Spectral inequality and optimal cost of controllability for the Stokes system

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 $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) is a bounded connected open set, with $\partial \Omega$ regular. Let T > 0 and ω be a nonempty subset of Ω . We consider the controlled Stokes system

$$y_t - \Delta y + \nabla p = f \mathbf{1}_{\omega} \quad \text{in } Q =]0, T[\times \Omega,$$

$$div \ y = 0 \qquad \qquad \text{in } Q,$$

$$y = 0 \qquad \qquad \text{on } \Sigma =]0, T[\times \partial \Omega,$$

$$y(0) = y_0 \qquad \qquad y_0 \in L^2(\Omega)^N, \text{ div } y_0 = 0, \ y_0 \cdot \nu|_{\partial \Omega} = 0.$$
(11)

Theorem

For every y_0 , there exists $f \in L^2(\omega \times (0, T))$ such that the associated solution of the Stokes system (1.1) satisfies

$$y(T) = 0. \tag{1.2}$$

Moreover, one has

$$\|f\|_{L^{2}(\omega\times(0,T))} \leq C_{stokes}(T)\|y_{0}\|_{L^{2}}, \quad C_{stokes}(T) \leq C_{1}e^{C_{2}/T}.$$
(1.3)

Previous results on the null controllability for the Stokes system have been obtained by means of global in time Carleman estimates:

E. Fernandez-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel (JMPA 2004) and O. Yu. Imanuvilov, J.-P. Puel, M. Yamamoto (Chin. Ann. Math. Ser. B 2009).

However, with this strategy of proof of null controllability, one obtains:

$$C_{stokes}(T) \leq C_1 e^{C_2/T^4}$$

Therefore our improvement is the behavior of the cost function for T > 0 small:

$$C_{stokes}(T) \leq C_1 e^{C_2/T}$$

Introduction and result

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Let V and H be the usual spaces in the context of fluid mechanics

$$\mathbf{V} = \{ u \in H_0^1(\Omega)^N; \text{ div } u = 0 \},$$
$$\mathbf{H} = \{ u \in L^2(\Omega)^N; \text{ div } u = 0, u \cdot \nu = 0 \text{ on } \partial\Omega \}$$

The Stokes operator A is the unbounded operator on **H** defined by

$$D(A) = \{ u \in \mathbf{V}, A(u) \in \mathbf{H} \}$$
$$A(u) = v \quad \text{iff} \quad \int_{\Omega} \nabla u \nabla g \, dx = \int_{\Omega} vg \, dx \quad \forall g \in \mathbf{V}$$

Therefore one has A(u) = v iff there exists a pressure p such that

$$-\triangle u + \nabla p = v$$

Observe that the pressure p is an harmonic function in Ω : $\triangle p = 0$.

The Stokes operator A is a selfadjoint positive operator with compact resolvant. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of **H**, given by the eigenvectors of the Stokes equation

$$\begin{aligned} -\Delta e_j + \nabla p_j &= \mu_j e_j & \text{in } \Omega, \\ div \ e_j &= 0 & \text{in } \Omega, \\ e_j &= 0 & \text{on } \partial \Omega, \end{aligned} \tag{2.1}$$

with the sequence of eigenvalues $\{\mu_j\}_{j=1}^\infty$ satisfying

$$0 < \mu_1 \le \mu_2 \le \dots$$
 $\lim_{j \to \infty} \mu_j = \infty$

Then $\{e_j\}_{i=1}^{\infty}$ is also an orthogonal basis of **V**, and one has

$$\|\operatorname{curl}(e_j)\|_{L^2}^2 = \mu_j$$

Theorem

Let $\omega \subset \Omega$ be a nonempty open set. There exists constants M > 0, K > 0 such that, for every sequence of complex numbers z_j and every real $\Lambda \ge 1$, we have

$$\sum_{\mu_j \leq \Lambda} |z_j|^2 = \int_{\Omega} \left| \sum_{\mu_j \leq \Lambda} z_j e_j(x) \right|^2 dx \leq M e^{K\sqrt{\Lambda}} \int_{\omega} \left| \sum_{\mu_j \leq \Lambda} z_j e_j(x) \right|^2 dx. \quad (2.2)$$

By an argument due to T.I. Seidman (J. Math. Anal. Appl. 2008) and revisited by L. Miller (Discrete Contin.Dyn. Syst. Ser. B, 2010), one deduce from this spectral inequality the null controllability result and the estimate on the cost.

$$C_{stokes}(T) \leq C_1 e^{C_2/T}$$

The spectral inequality has been proven for the eigenfunctions of the scalar Laplace operator with Dirichlet boundary conditions in:

D. Jerison and G. Lebeau. *Nodal sets of sums of eigenfunctions* Acts of the conference "Harmonic analysis and partial differential equations", 1996. Chicago Lectures in Mathematics.

It has been used by G. Lebeau and E. Zuazua in the study of null-controllability of a system of linear thermoelasticity (Arch. Rational Mech. Anal., 1998).

It has been recently extended to the bi-laplacian with Dirichlet boundary condition by J. Le Rousseau and L. Robbiano.

