Asymptotic properties of solutions to the elastic wave equation in a half-space

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Brief Summary of Results

- The elastic wave equation in a half space has
- Body waves : P, SV, SH waves ,
- Surface wave : Rayleigh wave .
- Any solution of $(L-\lambda)\mathbf{u}=0$ in the Agmon-Hormander space behaves like

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) \simeq\ & \frac{e^{i\sqrt{\lambda}r/c_P}}{r} f_P^{(+)} \mathbf{d}_P(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(+)} \mathbf{d}_{SV}(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(+)} \mathbf{d}_{SH}(\varphi) \\
& + \boxed{\sum_{\ell=1}^{2} \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} E_\ell f_R^{(+)} \mathbf{d}_R^{(\ell)}(\varphi_*)} \\
& - \frac{e^{-i\sqrt{\lambda}r/c_P}}{r} f_P^{(-)} \mathbf{d}_P^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(-)} \mathbf{d}_{SV}^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(-)} \mathbf{d}_{SH}^{(-)}(\varphi) \\
& - \boxed{\sum_{\ell=1}^{2} \frac{e^{-i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} \overline{E_\ell} f_R^{(-)} \mathbf{d}_R^{(\ell)}(-\varphi_*)}.
\end{aligned}$$

$$Lu = \left\{ -\frac{1}{\rho(x)} \sum_{j=i}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(u) \right\}_{1 \le i \le 3}$$

$$\sigma_{ij}(u) = \lambda(x)(\nabla \cdot u)\delta_{ij} + 2\mu(x)\mathcal{E}_{ij}(u)$$

$$u \simeq v \Longleftrightarrow \lim_{R \to \infty} \frac{1}{R} \int_{\Omega_R} |u - v|^2 \, dx = 0, \quad \Omega_R = \{x \in \Omega \,;\, |x| < R\}.$$

$$\mathcal{B}^* = \left\{ u \in L^2_{\mathrm{loc}}(\Omega) \,;\, \sup_{R > 1} \frac{1}{R} \int_{\Omega_R} |u|^2 \, dx < \infty \right\}.$$

$$r = |x|, \quad \varphi = \frac{x}{r}, \quad r_* = |x_*|, \quad \varphi_* = \frac{x_*}{r_*}$$

For the comparison, consider the wave equation in a stratified media

$$\big(-c(x)^2\Delta - \lambda\big)u = 0, \quad \text{in} \quad \mathbf{R}^3,$$

$$c(x) = \begin{cases} c_+, & x_3 > 0, \\ c_-, & x_3 < 0. \end{cases}$$

Then, the solution of the reduced wave equation in B^{¥ast} behaves like

$$u(x) \simeq \chi_+(x_3)\frac{e^{i\sqrt{\lambda}r/c_+}}{r}a_+(\omega) + \chi_-(x_3)\frac{e^{i\sqrt{\lambda}r/c_-}}{r}a_-(\omega), \quad \omega = x/r$$

- It is known that there exists an evanescent wave, which lives only near the interface ${x_3=0}$.
- However, as can be seen above, it cannot be observed in the topology of $B^{¥ast}$.
- So, in the case of the elastic equation, the appearance of the Rayleigh wave is not obvious.

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- Asymptotic expansion of integrals over the upper half sphere
- Asymptotic expansion of the resolvent

Motivation

Basic issue in the stationary scattering theory for PDE

Construct the Fourier transform

Describe the set of solutions to $(L \quad - \lambda)u = 0$ in terms of the Fourier transform

Expand the resolvent $R \quad (\lambda \pm i0)$ of L at $|x| \to \infty$

Expand the solutions to $(L \quad - \lambda)u = 0$ at $|x| \to \infty$

Motivation

Helmholtz equation

Let $\lambda > 0$. Suppose that u satisfies the Helmholtz equation

$$-\Delta u = \lambda u \quad \text{in } \mathbb{R}^{\mathrm{n}}.$$

Then u satisfies

$$\sup_{R>1} \frac{1}{R} \int_{|x|<R} |u(x)|^2 dx < \infty$$

if and only if u is written as $u(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\omega\cdot x} A(\omega) d\omega$

for some $A \in L^2(\mathbb{S}^{n-1})$. Moreover,

$$u(x) \simeq C(\lambda) \frac{e^{i\sqrt{\lambda}r}}{r^{\frac{n-1}{2}}} A(\theta) + \overline{C(\lambda)} \frac{e^{-i\sqrt{\lambda}r}}{r^{\frac{n-1}{2}}} A(-\theta)$$

$r = |x|, \theta = \frac{x}{r}$

Agmon & Hörmander 1976

$$u \simeq v$$

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|<R} |u(x) - v(x)|^2 dx = 0$$

Motivation

2-body Schrödinger Operator

Let $\lambda > 0$. Suppose that u satisfies the 2-body Schrödinger equation

$$\big(-\Delta + V(x)\big)u = \lambda u \quad \text{in } \mathbb{R}^n.$$

Then u satisfies

$$\sup_{R>1} \frac{1}{R} \int_{|x|<R} |u(x)|^2 dx < \infty$$

if and only if u is written as $u(x) = \mathcal{F}^*(\lambda)\varphi$

for some $\varphi \in L^2(\mathbb{S}^{n-1})$, where $\mathcal{F}(\lambda)$ is the Fourier transform associated with the 2-body Schrödinger operator. Moreover,

$$u(x) \simeq C(\lambda) \frac{e^{i\sqrt{\lambda}r}}{r^{\frac{n-1}{2}}} \varphi_+(\theta) + \overline{C(\lambda)} \frac{e^{-i\sqrt{\lambda}r}}{r^{\frac{n-1}{2}}} \varphi_-(\theta)$$

$r = |x|, \theta = \frac{x}{r}$

and $\varphi_\pm$ are related as follows:

$$\varphi_+ = \hat{S}(\lambda) J \varphi_-, \quad (J\varphi)(\omega) = \varphi(-\omega)$$

