \|Ly\|_{L^2(Q_T)}^2 \right), \quad \forall y \in Z \quad (15)$$

It suffices to re-define the form a in by $a(y, y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} \frac{\partial \bar{y}}{\partial \nu} d\sigma dx$ and the form l by $l(y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} y_{obs} d\sigma dx$ for all $y, \bar{y} \in Z$.

Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$$f(x, t) = \sigma(t)\mu(x)$$

$$c := 1, d(x, t) = d(x) \in L^p(\Omega), \sigma \in C^1([0, T]), \sigma(0) \neq 0, \mu \in H^{-1}(\Omega)$$

Theorem (Yamamoto-Zhang 2001)

Assume that (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ be the weak solution of (1) with $c := 1$ and $(y_0, y_1) = (0, 0)$. $\exists C > 0$ s.t.

$$C^{-1} \|\mu\|_{H^{-1}(\Omega)} \leq \|c(x) \partial_\nu y\|_{L^2(\Gamma_T)} \leq C \|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\left\{ \begin{array}{l} \inf J(y, \mu) := \frac{1}{2} \|c(x)(\partial_\nu y - y_{\nu, obs})\|_{L^2(\Gamma_T)}^2 + \frac{r}{2} \iint_{Q_T} (Ly - \sigma\mu)^2 dxdt, \\ \text{subject to } (y, \mu) \in W := \left\{ (y, \mu); y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \right. \\ \left. \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot, 0) = y_t(\cdot, 0) = 0 \right\}. \end{array} \right. \quad (\mathcal{P}_{y, \mu})$$

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Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \left\{ (y, \mu); y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \right. \\ \left. Ly - \sigma\mu \in L^2(Q_T), y(\cdot, 0) = y_t(\cdot, 0) = 0 \right\}. \quad (16)$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\bar{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ s.t. :

$$\|\mu\|_{H^{-1}(\Omega)}^2 \leq C_{obs} \left(\|c(x) \partial_\nu y\|_{L^2(\Gamma_T)}^2 + \|Ly - \sigma\mu\|_{L^2(Q_T)}^2 \right), \quad \forall (y, \mu) \in Y. \quad (\mathcal{H}_2)$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y, \mu), (\bar{y}, \bar{\mu}) \rangle_Y := \iint_{\Gamma_T} (c(x))^2 \partial_\nu y \partial_\nu \bar{y} d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)(L\bar{y} - \sigma\bar{\mu}) dx dt \quad \forall y, \bar{y} \in Z. \quad (17)$$

$$\|(y, z)\|_Y := \sqrt{\langle (y, \mu), (y, \mu) \rangle_Y}.$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \left\{ (y, \mu); y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \right. \\ \left. Ly - \sigma\mu \in L^2(Q_T), y(\cdot, 0) = y_t(\cdot, 0) = 0 \right\}. \quad (16)$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\bar{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ s.t. :

$$\|\mu\|_{H^{-1}(\Omega)}^2 \leq C_{obs} \left(\|c(x) \partial_\nu y\|_{L^2(\Gamma_T)}^2 + \|Ly - \sigma\mu\|_{L^2(Q_T)}^2 \right), \quad \forall (y, \mu) \in Y. \quad (\mathcal{H}_2)$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y, \mu), (\bar{y}, \bar{\mu}) \rangle_Y := \iint_{\Gamma_T} (c(x))^2 \partial_\nu y \partial_\nu \bar{y} d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)(L\bar{y} - \sigma\bar{\mu}) dx dt \quad \forall y, \bar{y} \in Z. \quad (17)$$

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$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\bar{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ s.t. :

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$$\langle (y, \mu), (\bar{y}, \bar{\mu}) \rangle_Y := \iint_{\Gamma_T} (c(x))^2 \partial_\nu y \partial_\nu \bar{y} d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)(L\bar{y} - \sigma\bar{\mu}) dx dt \quad \forall y, \bar{y} \in Z. \quad (17)$$

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Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f : mixed formulation

Find $((y, \mu), \lambda) \in Y \times L^2(Q_T)$ solution of

$$\begin{cases} a_r((y, \mu), (\bar{y}, \bar{\mu})) + b((\bar{y}, \bar{\mu}), \lambda) &= l(\bar{y}, \bar{\mu}), & \forall (\bar{y}, \bar{\mu}) \in Y \\ b((y, \mu), \bar{\lambda}) &= 0, & \forall \bar{\lambda} \in L^2(Q_T), \end{cases} \quad (18)$$

where

$$a_r : Y \times Y \rightarrow \mathbb{R}, \quad a_r((y, \mu), (\bar{y}, \bar{\mu})) := \iint_{\Gamma_T} c^2(x) \partial_\nu y \partial_\nu \bar{y} \, d\sigma dt \\ + r \iint_{Q_T} (Ly - \sigma\mu)(L\bar{y} - \sigma\bar{\mu}) \, dx dt, \quad r \geq 0$$

$$b : Y \times L^2(Q_T) \rightarrow \mathbb{R}, \quad b((y, \mu), \lambda) := \iint_{Q_T} \lambda (Ly - \sigma\mu) \, dx \, dt,$$

$$l : Y \rightarrow \mathbb{R}, \quad l(y, \mu) := \iint_{\Gamma_T} c^2(x) \partial_\nu y \, y_{\nu, obs} \, d\sigma dt.$$

PARABOLIC SITUATION

$\Omega \subset \mathbb{R}^N$ ($N \geq 1$) - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $y_0 \in \mathbf{H}$

$$\begin{cases} Ly := y_t - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ y(\cdot, 0) = y_0, & \Omega. \end{cases} \quad (19)$$

► Inverse Problem : **Distributed observation** on $q_T = \omega \times (0, T)$, $\omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } y \text{ s.t. } \{(19) \text{ and } y - y_{obs} = 0 \text{ on } q_T\} \end{cases}$$

WELL-KNOWN DIFFICULTY:

$$\left(Ly \in L^2(Q_T), y \in L^2(q_T), y|_{\Sigma_T} = 0 \right) \implies y \in C([\delta, T], H_0^1(\Omega)), \quad \forall \delta > 0$$