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We introduce the sets

$$Z = (0,1) \times \Omega$$
 and $W = (1/3,2/3) \times \Omega$.

Let $\Lambda \geq 1$. For a given sequence $(z_j)_j$ of complex numbers, set

$$u(s,x) = \sum_{\mu_j \leq \Lambda} z_j \frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} e_j(x),$$

The "classical proof" of the spectral inequality will be to glue together local Carleman estimates for the pair $(u, p) = \sum_{\mu_j \leq \Lambda} a_j(s)(e_j(x), p_j(x))$,

 $a_j(s)=z_jrac{sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}}$, which is a solution of

$$\begin{aligned} &-\partial_{ss}^{2}u - \Delta_{x}u + \nabla_{x}p = 0 & \text{in } Z, \\ & div_{x} \ u = 0 & \text{in } Z, \\ & u(s,x) = 0 & \text{on } (0,1) \times \partial\Omega, \\ & u(0,x) = 0 & \text{in } \Omega. \end{aligned}$$
(3.1)

Unfortunately, system (3.1) does not have local unique continuation property. Indeed, for any function q = q(s, x) such that $\Delta_x q = 0$, the pair

$$(u,p) = (\nabla_x q, \partial_{ss}^2 q)$$

is a solution of the two first lines of (3.1). Therefore, we cannot obtain a local Carleman estimate for system (3.1).

Thus we will work with v = curl u which is a solution of the system

$$\begin{aligned} -\partial_{ss}^2 v - \Delta_x v &= 0 \quad \text{in } Z, \\ v(0, x) &= 0 \qquad \text{in } \Omega. \end{aligned} \tag{3.2}$$

Now, the **major problem** is that v = curl(u) is equal to

$$v(s,x) = \sum_{\mu_j \leq \Lambda} z_j \frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} curl(e_j)(x).$$

v satisfies no boundary condition on the lateral boundary $(0,1) \times \partial \Omega$.

To overcome the preceding lack of boundary condition for v on $(0,1) \times \partial \Omega$, we introduce a small parameter h related to Λ by

$$h = rac{\delta}{\sqrt{\Lambda}}$$
 with $\delta > 0$ small.

Informally, $x \mapsto v(s, x)$ is concentrated at frequencies $\leq \sqrt{\Lambda}$. Thus, for the semiclassical analysis with semiclassical parameter h, the spectrum of v is concentrated near

$$|\xi| \le h\sqrt{\Lambda} = \delta$$

Therefore we force localization near $\xi = 0$

The Main Inequality

Theorem

There exists $h_0 > 0$, $\mu > 0$, A > 0, $\alpha \in (0, 1)$, and for $\delta > 0$ small enough $C_{\delta} > 0$, such that $v = curl \ u$ satisfies for all $h \in]0, h_0]$, all $(z_j)_j$ and all $\Lambda \in [1, (\delta/h)^2]$

$$\|v\|_{H^{1}(W)^{N}} \leq C_{\delta} \Big(e^{-\mu/h} \|v\|_{H^{1}(Z)^{N}} + e^{A/h} \|v\|_{H^{1}(Z)^{N}}^{1-\alpha} \|\partial_{s}v(0,x)\|_{L^{2}(\omega)^{N}}^{\alpha} \Big)$$
(3.3)

$$(\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} \leq C \, \|v\|_{H^1(W)^N} \,, \quad \|v\|_{H^1(Z)^N} \leq C \Lambda^M (\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} e^{\sqrt{\Lambda}}$$

$$\|\partial_s v(0,x)\|_{L^2(\omega)^N}^2 \leq C \Lambda^M (\sum_{\mu_j \leq \Lambda} |z_j|^2)^{1/2} \|\partial_s v(0,x)\|_{L^2(\tilde{\omega})^N} \quad \omega \subset \subset \tilde{\omega}$$

Main Inequality \implies Spectral inequality

In order to estimate $||v||_{H^1(W)^N}$, we will estimate $||v||_{H^1(B_r(z)\cap Z)^N}$ for any $z \in (0, 1) \times \overline{\Omega}$. There is three cases:

1)
$$z = (s, x) \in (0, 1) \times \omega$$
, with $s > 0$ small
2) $z = (s, x) \in (0, 1) \times \Omega$;
3) $z = (s, x) \in (0, 1) \times \partial \Omega$.

Each one of the cases above are proved using appropriate Carleman inequalities. Actually, since in *cases* 1 and 2 we are away from the lateral boundary, the proof for such cases is performed exactly as in the case of the heat equation (see G. Lebeau, L. Robbiano , CPDE 1995), and we obtain the better **pure interpolation estimate**: there exists $r = r_z > 0$, C_z , and $\alpha_z \in]0, 1[$ such that

$$\|v\|_{H^{1}(B_{r}(z)\cap Z)^{N}} \leq C_{z} \|v\|_{H^{1}(Z)^{N}}^{1-\alpha_{z}} \|\partial_{s}v(0,x)\|_{L^{2}(\omega)^{N}}^{\alpha_{z}}$$
(3.4)