Yafaev 1991

$\hat{S}(\lambda)$ is the scattering matrix

Motivation

3-body Schrödinger Operator

Let $\lambda \in \sigma_{cont}(H) \setminus \mathcal{T}'$. Suppose that u satisfies the 3-body Schrödinger equation

$$Hu = \lambda u.$$

Then $u \in \mathcal{B}^*$ if and only if u is written as $u(x) = \mathcal{F}^*(\lambda)\varphi$

for some $\varphi \in L^2(\mathbb{S}^5) \oplus \bigoplus_{a,n} L^2(\mathbb{S}^2)$,

where $\mathcal{F}(\lambda)$ is the Fourier transform associated with the 3-body Schrödinger operator. Moreover,

$$u(x) \simeq C(\lambda) \frac{e^{i\sqrt{\lambda}r}}{r^{\frac{5}{2}}} \varphi_0^{(+)}(\theta) + \overline{C(\lambda)} \frac{e^{-i\sqrt{\lambda}r}}{r^{\frac{5}{2}}} \varphi_0^{(-)}(\theta)$$

$$+ \sum_{a,n} \left[C_{a,n}(\lambda) \frac{e^{i\sqrt{\lambda-\lambda^{a,n}}r_a}}{r_a} \varphi_{a,n}^{(+)}(\omega_a) \otimes \varphi^{a,n}(x^a) \right)$$

and $\varphi^{(\pm)} = \left(\varphi_0^{(\pm)}, \varphi_{a,1}^{(\pm)}, \cdots\right)$ are related as follows:

$$\varphi^{+} = \widehat{\boldsymbol{S}}(\lambda)\boldsymbol{J}\varphi^{-}$$

where $\widehat{\boldsymbol{S}}(\lambda)$ is the scattering matrix for H and $\boldsymbol{J}$ is the space inversion.

Isozaki 2001

Here, we are using the following coordinate system

The configuration space of 3 particles $x^{(i)} \in \mathbf{R}^3$, $i = 1, 2, 3$, with center of mass removed:

$$\{(x^{(1)}, x^{(2)}, x^{(3)})\,;\, \sum_{i=1}^{3} m_i x^{(i)} = 0\} \simeq \mathbf{R}^6.$$

For a pair $a = (i, j)$, we put

$$x^a = x^{(i)} - x^{(j)}, \quad r^a = |x^a|,$$

$$x_a = x^{(k)} - \frac{m_i x^{(i)} + m_j x^{(j)}}{m_i + m_j}, \quad r_a = |x_a|,$$

$$\omega_a = x_a / r_a.$$

Motivation

Seismic Wave

Homogeneous

Isotropic

Elastic half-space $\mathbb{R}_+^3$

Free boundary

Elastic Wave Equation

$$-({c_P}^2 - {c_S}^2)\nabla(\nabla\cdot\boldsymbol{u}) - {c_S}^2\Delta\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{u}$$

P-wave velocity

S-wave velocity

$$\boldsymbol{u}(\boldsymbol{x}) = \big(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), u_3(\boldsymbol{x})\big)$$

$$\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}_+^{\mathbf{3}}$$

Motivation *Similarity to the 3-body Schrödinger Operator*

$$\boldsymbol{u}(\boldsymbol{x}) \sim \; ? \quad \textbf{as } |\boldsymbol{x}| \to \infty$$

3-cluster scattering	P-wave	S-wave	$\sim \dfrac{e^{\pm i\sqrt{\lambda}r}}{r}$
	Body wave		
2-cluster scattering	Rayleigh surface wave		$\sim \dfrac{e^{\pm i\sqrt{\lambda}r_*}}{\sqrt{r_*}} \otimes e^{-C_R\sqrt{\lambda_R}x_3}$

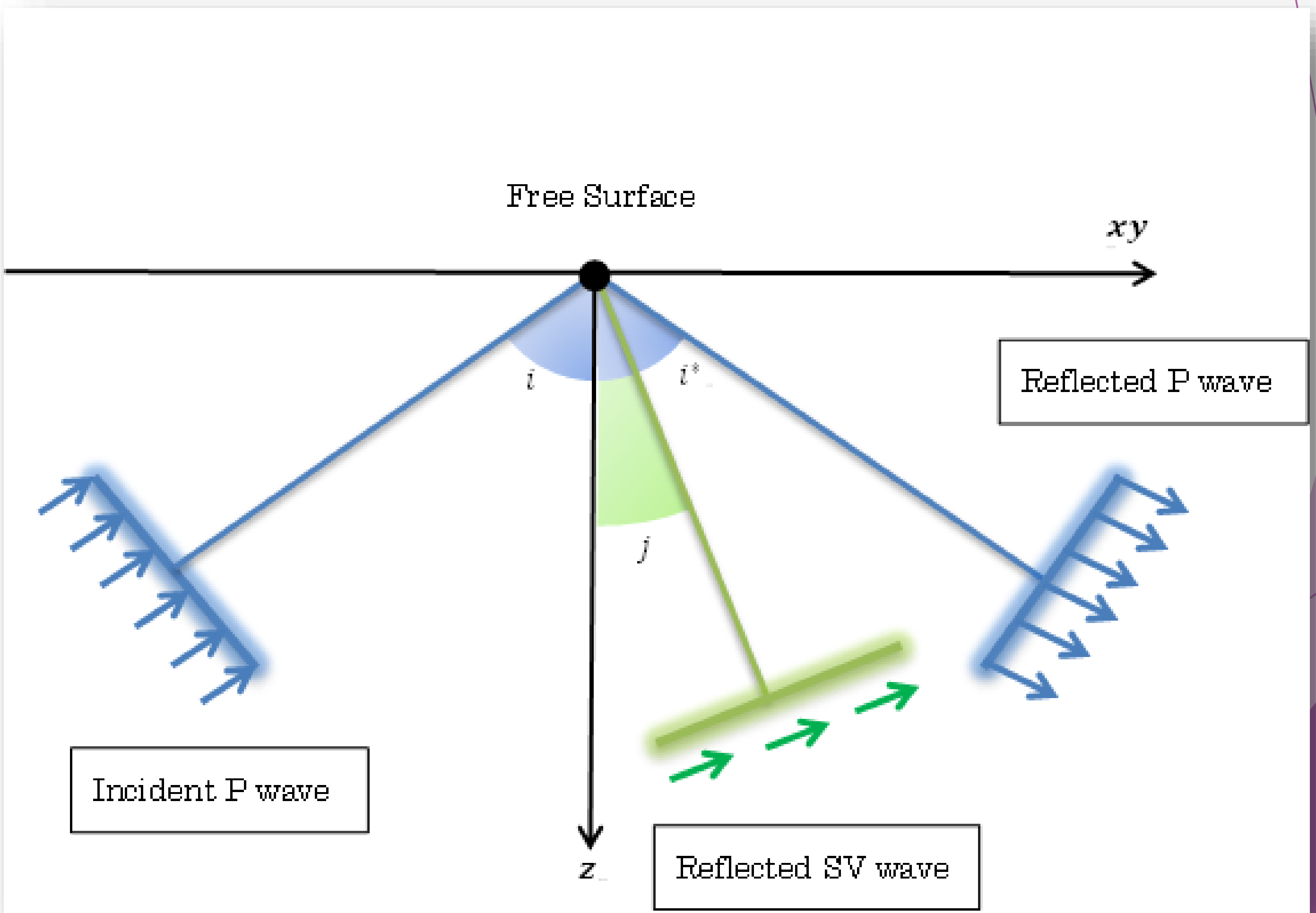
The solution to the stationary elastic wave equation possesses