Second order mixed formulation as in the previous part

We then define the following extremal problem :

$$\left\{ \begin{array}{l} \text{Minimize } J(y) := \frac{1}{2} \iint_{q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 dx dt + r \iint_{Q_T} (\rho^{-1} L y)^2 dx dt \\ \text{Subject to } y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1} L y = 0 \text{ in } L^2(Q_T) \right\} \end{array} \right\} \quad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where $(\rho_\star \in \mathbb{R}_\star^+)$

$$\mathcal{R} := \{w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T)) \forall \delta > 0\}$$

Let $\mathcal{Y}_0 := \left\{ y \in C^2(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T \right\}$ and for $\eta > 0, \rho \in \mathcal{R}$, the bilinear form by

$$(y, \bar{y})_{\mathcal{Y}_0} := \iint_{q_T} \rho_0^{-2} y \bar{y} dx dt + \eta \iint_{Q_T} \rho^{-2} L y L \bar{y} dx dt, \quad \forall y, \bar{y} \in \mathcal{Y}_0.$$

Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta \|\rho^{-1} L y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.$$

Second order mixed formulation as in the previous part

We then define the following extremal problem :

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Mixed formulation

Find $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(y, \bar{y}) + b(\bar{y}, \lambda) &= l(\bar{y}) & \forall \bar{y} \in \mathcal{Y}, \\ b(y, \bar{\lambda}) &= 0 & \forall \bar{\lambda} \in L^2(Q_T), \end{cases} \quad (20)$$

where

$$a_r : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}, \quad a(y, \bar{y}) := \iint_{q_T} \rho_0^{-2} y \bar{y} \, dx \, dt + r \iint_{Q_T} \rho^{-2} L y L \bar{y} \, dx \, dt$$

$$b : \mathcal{Y} \times L^2(Q_T) \rightarrow \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \rho^{-1} L y \lambda \, dx \, dt$$

$$l : \mathcal{Y} \rightarrow \mathbb{R}, \quad l(y) := \iint_{q_T} \rho_0^{-2} y y_{obs} \, dx \, dt.$$

Mixed formulation

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$.

1. The mixed formulation (20) is well-posed.
2. The unique solution $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \mathcal{Y} \times L^2(Q_T) \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}_r(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. The solution (y, λ) satisfies the estimates

$$\|y\|_{\mathcal{Y}} \leq \|\rho_0^{-1} y_{\text{obs}}\|_{L^2(Q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \leq 2\sqrt{\rho_\star^{-2} \|\rho\|_{L^\infty(Q_T)}^2 + \eta} \|\rho_0^{-1} y_{\text{obs}}\|_{L^2(Q_T)}.$$

Corollary

Let $\rho_0 \in \mathcal{R}$, $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume $\exists K$ s.t.

$$\rho_0 \leq K \rho_{c,0}, \quad \rho \leq K \rho_c \quad \text{in } Q_T.$$

If (y, λ) is the solution of the mixed formulation (20), then $\exists C > 0$ such that

$$\|\rho_{c,0}^{-1} y\|_{L^2(Q_T)} \leq C \|y\|_{\mathcal{Y}}.$$

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Break ! Open question

Remark

The inf-sup property is open if the weight ρ is simply in $\mathcal{R}$: in that case, the weight may blow up at time $t = 0$. It suffices to prove that the function $z := \rho_0^{-1} y$ solution of

$$\rho^{-1} L(\rho_0 z) = \lambda^0 \text{ in } Q_T, \quad z = 0 \text{ on } \Sigma_T, \quad z(\cdot, 0) = 0 \text{ in } \Omega$$

for any $\lambda^0 \in L^2(Q_T)$ satisfies the following estimate for some positive constant C

$$\|z\|_{L^2(Q_T)} \leq C \|\rho^{-1} L(\rho_0 z)\|_{L^2(Q_T)}.$$

In the case of interest for which both ρ_0 and ρ blow up at $t \rightarrow 0^+$, for instance given by $\rho_c(x, t) = e^{\frac{\beta(x)}{t}} - \rho_{c,0}(x, t) = t^{3/2} \rho_c(x, t)$

$$\begin{aligned} \rho^{-1} L(\rho^0 z) &= t^{3/2} z_t \\ &+ \left((3/2 - c(x) \Delta \beta - \nabla \beta \cdot \nabla c) t^{1/2} - (\beta(x) + \nabla \beta \cdot \nabla c) t^{-1/2} + d t^{3/2} \right) z + \dots \end{aligned}$$

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Stabilization

The first equation of the mixed formulation (20) reads as follows:

$$\iint_{q_T} \rho_0^{-2} y \bar{y} \, dx \, dt + \iint_{Q_T} \rho^{-1} L \bar{y} \lambda \, dx \, dt = \iint_{q_T} \rho_0^{-2} y_{obs} \bar{y} \, dx \, dt \quad \forall \bar{y} \in \mathcal{Y}.$$

$\rho^{-1} \lambda \in L^2(Q_T)$ solves the parabolic equation in the transposition sense, i.e. $\rho^{-1} \lambda$ solves the problem :

$$\begin{cases} L^*(\rho^{-1} \lambda) = -\rho_0^{-2} (y - y_{obs}) 1_{q_T} & \text{in } Q_T, \\ \rho^{-1} \lambda = 0 & \text{on } \Sigma_T, \\ (\rho^{-1} \lambda)(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (21)$$

Therefore, $\rho^{-1} \lambda$ belongs to $C^0([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$.

$$\Lambda := \{ \lambda : \rho^{-1} \lambda \in C^0([0, T]; L^2(\Omega)), \rho_0 L^*(\rho^{-1} \lambda) \in L^2(Q_T), \\ \rho^{-1} \lambda = 0 \text{ on } \Sigma_T, (\rho^{-1} \lambda)(\cdot, T) = 0 \}. \quad (22)$$

$$\begin{cases} \sup_{\lambda \in \Lambda} \inf_{y \in \mathcal{Y}} \mathcal{L}_{r, \alpha}(y, \lambda), \\ \mathcal{L}_{r, \alpha}(y, \lambda) := \mathcal{L}_r(y, \lambda) - \frac{\alpha}{2} \left\| \rho_0 \left(L^*(\rho^{-1} \lambda) + \rho_0^{-2} (y - y_{obs}) 1_{\omega} \right) \right\|_{L^2(Q_T)}^2. \end{cases}$$