Let $r = dist(x, \partial \Omega)$. For $s_0 \in]0, 1[$ and $\epsilon > 0$ small, set

$$\psi(s,r)=r-\frac{4}{\epsilon}(s-s_0)^2$$

and consider $\varphi = e^{D\psi}$. For *D* large enough, φ satisfies Hörmander's sub-ellipticity condition for the operator *A* in the variables $(s, r, y), y \in \partial\Omega$ and $\partial_r \varphi > 0$ near r = 0. We also introduce a cutoff function $\chi(s, r) = \chi_0(s)\chi_1(r)$, with $\chi_0 \in C_0^{\infty}((s_0 - \epsilon, s_0 + \epsilon)), \chi_0 \equiv 1$ in $[s_0 - \epsilon/2, s_0 + \epsilon/2]$ and $\chi_1 \in C_0^{\infty}([0, \epsilon))$ with $\chi_1 \equiv 1$ on $[0, \epsilon/2]$.

For the proof of *case* 3, we use the following Carleman estimate for the operator $P = -\triangle_z = -(\partial_s^2 + \triangle_x)$. There exist *C* and $h_1 > 0$ such that for every $h \in (0, h_1]$, one has

$$h||e^{\varphi/h}\chi v||_{L^2}^2 + h^3||e^{\varphi/h}\nabla_{s,x}(\chi v)||_{L^2}^2$$

$$\leq C\left(h||e^{\varphi/h}\chi v||^{2}_{L^{2}(r=0)} + h^{3}||e^{\varphi/h}\nabla_{s,y}(\chi v)||^{2}_{L^{2}(r=0)} + h^{4}||e^{\varphi/h}P(\chi v)||^{2}_{L^{2}}\right)$$

One has P(v) = 0, and in *cases* 1 and 2 we have a pure interpolation estimate. Thus the third term on the right

$$h^4 ||e^{\varphi/h} P(\chi v)||_{L^2}^2 = h^4 ||e^{\varphi/h}[P,\chi]v||_{L^2}^2$$

will give at the end a contribution which will be easily estimated by the right hand side of the Main Inequality.

The main problem is thus the control of the two first term in the right hand side of the Carleman estimate:

$$h||e^{\varphi/h}\chi v||_{L^{2}(r=0)}^{2} + h^{3}||e^{\varphi/h}\nabla_{s,y}(\chi v)||_{L^{2}(r=0)}^{2}$$
(3.5)

Since φ is independent of y, the principal symbol p_{φ} of the conjugate operator

$$P_{\varphi}g = h^2 e^{\varphi/h} P(e^{-\varphi/h}g)$$

is equal to

$$p_{\varphi} = (\sigma + i\partial_{s}\varphi)^{2} + (\tau + i\partial_{r}\varphi)^{2} + \eta^{2}.$$

where $\xi = (\tau, \eta)$ is the dual variable of x = (r, y), and σ the dual variable of *s*. The roots of p_{φ} with respect to τ are given by

$$\tau_{\pm} = -i\partial_r \varphi \pm i\sqrt{\eta^2 + \left(\sigma + i\partial_s \varphi\right)^2}.$$

Close to $s = s_0$, $\sigma = 0$ and $\eta = 0$, since $\partial_r \varphi(s_0, 0) > 0$ and $\partial_s \varphi(s_0, 0) = 0$, we get $Im \tau_{\pm} < 0$. From elliptic boundary estimates, the localization near $\sigma = 0$ of (3.5) will be absorbed by the left hand side of the Carleman estimate. It remains to estimate the contribution of the localization near $\eta=0$ and $|\sigma|\geq c>0$ of

$$h||e^{\varphi/h}\chi v||_{L^{2}(r=0)}^{2} + h^{3}||e^{\varphi/h}\nabla_{s,y}(\chi v)||_{L^{2}(r=0)}^{2}$$
(3.6)

Thus we need an estimate on the Fourier transform:

$$J = \int e^{-it\sigma/h} \tilde{\chi}(t) e^{-\frac{1}{h}(\theta(t)\pm(t+s_0)\sqrt{\nu_j})} dt = \int e^{i\phi(t)/h} \tilde{\chi}(t) dt, \qquad (3.7)$$

with $\sqrt{\nu}_j = h \sqrt{\mu}_j \leq \delta$ and the phase

$$\phi = -t\sigma + i(\theta(t) \pm (t+s_0)\sqrt{\nu_j}), \quad \theta(t) = \varphi(t+s_0,0) - \varphi(s_0,0)$$

This is an **analytic function of** t and $|\phi'(0) + \sigma| \leq \delta$. This allows to prove that the contribution of the localization near $\eta = 0$ and $|\sigma| \geq c > 0$ of (3.6) is estimated for some $\mu = \mu_c$ by

$$e^{-2\mu/h} \|v\|_{H^1(Z)^N}^2$$

It is here that we use **analyticity in** *s* **of the coefficient of** $\partial_s^2 + \Delta_x$. This conclude the proof of the Main Inequality. Observe that we use the boundary condition

 $e_j|_{\partial\Omega} = 0$

in order to conclude

 $curl(e_j)|_{\partial\Omega}$ is concentrated near $\eta = 0$.

HAPPY BIRTHDAY

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