anisotropy

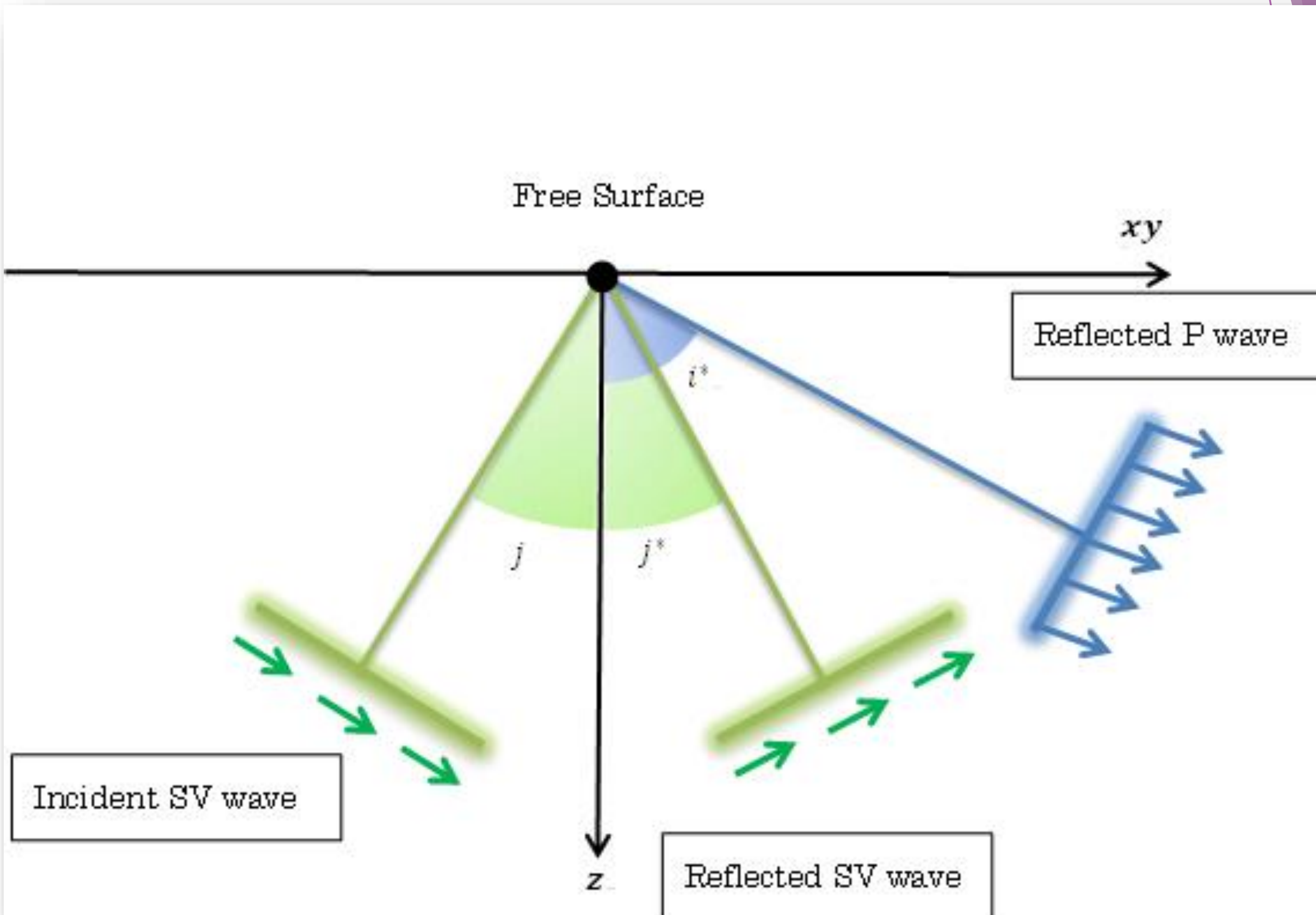
in its spatial asymptotic.

Reflection of P waves at the surface

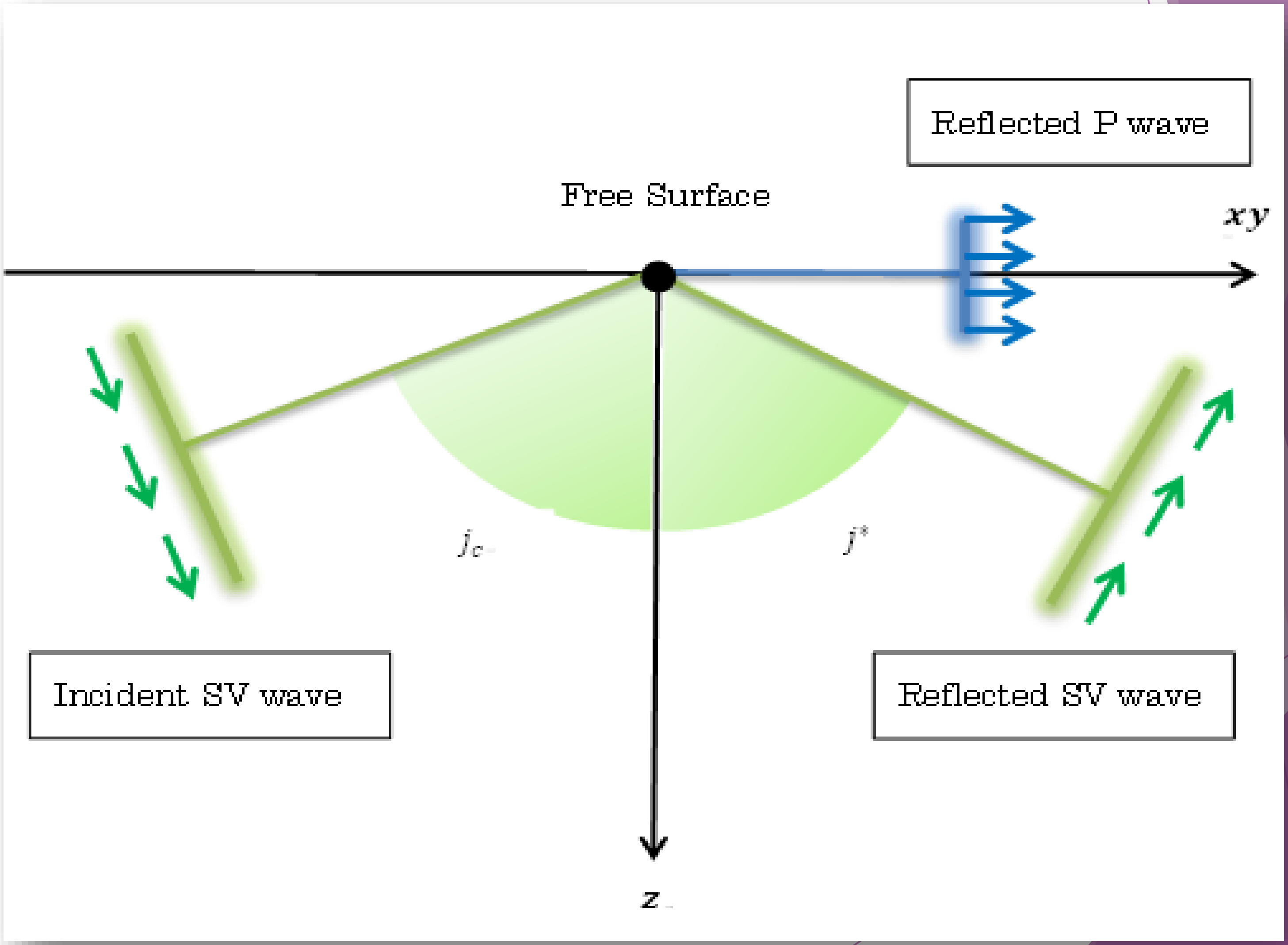
P wave is oscillating along the direction of propagation.
S wave is oscillating vertically to the direction of propagation