Dual formulation

For any $r > 0$, let us define the linear operator $\mathcal{T}_r$ from $L^2(Q_T)$ into $L^2(Q_T)$ by

$$\mathcal{T}_r \lambda := \rho^{-1} L y, \quad \forall \lambda \in L^2(Q_T)$$

where $y \in \mathcal{Y}$ is the unique solution to

$$a_r(y, \bar{y}) = b(\bar{y}, \lambda), \quad \forall \bar{y} \in \mathcal{Y}. \quad (23)$$

Lemma

For any $r > 0$, the operator $\mathcal{T}_r$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Proposition

For any $r > 0$, let $y_0 \in \mathcal{Y}$ be the unique solution of

$$a_r(y_0, \bar{y}) = l(\bar{y}), \quad \forall \bar{y} \in \mathcal{Y}$$

and let $J_r^{**} : L^2(Q_T) \rightarrow L^2(Q_T)$ be the functional defined by

$$J_r^{**}(\lambda) := \frac{1}{2} \iint_{Q_T} (\mathcal{T}_r \lambda) \lambda \, dx \, dt - b(y_0, \lambda).$$

The following equality holds :

$$\sup_{\lambda \in L^2(Q_T)} \inf_{y \in \mathcal{Y}} \mathcal{L}_r(y, \lambda) = - \inf_{\lambda \in L^2(Q_T)} J_r^{**}(\lambda) + \mathcal{L}_r(y_0, 0).$$

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$H_0^1 - L^2$ first order formulation

First order formulation involving y and the flux $\mathbf{p} = c(x)\nabla y$.

$$\begin{cases} \mathcal{I}(y, \mathbf{p}) := y_t - \nabla \cdot \mathbf{p} + d y = f, & \mathcal{J}(y, \mathbf{p}) := c(x)\nabla y - \mathbf{p} = \mathbf{0} & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases} \quad (24)$$

$$(y_0, f) \in L^2(\Omega) \times L^2(Q_T) \implies p \in \mathbf{L}^2(Q_T), y \in L^2(0, T, H_0^1(\Omega)), y_t \in L^2(0, T, H^{-1}(\Omega))$$

► Inverse Problem : **Distributed observation** on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } (y, \mathbf{p}) \text{ s.t. } \{(24) \text{ and } y - y_{obs} = 0 \text{ on } q_T\} \end{cases}$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\left\{ \begin{array}{l} \text{Minimize } J(y, \mathbf{p}) := \frac{1}{2} \iint_{q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 dx dt + \mathbf{r} \dots \\ (y, \mathbf{p}) \in \mathcal{V} := \left\{ (y, \mathbf{p}) \in \mathcal{U} : \rho_1^{-1} \mathcal{J}(y, \mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1} \mathcal{I}(y, \mathbf{p}) = 0 \text{ in } L^2(Q_T) \right\} \end{array} \right\}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \left\{ (y, \mathbf{p}) \in C^1(\overline{Q_T}) \times \mathbf{C}^1(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T \right\}$ for

$$\begin{aligned} ((y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}}))_{\mathcal{U}_0} &= \iint_{q_T} \rho_0^{-2} y \bar{y} dx dt + \eta_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y, \mathbf{p}) \cdot \mathcal{J}(\bar{y}, \bar{\mathbf{p}}) dx dt \\ &\quad + \eta_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y, \mathbf{p}) \mathcal{I}(\bar{y}, \bar{\mathbf{p}}) dx dt \quad \forall (y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}}) \in \mathcal{U}_0. \end{aligned}$$

for any $\eta_1, \eta_2 > 0$ and any $\rho, \rho_0, \rho_1 \in \mathcal{R}$

$$\|(y, \mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta_1 \|\rho_1^{-1} \mathcal{J}(y, \mathbf{p})\|_{L^2(Q_T)}^2 + \eta_2 \|\rho^{-1} \mathcal{I}(y, \mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\left\{ \begin{array}{l} \text{Minimize } J(y, \mathbf{p}) := \frac{1}{2} \iint_{Q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 dx dt + \textcolor{red}{r} \dots \\ (y, \mathbf{p}) \in \mathcal{V} := \left\{ (y, \mathbf{p}) \in \mathcal{U} : \rho_1^{-1} \mathcal{J}(y, \mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1} \mathcal{I}(y, \mathbf{p}) = 0 \text{ in } L^2(Q_T) \right\} \end{array} \right\}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \left\{ (y, \mathbf{p}) \in C^1(\overline{Q_T}) \times \mathbf{C}^1(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T \right\}$ for

$$\begin{aligned} ((y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}}))_{\mathcal{U}_0} &= \iint_{Q_T} \rho_0^{-2} y \bar{y} dx dt + \eta_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y, \mathbf{p}) \cdot \mathcal{J}(\bar{y}, \bar{\mathbf{p}}) dx dt \\ &\quad + \eta_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y, \mathbf{p}) \mathcal{I}(\bar{y}, \bar{\mathbf{p}}) dx dt \quad \forall (y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}}) \in \mathcal{U}_0. \end{aligned}$$

for any $\eta_1, \eta_2 > 0$ and any $\rho, \rho_0, \rho_1 \in \mathcal{R}$

$$\|(y, \mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1} y\|_{L^2(Q_T)}^2 + \eta_1 \|\rho_1^{-1} \mathcal{J}(y, \mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2 \|\rho^{-1} \mathcal{I}(y, \mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation - Mixed formulation

Precisely, we set $\mathcal{X} := L^2(Q_T) \times \mathbf{L}^2(Q_T)$ and then we consider the following mixed formulation : find $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in \mathcal{U} \times \mathcal{X}$ solution of

$$\begin{cases} a_r((y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}})) + b((\bar{y}, \bar{\mathbf{p}}), (\lambda, \boldsymbol{\mu})) &= l(\bar{y}, \bar{\mathbf{p}}) & \forall (\bar{y}, \bar{\mathbf{p}}) \in \mathcal{U}, \\ b((y, \mathbf{p}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) &= 0 & \forall (\bar{\lambda}, \bar{\boldsymbol{\mu}}) \in \mathcal{X}, \end{cases} \quad (25)$$