Reflection of SV-waves at the free surface



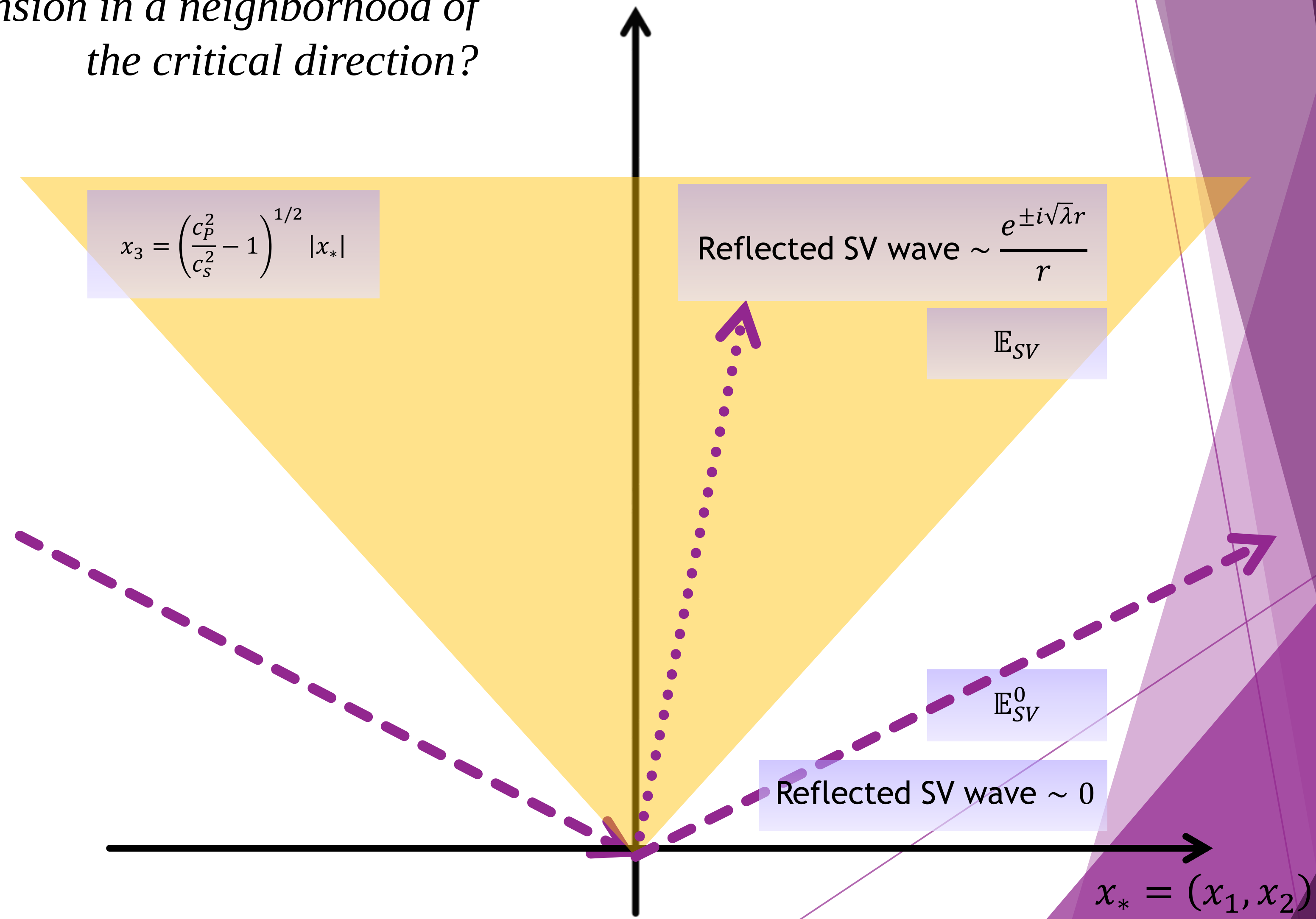
Critical angle of reflection for SV-waves



Motivation

Can we obtain the asymptotic expansion in a neighborhood of the critical direction?

So, there are two types of S waves:
Above the critical cone : SV wave
Below the critical cone : SV^0 wave



Moreover, there is another S wave

- SH wave : oscillating horizontally to the surface

5 types of seismic waves

- P wave : (primary)
- S wave : (secondly)
  - SV wave
  - SV^0 wave
  - SH wave
- R wave : (Rayleigh)

- SV wave and SV^0 wave have singularities along the critical cone.

Our tool is classical.

- The main tool is the stationary phase method.
- Usually, the 2nd order derivatives of the phase and amplitudes are used.
- However, in this case, we can use only the 1st order derivatives.
- We employ the classical computation of

E. T. Copson, Asymptotic expansions, Reprint of the 1965 original. Cambridge Tracts in Mathematics, 55, Cambridge University Press (2004)

Another problem is : We evaluate integrals over the hemisphere, and the stationary phase point appears on the boundary circle.

- The Fresnel integral plays a role.

R. M. Lewis, Asymptotic theory of transients, “Electromagnetic wave theory”, URSI Symposium,

Proceedings, Delft, Pergamon Press (1967)

Free operator

$$\mathrm{u}(x) = {}^{t}(u_1(x), u_2(x), u_3(x)) \in \mathcal{H}_0 = L^2\left(R^3_+, C^3; \rho_0\, dx\right)$$

$$\boldsymbol{L_0 u = -\rho_0^{-1}\{(\lambda_0 + \mu_0)\nabla(\nabla \cdot u) + \mu_0 \Delta u\}}$$

$$u \in D(L_0) \Leftrightarrow \begin{cases} u \in H^1(R^3_+, C^3),\, L_0 u \in \mathcal{H}_0 \\ \lambda_0 \nabla \cdot u \delta_{j3} + \mu_0 \left(\dfrac{\partial u_j}{\partial x_3} + \dfrac{\partial u_3}{\partial x_j}\right)\Big|_{x_3=0} = 0 \ (j = 1,2,3) \end{cases}$$

We state our results for the free case. However, they are easily extended to the locally perturbed case.

Reference

- Y. Dermenjian and J. C. Guillot, Scattering of elastic waves in a perturbed isotropic half space with a free boundary, The limiting absorption principle, Math. Methods Appl. Sci. 10 (1988), 87-124
- Y. Dermenjian and P. Gaitan, Study of generalized eigenfunctions of a perturbed isotropic elastic half-space, Math. Methods Appl. Sci, 23 (2000), 685-708

Recall the structure of generalized eigenfunctions

$$\Psi_P(x,k):\ incident\ P\ wave + reflected\ S\ wave + reflected\ P\ wave$$

$$\Psi_{SV}(x,k): incident\ S\ wave + reflected\ S\ wave + reflected\ P\ wave$$

$$\Psi_{SH}(x,k): incident\ S\ wave + reflected\ S\ wave$$

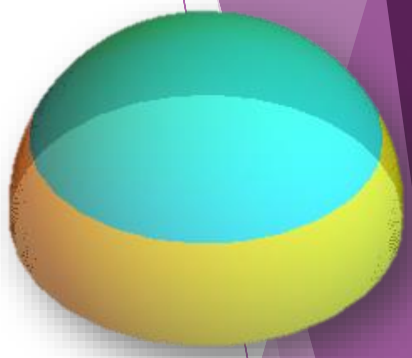
$$\Psi_R(x,p): Rayleigh\ wave$$

$$x,k \in R^3_+, \qquad p \in R^2$$

Be careful

$\Psi_{SV}(x,k)$ is Lipshiz continuous,
not C^2

$$k = |k|\frac{k}{|k|} = |k|\omega = |k|(\omega_1,\omega_2,\omega_3) = |k|(\widetilde{\omega},\omega_3)$$



$$\gamma_{SP}(\widetilde{\omega}) = \left(\frac{c_S^2}{c_P^2} - |\widetilde{\omega}|^2\right)^{1/2} \qquad \gamma'_{SP}(\widetilde{\omega}) = \left(|\widetilde{\omega}|^2 - \frac{c_S^2}{c_P^2}\right)^{1/2}$$

$$k_{SP} = \begin{cases} |k|\big(\widetilde{\omega},\gamma_{SP}(\widetilde{\omega})\big) & (k = |k|\omega \in E_{SV}) \\ |k|\big(\widetilde{\omega},i\gamma'_{SP}(\widetilde{\omega})\big) & (k = |k|\omega \in E_{SV}^0) \end{cases}$$

$$\Psi_{SV}(x,k) = \frac{1}{(2\pi)^{\frac{3}{2}}\rho_0{}^{\frac{1}{2}}}\left\{ e^{ix\cdot\hat{k}}\,\boldsymbol{a}_{SV}^{(1)}(\omega) + e^{ix\cdot k}\boldsymbol{a}_{SV}^{(2)}(\omega) + e^{ix\cdot k_{SP}}\boldsymbol{a}_{SV}^{(3)}(\omega)\right\}$$

Results

Generalized Fourier transform

Making use of these generalized eigenfunctions, we define the Fourier transform associated with L_0.

$$F_P\mathbf{u}(k) = \int_{\mathbb{R}^3_+} \overline{\Psi_P(x,k)} \cdot \mathbf{u}(x)\rho_0 dx,$$

$$F_{SV}\mathbf{u}(k) = \int_{\mathbb{R}^3_+} \{\chi_{\tilde{E}_{SV}}(k)\overline{\Psi_{SV}(x,k)} + \chi_{\tilde{E}^0_{SV}}(k)\overline{\Psi^0_{SV}(x,k)}\} \cdot \mathbf{u}(x)\rho_0 dx,$$

$$F_{SH}\mathbf{u}(k) = \int_{\mathbb{R}^3_+} \overline{\Psi_{SH}(x,k)} \cdot \mathbf{u}(x)\rho_0 dx,$$

$$F_R\mathbf{u}(p) = \int_{\mathbb{R}^3_+} \overline{\Psi_R(x,p)} \cdot \mathbf{u}(x)\rho_0 dx$$

for $\mathbf{u} \in C_0^\infty(\mathbb{R}^3_+, \mathbb{C}^3)$, then F_j $(j = P, SV, SH)$ are extended as a partial isometric operator from $\mathcal{H}$ to $L^2(\mathbb{R}^3_+)$. Similarly, F_R is extended as an partially isometric operator from $\mathcal{H}$ to $L^2(\mathbb{R}^2)$. Thus the Fourier transform $\mathcal{F}$ associated with the elastic operator L_0 can be defined as

$$\mathcal{F}\mathbf{u} = (F_P\mathbf{u}, F_{SV}\mathbf{u}, F_{SH}\mathbf{u}, F_R\mathbf{u}), \qquad \mathbf{u} \in \mathcal{H}.$$

The Fourier transform $\mathcal{F}$ is a unitary operator from $\mathcal{H}$ into

$$\hat{\mathcal{H}} = L^2(\mathbb{R}^3_+) \oplus L^2(\mathbb{R}^3_+) \oplus L^2(\mathbb{R}^3_+) \oplus L^2(\mathbb{R}^2)$$

such that

$$\mathcal{F}(L_0\mathbf{u}) = \left(c_P^2|k|^2F_P\mathbf{u},\ c_S^2|k|^2F_{SV}\mathbf{u},\ c_S^2|k|^2F_{SH}\mathbf{u},\ c_R^2|k|^2F_R\mathbf{u}\right)$$

for $\mathbf{u} \in D(L_0)$.