where

$$\begin{aligned} a_r : \mathcal{U} \times \mathcal{U} &\rightarrow \mathbb{R}, \quad a_r((y, \mathbf{p}), (\bar{y}, \bar{\mathbf{p}})) := \iint_{q_T} \rho_0^{-2} y \bar{y} \, dx \, dt \\ &+ r_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y, \mathbf{p}) \cdot \mathcal{J}(\bar{y}, \bar{\mathbf{p}}) \, dx \, dt + r_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y, \mathbf{p}) \mathcal{I}(\bar{y}, \bar{\mathbf{p}}) \, dx \, dt \\ b : \mathcal{U} \times \mathcal{X} &\rightarrow \mathbb{R}, \quad b((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) := \iint_{Q_T} \rho_1^{-1} \mathcal{J}(y, \mathbf{p}) \cdot \boldsymbol{\mu} \, dx \, dt + \iint_{Q_T} \rho^{-1} \mathcal{I}(y, \mathbf{p}) \lambda \, dx \, dt \\ l : \mathcal{U} &\rightarrow \mathbb{R}, \quad l(y, \mathbf{p}) := \iint_{q_T} \rho_0^{-2} y y_{obs} \, dx \, dt. \end{aligned}$$

$$\forall \mathbf{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$$

Parabolic case: $H_0^1 - L^2$ first order formulation - Global stability

Proposition (Imanuvilov-Puel-Yamamoto, 2010)

$$\rho_p(x, t) := \exp\left(\frac{\beta(x)}{t^2}\right), \quad \beta(x) := K_1 \left(e^{K_2} - e^{\beta_0(x)} \right),$$

$$\rho_{p,0}(x, t) := t\rho_p(x, t), \quad \rho_{p,1}(x, t) := t^{-1}\rho_p(x, t), \quad \rho_{p,2}(x, t) := t^{-2}\rho_p(x, t)$$

$$\exists C = C(\omega, T) > 0 \text{ s.t.}$$

$$\|\rho_{p,0}^{-1}y\|_{L^2(Q_T)}^2 + \|\rho_{p,1}^{-1}\nabla y\|_{L^2(Q_T)}^2 \leq C \left(\|\rho_p^{-1}\mathbf{G}\|_{L^2(Q_T)}^2 + \|\rho_{p,2}^{-1}g\|_{L^2(Q_T)}^2 + \|\rho_{p,0}^{-1}y\|_{L^2(Q_T)}^2 \right),$$

for any

$$\begin{cases} y \in \mathcal{K} := \left\{ y \in L^2(0, T; H_0^1(\Omega)) : y_t \in L^2(0, T; H^{-1}(\Omega)) \right\}, \\ Ly = g + \nabla \cdot \mathbf{G} \text{ in } Q_T, \quad (g, \mathbf{G}) \in L^2(Q_T) \times \mathbf{L}^2(Q_T). \end{cases}$$

$$\begin{cases} Ly = \mathcal{I}(y, \mathbf{p}) - \nabla \cdot \mathcal{J}(y, \mathbf{p}), \\ \mathcal{J}(y, \mathbf{p}) := c(x)\nabla y - \mathbf{p}, \quad \mathcal{I}(y, \mathbf{p}) := y_t - \nabla \cdot \mathbf{p} + dy \end{cases}$$

FINITE DIMENSIONAL APPROXIMATION

Conformal approximation of the mixed formulation

(boundary observation case, to fix idea)

Let $Z_h \subset Z$ and $\Lambda_h \subset L^2(Q_T)$, $\dim(Z_h), \dim(\Lambda_h) < \infty, \forall h > 0$. Find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(y_h, \bar{y}_h) + b(\bar{y}_h, \lambda_h) &= I(\bar{y}_h), & \forall \bar{y}_h \in Z_h \\ b(y_h, \bar{\lambda}_h) &= 0, & \forall \bar{\lambda}_h \in \Lambda_h. \end{cases} \quad (26)$$

if $r > 0$, a_r is coercive on Z : $a_r(y, y) \geq \frac{r}{\eta} \|y\|_Z^2 \quad \forall y \in Z$.

If there $\delta > 0$ such that

$$\forall h > 0 \quad \delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)} \|y_h\|_Z} > \delta. \quad (27)$$

then, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (26).

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then, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (26).

First estimate

Proposition

Let $h > 0$. Let (y, λ) and (y_h, λ_h) be the solution of (20) and of (26) respectively. Let δ_h the discrete inf-sup constant defined by (27). Then,

$$\begin{aligned}\|y - y_h\|_Z &\leq 2 \left(1 + \frac{1}{\sqrt{\eta}\delta_h} \right) d(y, Z_h) + \frac{1}{\sqrt{\eta}} d(\lambda, \Lambda_h), \\ \|\lambda - \lambda_h\|_{L^2(Q_T)} &\leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h} \right) \frac{1}{\delta_h} d(y, Z_h) + \frac{3}{\sqrt{\eta}\delta_h} d(\lambda, \Lambda_h)\end{aligned}$$

$$d(\lambda, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}$$

$\mathcal{T}_h$ -triangulation s.t. $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$. $h := \max\{\text{diam}(K), K \in \mathcal{T}_h\}$.

$$Z_h = \{y_h \in Z \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h, \quad z_h = 0 \text{ on } \Sigma_T\},$$

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N = 1$ - Convergence in Z)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z \leq K \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}} \right) h^k, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K \left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h} \right) \frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h} \right) h^k.$$

Writing the ineq. obs. for $y - y_h \in Z$ and using that $L(y - y_h) = -Ly_h$, we get

$$\begin{aligned} \|y - y_h\|_{L^2(Q_T)}^2 &\leq C_{\Omega, T} (C_{obs} + 1) (\|\partial_\nu(y - y_h)\|_{L^2(\Gamma_T)}^2 + \|Ly_h\|_{L^2(Q_T)}^2) \\ &\leq C_{\Omega, T} (C_{obs} + 1) \max(1, \frac{2}{\sqrt{\eta}}) \|y - y_h\|_Z \end{aligned}$$

Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{\eta}}) \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}} \right) h^k.$$

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Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{\eta}}) \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}} \right) h^k.$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N = 1$ - Convergence in Z)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z \leq K \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}} \right) h^k, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K \left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h} \right) \frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h} \right) h^k.$$

Writing the ineq. obs. for $y - y_h \in Z$ and using that $L(y - y_h) = -Ly_h$, we get

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Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{\eta}}) \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}} \right) h^k.$$

Choice of r versus δ_h

$$(\eta = r)$$

$$\delta_{r,h} \approx C_r \frac{h}{\sqrt{r}} \quad \text{as } h \rightarrow 0^+, \quad C_r > 0 \quad (28)$$

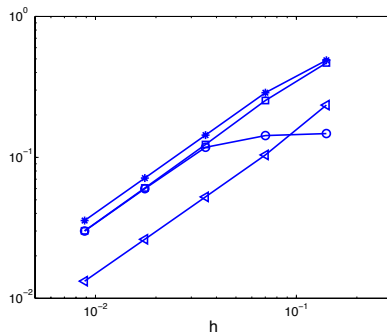


Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($\triangleleft$).