Dermenjian & Guillot 1988

Results

Restriction of the Fourier transform

We define a operator $\mathcal{F}(\lambda)$ as

$$\mathcal{F}(\lambda)\mathbf{u} = (F_P\mathbf{u}(\sqrt{\lambda}\omega), F_{SV}\mathbf{u}(\sqrt{\lambda}\omega), F_{SH}\mathbf{u}(\sqrt{\lambda}\omega), F_R\mathbf{u}(\sqrt{\lambda}\mu)),$$

for $\lambda > 0$ and $\mathbf{u} \in \mathcal{H}$, where $\omega \in \mathbb{S}^2_+$ and $\mu \in \mathbb{S}^1$. The adjoint operator to $\mathcal{F}(\lambda)$ is denoted by $\mathcal{F}^*(\lambda)$. It will be shown that the operator $\mathcal{F}(\lambda)$ is a bounded operator from $\mathcal{B}$ to

$$\hat{\boldsymbol{h}} = L^2(\mathbb{S}^2_+) \oplus L^2(\mathbb{S}^2_+) \oplus L^2(\mathbb{S}^2_+) \oplus L^2(\mathbb{S}^1)$$

and $\mathcal{F}^*(\lambda)$ is a bounded operator from $\hat{\boldsymbol{h}}$ to $\mathcal{B}$.

Results

$\mathcal{B} - \mathcal{B}^*$ *space*

Let

$$\mathcal{B} = \mathcal{B}(\mathbb{R}^3_+, \mathbb{C}^3) = \{\ \mathbf{u}(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{C}^3)\ ;\ \|\mathbf{u}\|_{\mathcal{B}} < \infty \},$$

where

$$\|\mathbf{u}\|_{\mathcal{B}} = \left(\int_{|x|<1, x_3>0} |\mathbf{u}(x)|^2 dx \right)^{1/2} + \sum_{n=0}^{\infty} 2^{n/2} \left(\int_{2^n<|x|<2^{n+1}, x_3>0} |\mathbf{u}(x)|^2 dx \right)^{1/2}.$$

Then the norm of the dual space $\mathcal{B}^*$ is equivalent to

$$\mathbf{u} \in \mathcal{B}^* \iff \|\mathbf{u}\|_{\mathcal{B}^*} = \sup_{R \geqq 1} \left(\frac{1}{R} \int_{|x|<R, x_3>0} |\mathbf{u}(x)|^2 dx \right)^{1/2} < \infty.$$

The closure $\mathcal{B}^*_0$ of $L^2(\mathbb{R}^3_+)$ in the norm of $\mathcal{B}^*$ consists of functions $f(x)$ satisfying

$$\lim_{R \to \infty} \frac{1}{R} \int_{|x| \le R,\ x_3>0} |f(x)|^2\, dx = 0.$$

The space $\mathcal{B}$ has following properties:

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^*_0 \subset \mathcal{B}^* \subset L^{2,-s}$$

for $s > 1/2$, where $L^{2,s}$ for $s \in \mathbb{R}$ is the weighted L^2 space

Main Theorem

Describe the set of solutions to $(L_0 - \lambda)u = 0$ in terms of the Fourier transform.

Let $\lambda > 0$. Suppose that $\boldsymbol{u}$ satisfy

$$(L_0 - \lambda)\boldsymbol{u} = \boldsymbol{0}.$$

Then $\boldsymbol{u} \in \mathcal{B}^*$ if and only if

$$\boldsymbol{u} = \mathcal{F}^*(\lambda)\Theta$$

for some $\Theta \in (L^2(\mathbb{S}^2_+), L^2(\mathbb{S}^2_+), L^2(\mathbb{S}^2_+), L^2(\mathbb{S}^1))$

$$\boldsymbol{u} = \mathcal{F}^*(\lambda)\Theta$$
$$= F_P^*(\lambda)\Theta_1 + F_{SV}^*(\lambda)\Theta_2 + F_{SH}^*(\lambda)\Theta_3 + F_R^*(\lambda)\Theta_4$$

$$F_P^*(\lambda)\Theta_1 = \int_{\mathbb{S}_+^2} \boldsymbol{\Psi}_P(x, \sqrt{\lambda}\,\omega)\,\Theta_1(\omega)\,d\omega$$

Reflected P wave

$$\boldsymbol{\Psi}_P(x,k) = \frac{\rho_0^{-1/2}}{(2\pi)^{3/2}} e^{i\bar{k}\bar{x}} \left\{ e^{-ik_3x_3}\boldsymbol{a}_p^{(1)}(\omega) + e^{i\xi_{PS}(k)x_3}\boldsymbol{a}_p^{(2)}(\omega) + e^{ik_3x_3}\boldsymbol{a}_p^{(3)}(\omega) \right\}$$

Incident P wave

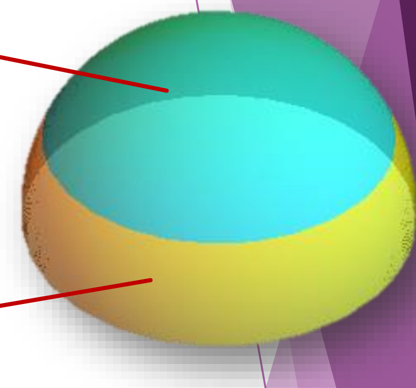
Reflected S wave

$\omega = k/|k|$

Corollary

Let $\lambda > 0$. Suppose that $\mathbf{u} \in \mathcal{B}^*$ satisfies $L_0\mathbf{u} = \lambda\mathbf{u}$. Then for some $(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \in \hat{\boldsymbol{h}}$,

$$
\begin{aligned}
&\boldsymbol{u}(x) \simeq \\
&\quad \frac{\overline{C}}{\sqrt{\lambda_P}} \frac{e^{-i\sqrt{\lambda_P}r}}{r}(\Theta_1 \boldsymbol{a}_P^{(1)})(\varphi_-) + \frac{c_{PS}C}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_S}r}}{r}\chi(\mathbb{S}_{SV})\frac{[\varphi_*]}{[c_{PS}\varphi_*]}(\Theta_1 \boldsymbol{a}_P^{(2)})(\varphi_{PS}) \\
&+ \frac{C}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_P}r}}{r}(\Theta_1 \boldsymbol{a}_P^{(3)})(\varphi) \\
&+ \frac{\overline{C}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r}\chi(\mathbb{S}_{SV})(\Theta_2 \boldsymbol{a}_{SV}^{(1)})(\varphi_-) + \frac{C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r}\chi(\mathbb{S}_{SV})(\Theta_2 \boldsymbol{a}_{SV}^{(2)})(\varphi) \\
&+ \frac{c_{SP}C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_P}r}}{r}\frac{[\varphi_*]}{[c_{SP}\varphi_*]}(\Theta_2 \boldsymbol{a}_{SV}^{(3)})(\varphi_{SP}) \\
&+ \frac{\overline{C}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r}\chi(\mathbb{S}_{SV}^0)(\Theta_2 \boldsymbol{a}_{SV}^{(4)})(\varphi_-) + \frac{C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r}\chi(\mathbb{S}_{SV}^0)(\Theta_2 \boldsymbol{a}_{SV}^{(5)})(\varphi) \\
&+ \frac{\overline{C}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r}(\Theta_3 \boldsymbol{a}_{SH})(\varphi_-) + \frac{C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r}(\Theta_3 \boldsymbol{a}_{SH})(\varphi) \\
&+ \overline{D}_1 \frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}(\Theta_4 \boldsymbol{a}_R^{(1)})(-\hat{\varphi}_*)e^{-\sqrt{\lambda_R}[c_{RP}]x_3} + D_1 \frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}(\Theta_4 \boldsymbol{a}_R^{(1)})(\hat{\varphi}_*)e^{-\sqrt{\lambda_R}[c_{RP}]x_3} \\
&+ \overline{D}_2 \frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}(\Theta_4 \boldsymbol{a}_R^{(2)})(-\hat{\varphi}_*)e^{-\sqrt{\lambda_R}[c_{RS}]x_3} + D_2 \frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}(\Theta_4 \boldsymbol{a}_R^{(2)})(\hat{\varphi}_*)e^{-\sqrt{\lambda_R}[c_{RS}]x_3},
\end{aligned}
$$



where $r = |x|$, $\varphi = x/r$, $r_* = |x_*|$, $\hat{\varphi}_* = x_*/r_*$ and

$$
C = \frac{e^{-\pi i/2}}{\sqrt{2\pi\rho_0}}, \quad D_1 = \frac{\tilde{C}}{\sqrt{2\pi}}(2 - c_{RS}^2)e^{-\pi i/4}, \quad D_2 = \frac{\tilde{C}}{\sqrt{2\pi}}2[c_{RP}]e^{-\pi i/4}.
$$

This corollary dose not provide a representation of the

Our next result provides an asymptotic representation of the $\mathcal{B}^*$-solution uniform in a neighborhood of the critical circle $\partial\mathbb{S}_{SV}$. We see that the smooth transition near the critical circle is described by the Fresnel type integral

$$\mathrm{Fr}(x) = \int_x^\infty e^{is^2}\, ds.$$

In order to state our result, we define functions as follows:

$$\begin{aligned}
\psi(s,\varphi) &= |\varphi_*|s + \varphi_3\sqrt{1-s^2}, \quad \varphi = (\varphi_*, \varphi_3) \in \mathbb{S}^2_+ \\
\alpha(|\varphi_*|) &= \mathrm{sgn}(c_{SP} - |\varphi_*|)\sqrt{1-\psi(c_{SP},\varphi)}, \quad \beta(s) = \sqrt{1-s}, \\
\mathfrak{Fr}(x,s) &= \int_{x\alpha(s)}^{x\beta(s)} e^{it^2}\, dt
\end{aligned}$$

for $0 < s < 1$, where $\mathrm{sgn}(x)$ denotes the signum function.

Let $\lambda > 0$. Suppose that $\mathbf{u} \in \mathcal{B}^*$ satisfies $L_0\mathbf{u} - \lambda\mathbf{u} = \mathbf{0}$. Then for some $(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \in \hat{\boldsymbol{h}}$,

$$
\begin{aligned}
\mathbf{u}(x) \simeq & \frac{\overline{C}}{\sqrt{\lambda_P}} \frac{e^{-i\sqrt{\lambda_P}r}}{r} (\Theta_1 \boldsymbol{a}_P^{(1)})(\varphi_-) \\
&+ \frac{c_{PS}C_-}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_S}r}}{r} \overline{\mathrm{Fr}(-\sqrt{r}\lambda_P^{1/4}\alpha(|\varphi_*|))} \frac{[\varphi_*]}{[c_{PS}\varphi_*]} (\Theta_1 \boldsymbol{a}_P^{(2)})(\varphi_{PS}) \\
&+ \frac{C}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_P}r}}{r} (\Theta_1 \boldsymbol{a}_P^{(3)})(\varphi) \\
&+ \frac{C_+}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r} \mathrm{Fr}(-\sqrt{r}\lambda_S^{1/4}\alpha(|\varphi_*|)) (\Theta_2 \boldsymbol{a}_{SV}^{(1)})(\varphi_-) \\
&+ \frac{C_-}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r} \overline{\mathrm{Fr}(-\sqrt{r}\lambda_S^{1/4}\alpha(|\varphi_*|))} (\Theta_2 \boldsymbol{a}_{SV}^{(2)})(\varphi) \\
&+ \frac{c_{SP}C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_P}r}}{r} \frac{[\varphi_*]}{[c_{SP}\varphi_*]} (\Theta_2 \boldsymbol{a}_{SV}^{(3)})(\varphi_{SP}) \\
&+ \frac{C_+}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r} \mathfrak{Fr}(\sqrt{r}\lambda_S^{1/4}, \varphi_*) (\Theta_2 \boldsymbol{a}_{SV}^{(4)})(\varphi_-) \\
&+ \frac{C_-}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r} \overline{\mathfrak{Fr}(\sqrt{r}\lambda_S^{1/4}, \varphi_*)} (\Theta_2 \boldsymbol{a}_{SV}^{(5)})(\varphi) \\
&+ \frac{\overline{C}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r} (\Theta_3 \boldsymbol{a}_{SH})(\varphi_-) + \frac{C}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r} (\Theta_3 \boldsymbol{a}_{SH})(\varphi) \\
&+ \overline{D}_1 \frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} (\Theta_4 \boldsymbol{a}_R^{(1)})(-\hat{\varphi}_*) e^{-\sqrt{\lambda_R}[c_{RP}]x_3} + D_1 \frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} (\Theta_4 \boldsymbol{a}_R^{(1)})(\hat{\varphi}_*) e^{-\sqrt{\lambda_R}[c_{RP}]x_3} \\
&+ \overline{D}_2 \frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} (\Theta_4 \boldsymbol{a}_R^{(2)})(-\hat{\varphi}_*) e^{-\sqrt{\lambda_R}[c_{RS}]x_3} + D_2 \frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} (\Theta_4 \boldsymbol{a}_R^{(2)})(\hat{\varphi}_*) e^{-\sqrt{\lambda_R}[c_{RS}]x_3}
\end{aligned}
$$

where $r = |x|$, $\varphi = x/r$, $r_* = |x_*|$, $\hat{\varphi}_* = x_*/r_*$ and

$$
C = \frac{e^{-\pi i/2}}{\sqrt{2\pi\rho_0}}, \quad C_\pm = \frac{e^{\pm\pi i/4}}{\sqrt{2\rho_0\pi}}, \quad D_1 = \frac{\tilde{C}}{\sqrt{2\pi}}(2 - c_{RS}^2)e^{-\pi i/4}, \quad D_2 = \frac{\tilde{C}}{\sqrt{2\pi}} 2[c_{RP}]e^{-\pi i/4}.
$$