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}}) \left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \leq Kh^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h} \left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}}) \left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \leq Kh^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h} \left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

$\alpha \in (0, 1)$ - Stabilized mixed formulation

The problem (10) becomes : find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_{r,\alpha}(y_h, \bar{y}_h) + b_\alpha(\lambda_h, \bar{y}_h) &= l_{1,\alpha}(\bar{y}_h), & \forall \bar{y}_h \in Z_h \\ b_\alpha(\bar{\lambda}_h, y_h) - c_\alpha(\lambda_h, \bar{\lambda}_h) &= l_{2,\alpha}(\bar{\lambda}_h), & \forall \bar{\lambda}_h \in \tilde{\Lambda}_h, \end{cases} \quad (29)$$

$$\Lambda_h = \{\lambda \in Z_h; \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0\}.$$

Proposition (BFS element for $N = 1$ - Rates of convergence)

Let $h > 0$, let $k \in \{0, 2\}$. If the solution $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z + \|\lambda - \lambda_h\|_\Lambda \leq Kh^k.$$

Recovering the solution and the source $\mu \in H^{-1}(\Omega)$

$$\begin{cases} a_r((y_h, \mu_h), (\bar{y}_h, \bar{\mu}_h)) + b(\bar{y}_h, \lambda_h) &= l(\bar{y}_h), & \forall (\bar{y}_h, \bar{\mu}_h) \in Y_h \\ b((y_h, \mu_h), \bar{\lambda}_h) &= 0, & \forall \bar{\lambda}_h \in \Lambda_h. \end{cases} \quad (30)$$

Theorem (BFS element for $N = 1$ - Rate of convergence $L^2(Q_T)$)

Let $h > 0$, let $k, q \in \{0, 2\}$ be two nonnegative integers. If $((y, \mu), \lambda) \in H^{k+2}(Q_T) \times H^q(\Omega) \times H^k(Q_T)$, $\exists$

$$K = K(\|y\|_{H^{k+2}(Q_T)}, \|\mu\|_{H^k(\Omega)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}),$$

independent of h , such that

$$\|y - y_h\|_{L^2(Q_T)} \leq KC_{\Omega, T}(1 + \|\sigma\|_{L^2(0, T)} \sqrt{C_{obs}}) \max(1, \frac{1}{\sqrt{\eta}}) \left[\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right) h^k + \left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right) (\Delta x)^q \right].$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

$$\mathbb{Q}_3 \times \mathbb{Q}_1 \text{ approximation} \implies \delta_h \approx \frac{C_r}{\sqrt{r}}$$

$$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \implies \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \mu))$;

$$(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1) \text{ approximation; } \implies \delta_h = 0 \quad \forall r, h > 0 \quad (\text{Ker}(B_h^*) \neq \emptyset)$$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

$$\text{If } ((y, \mathbf{p}), (\lambda, \mu)) \in (H^2(Q_T) \times \mathbf{H}^2(Q_T)) \times (H^1(Q_T) \times \mathbf{H}^1(Q_T))$$

$$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

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$$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

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- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

$$\text{If } ((y, \mathbf{p}), (\lambda, \mu)) \in (H^2(Q_T) \times \mathbf{H}^2(Q_T)) \times (H^1(Q_T) \times \mathbf{H}^1(Q_T))$$

$$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

EXPERIMENTS

Numerical illustration - $N = 1$

$$\text{(EX1)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3, 2/3)}(x), \quad x \in (0, 1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k} (\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

$f = 0$. $T = 2$ - The corresponding solution of (1) with $c \equiv 1$, $d \equiv 0$ is given by

$$y(x, t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

Example 1 - $N = 1$ - Observation on q_T

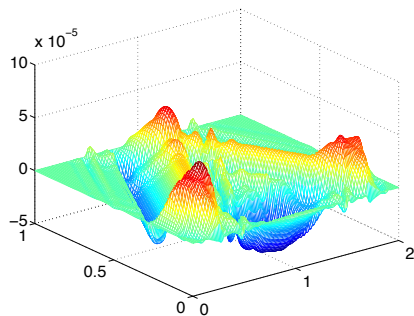
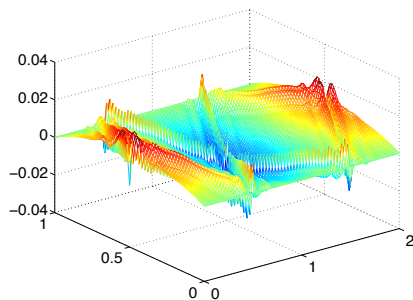
$$q_T = (0.1, 0.3) \times (0, T)$$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\ y - y_h\ _{L^2(Q_T)}}{\ y\ _{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\ y - y_h\ _{L^2(q_T)}}{\ y\ _{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\ Ly_h\ _{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\ \lambda_h\ _{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
# CG iterates	29	46	83	133	201

$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{\|y - y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}} = \mathcal{O}(h^{0.94}). \quad (31)$$

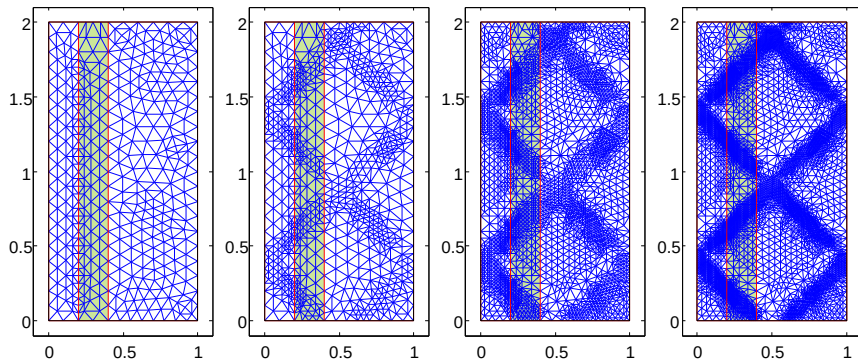
$$\|Ly_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \quad (32)$$

Example 2 - $N = 1$ - Observation on q_T



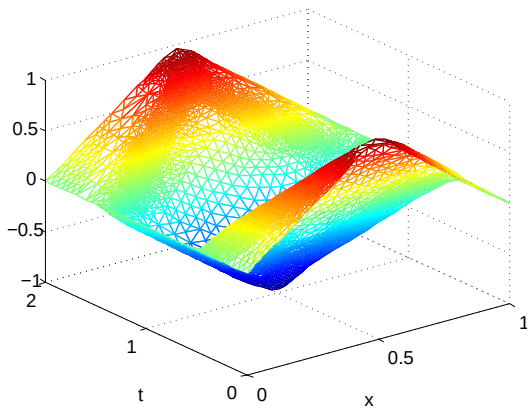
$y - y_h$ and λ_h in Q_T

Example 1 - $N = 1$ - Mesh adaptation



Iterative local refinement of the mesh according to the gradient of y_h

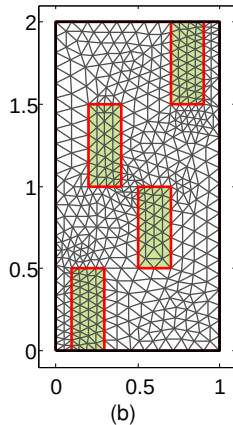
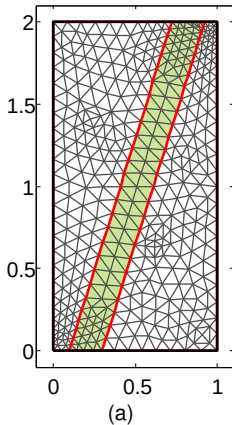
Example 1 - $N = 1$ - Mesh adaptation



Reconstructed state y_h on the adapted mesh

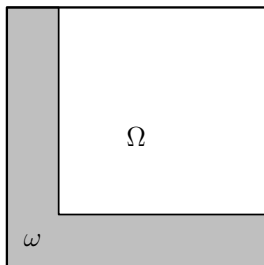
Example 2 : $N = 1$ - Non cylindrical domain q_T

Triangular meshes - reduced HCT elements

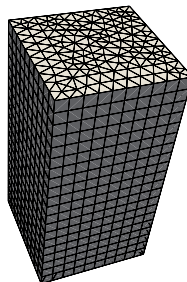


Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.

2D example: $\Omega = (0, 1)^2$ - Observation on q_T



(a)



(b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T .

2D example: $\Omega = (0, 1)^2$ - Observation on q_T

$$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega):$$

$$(\mathbf{EX2-2D}) \quad \begin{cases} y_0(x_1, x_2) = (1 - |2x_1 - 1|)(1 - |2x_2 - 1|) \\ y_1(x_1, x_2) = \mathbf{1}_{(\frac{1}{3}, \frac{2}{3})^2}(x_1, x_2) \end{cases} \quad (x_1, x_2) \in \Omega. \quad (33)$$

The Fourier coefficients of the corresponding solution are

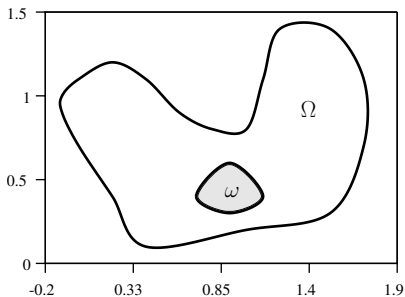
$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin \frac{\pi k}{2} \sin \frac{\pi l}{2}$$

$$b_{kl} = \frac{1}{\pi^2 kl} \left(\cos \frac{\pi k}{3} - \cos \frac{2\pi k}{3} \right) \left(\cos \frac{\pi l}{3} - \cos \frac{2\pi l}{3} \right).$$

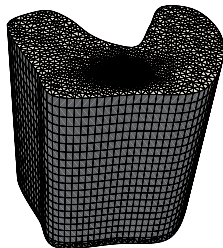
Mesh number	0	1	2	3
$\frac{\ y - y_h\ _{L^2(Q_T)}}{\ y\ _{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\ Ly_h\ _{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\ \lambda_h\ _{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2-2D** – $r = h^2$

2D example - Observation on q_T



(a)



(b)

Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T .

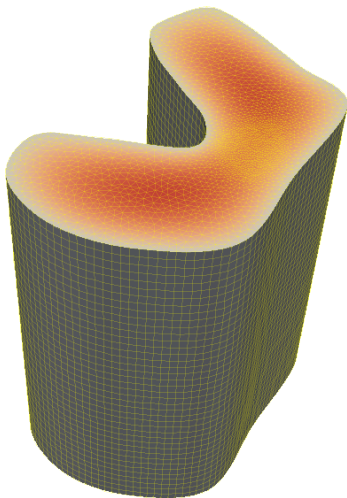
2D example - Observation on q_T

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \quad y_1 = 0. \quad (34)$$

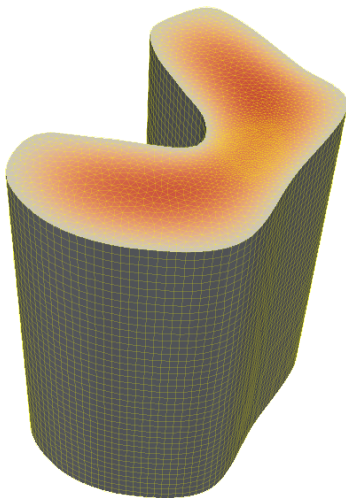
Mesh number	0	1	2
$\frac{\ \bar{y}_h - y_h\ _{L^2(Q_T)}}{\ \bar{y}_h\ _{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\ Ly_h\ _{L^2(Q_T)}$	3.21	2.01	1.17
$\ \lambda_h\ _{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$

2D example - Observation on q_T



(a)



(b)

y and y_h in Q_T

Numerical illustration - $N = 1$ - Observation on Γ_T

$$f = 0 - T = 2$$

$$(\mathbf{EX2}) \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3, 2/3)}(x), \quad x \in (0, 1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

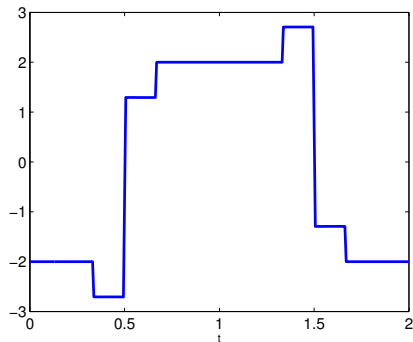


Figure: The observation $y_{\nu, obs}$ on $\{1\} \times (0, T)$ associated to initial data **EX1**.

Numerical illustration - $N = 1$ - Observation on Γ_T

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\ y - y_h\ _{L^2(Q_T)}}{\ y\ _{L^2(Q_T)}}$	1.63×10^{-2}	6.63×10^{-3}	2.78×10^{-3}	1.29×10^{-3}	5.72×10^{-4}
$\frac{\ \partial_\nu(y - y_h)\ _{L^2(\Gamma_T)}}{\ \partial_\nu y\ _{L^2(\Gamma_T)}}$	7.67×10^{-3}	4.95×10^{-3}	3.24×10^{-3}	2.16×10^{-3}	1.48×10^{-3}
$\ Ly_h\ _{L^2(Q_T)}$	0.937	1.204	1.496	1.798	2.135
$\ \lambda_h\ _{L^2(Q_T)}$	7.74×10^{-3}	3.74×10^{-3}	1.72×10^{-3}	7.90×10^{-4}	3.60×10^{-4}
$\text{card}(\{\lambda_h\})$	861	3 321	13 041	51 681	205 761
$\# \text{ CG iterates}$	57	103	172	337	591

$$r = h^2 : \quad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{1.20}), \quad \frac{\|\partial_\nu(y - y_h)\|_{L^2(\Gamma_T)}}{\|\partial_\nu y\|_{L^2(\Gamma_T)}} = \mathcal{O}(h^{0.59}), \quad (35)$$

$$\|\lambda_h\|_{L^2(Q_T)} = \mathcal{O}(h^{1.11}), \quad \|Ly_h\|_{L^2(Q_T)} = \mathcal{O}(h^{-0.29}).$$

Example 2 - $N = 2$ - The stadium

$$T = 3$$

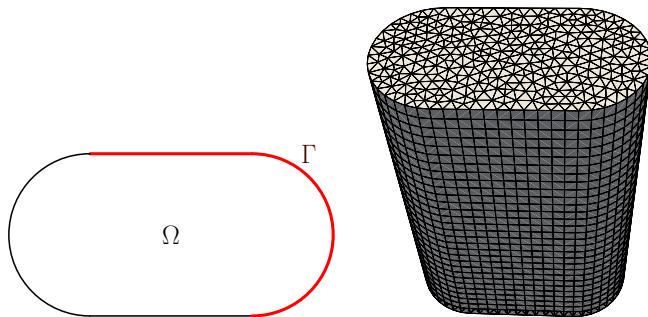


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T .

Example 2 - $N = 2$ - Recovering of the initial data

$$T = 3$$

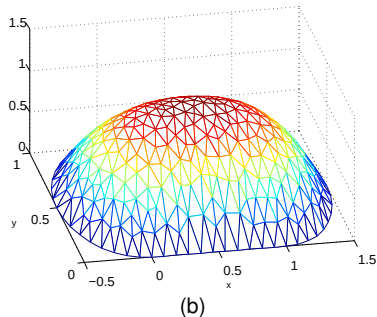
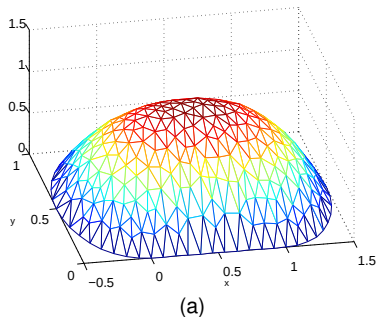


Figure: (a) Initial data y_0 given by (34). (b) Reconstructed initial data $y_h(\cdot, 0)$.

$N = 1$ - Reconstruction of y and μ from the boundary

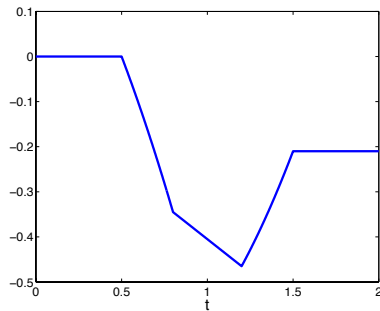
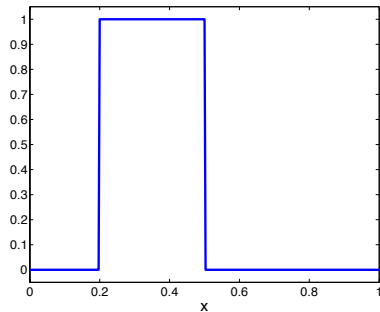


Figure: $\mu(x)$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$$\Delta x = \Delta t = 1/160$$

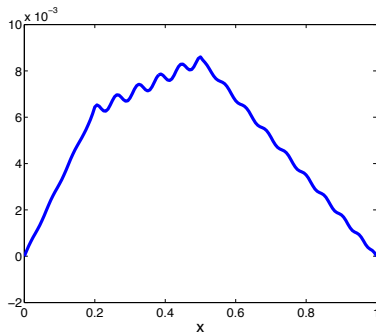
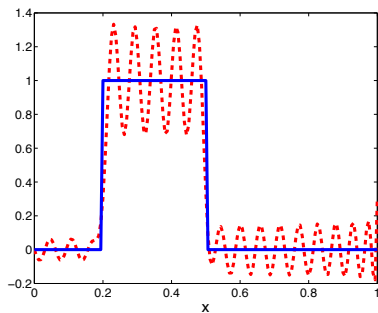


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu - \mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 7.18 \times 10^{-2}, \quad \|y - y_h\|_{L^2(Q_T)} \approx 8.68 \times 10^{-4}$$

$N = 1$ - Reconstruction of y and μ from the boundary

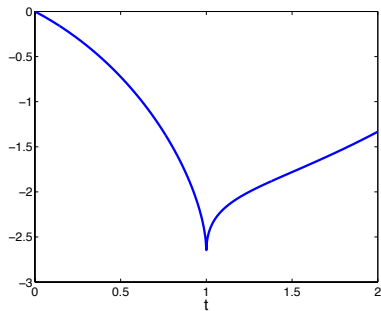
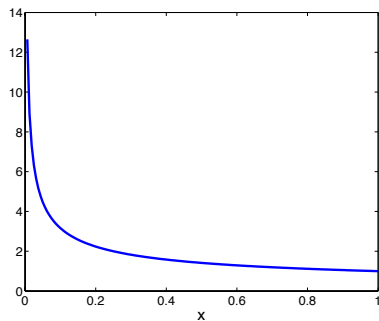


Figure: $\mu(x) = \frac{1}{\sqrt{x}}$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$$\Delta x = \Delta t = \frac{1}{160}$$

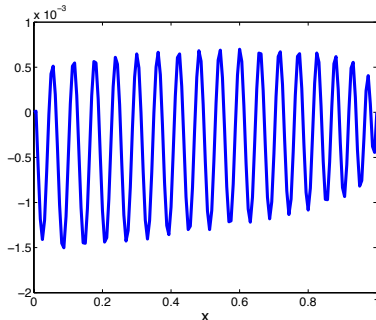
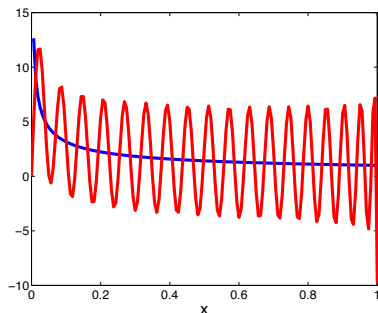


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu - \mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 2.21 \times 10^{-2}, \quad \|y - y_h\|_{L^2(Q_T)} \approx 3.56 \times 10^{-5}$$

$N = 1$ - Reconstruction of y and μ from the boundary

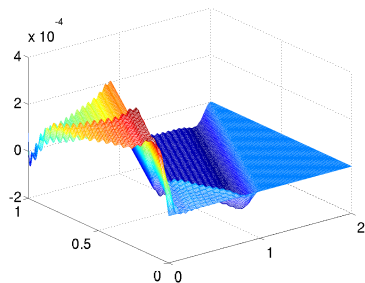
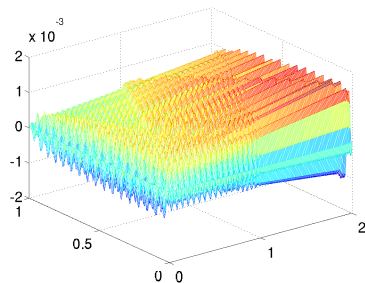
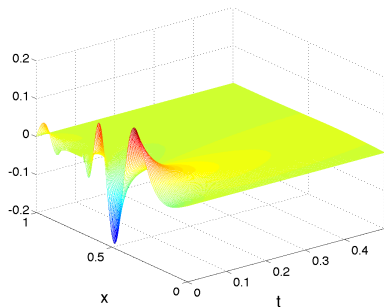


Figure: $y - y_h$ and λ_h

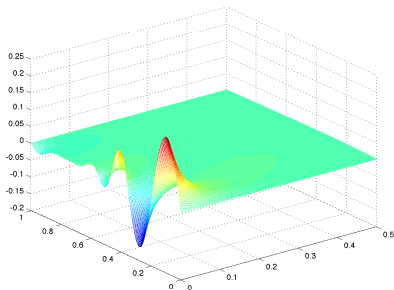
$N = 1$ - Heat eq. Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0, 1) \times (0, T), \quad q_T = (0.7, 0.8) \times (0, T), \quad T = 1/2$$

$$\min_{y_{0h}} \left(J_h(y_{0h}) + \frac{h^2}{2} \|y_{0h}\|_{L^2(\Omega)}^2 \right) \quad \text{vs.} \quad \min_{\lambda_h} J^{**}(\lambda_h) \quad \text{over } \Lambda_h \quad (36)$$



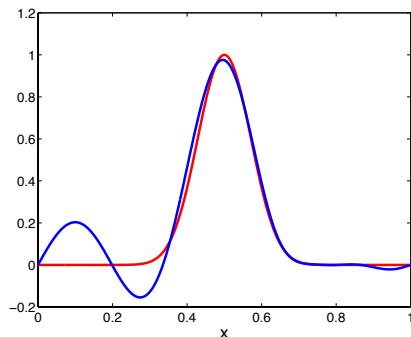
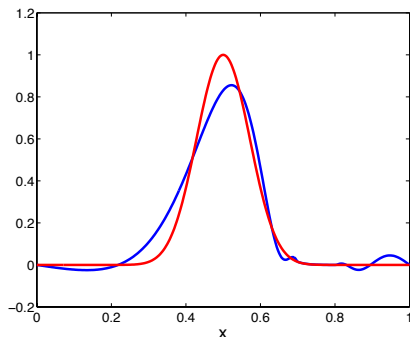
$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2},$$



$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$

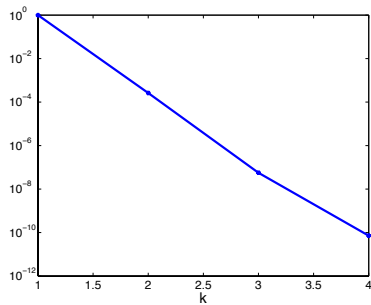
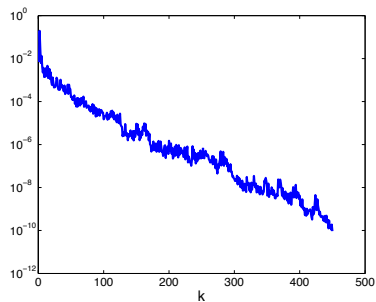
$N = 1$ - Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0, 1) \times (0, T), \quad q_T = (0.7, 0.8) \times (0, T), \quad T = 1/2$$



Restriction at $(0, 1) \times \{0\}$

$N = 1$ - Comparison with the standard method



Residu w.r.t. iterates

Concluding remarks

MIXED FORMULATION ALLOWS TO RECONSTRUCT SOLUTION AND SOURCE

DIRECT AND ROBUST METHOD - STRONG CONVERGENCE

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The end

THANK YOU FOR YOUR ATTENTION