Theorem

Expand the solutions to $(L_0 - \lambda)u = 0$ at $|x| \to \infty$.

$$= \sqrt{\pi} e^{-\frac{i\pi}{4}} \chi(\mathbb{S}_{SV}) + O(r^{-1}), r \to \infty$$

Thus this Theorem implies Corollary. Moreover, if $\varphi \in \partial \mathbb{S}_{SV}$, then $\alpha(|\varphi_*|) = 0$. This shows that

$$\overline{\mathrm{Fr}(-\sqrt{r}\lambda_P^{1/4}\alpha(|\varphi_*|))} = \overline{\mathrm{Fr}(0)} = \frac{\sqrt{\pi}}{2}e^{-i\pi/4}$$

for $\varphi \in \partial \mathbb{S}_{SV}$, which describes the asymptotic approximation on the critical circle for the reflected S-wave generated by incident P-wave.

Proof

Limiting absorption principle (LAP) on $\mathcal{B}-\mathcal{B}^*$ space

Asymptotic expansion of the resolvent $\boldsymbol{R_0(\lambda \pm i0)}$

Closed range theorem

$$u \in \mathcal{B}^*, (L_0-\lambda)u = 0 \Rightarrow \ u \in \mathrm{Ker}\big(\mathcal{F}(\lambda)\big)^{\perp}$$

$$\therefore \quad u \in \mathrm{Ran}\big(\mathcal{F}^*(\lambda)\big)$$

Proof *Fourier transform restricted to the sphere*

Let $\lambda > 0$. Then we have

$$F(\lambda) \in \mathbf{B}(\mathbf{B};\hat{\mathbf{h}}), \qquad \mathbf{F}^*(\lambda) \in \mathbf{B}(\hat{\mathbf{h}};\mathbf{B}^*).$$

Moreover, estimates

$$\|\mathcal{F}_\sharp(\lambda)\mathbf{u}\|_{L^2(\mathbb{S}^2_+)} \leqq C\lambda^{-1/2}\|\mathbf{u}\|_B, \tag{1}$$

$$\|\mathcal{F}_R(\lambda)\mathbf{u}\|_{L^2(\mathbb{S}^1)} \leqq C\lambda^{-1/4}\|\mathbf{u}\|_B \tag{2}$$

and

$$\|\mathcal{F}_\sharp(\lambda)^*\tilde{f}\|_{B^*} \leqq C\lambda^{-1/2}\|\tilde{f}\|_{L^2(\mathbb{S}^2_+)},$$

$$\|\mathcal{F}_R(\lambda)^*\tilde{g}\|_{B^*} \leqq C\lambda^{-1/4}\|\tilde{g}\|_{L^2(\mathbb{S}^1)}$$

hold , where the positive constant C does not depend on $\lambda > 0$.

Proof *LAP on $\mathcal{B}-\mathcal{B}^*$ space*

Let $0 < a < \lambda < b$. Then the norm of the operator $R_0(z) : B \to B^*$ is uniformly bounded:

$$\|R_0(z)\|_{B\to B^*} \leq C, \tag{1}$$

where the constant $C > 0$ does not depend on $z = \lambda + i\varepsilon$ with $\neq 0$.

by using the **Mourre** method

Proof

Asymptotic expansion of the resolvent

$$(R_0(z)\boldsymbol{f})(x) = B_P(z)\boldsymbol{f} + B_{SV}(z)\boldsymbol{f} + B_{SH}(z)\boldsymbol{f} + B_R(z)\boldsymbol{f}$$

$$(B_P(z)\boldsymbol{f})(x) = F_P^*(c_P^2|k|^2 - z)^{-1}F_P\boldsymbol{f}$$

$$= \int_{\mathbb{R}^3_+} \overline{\boldsymbol{\Psi}_P(x,k)} \frac{F_P\boldsymbol{f}(k)}{(c_P^2|k|^2 - z)} dk \qquad k = \mu\omega$$

$$= C_\rho \sum_{l=1}^{3} \int_0^\infty \frac{\mu^2}{c_P^2\mu^2 - z} \underline{\big(J_{P,l}(\mu)F_P\boldsymbol{f}\big)(x)d\mu}$$

$\sim$? as $|x| \to \infty$

Evaluate asymptotic expansion

Proof

Asymptotic expansion of integrals over the upper half sphere

$$\hat{\omega} = (\omega_1, \omega_2, -\omega_3)$$

$$(J_{SV,1}(\mu)\widehat{\boldsymbol{f}_{SV}})(x) = \int_{\mathbb{S}_{SV}} e^{i\mu x\cdot\hat{\omega}}\,\widehat{\boldsymbol{f}_{SV}}(\mu\omega)\mathbf{a}_{SV}^{(1)}(\omega)d\omega$$

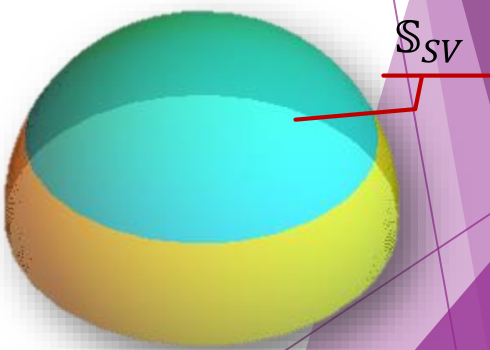
$$= \int_{\mathbb{S}_{SV}} e^{-i\mu x\cdot\kappa}\,\widehat{\boldsymbol{f}_{SV}}(\mu\kappa_-)\mathbf{a}_{SV}^{(1)}(\kappa_-)d\kappa$$

$$\kappa_- = (-\kappa_1, -\kappa_2, \kappa_3)$$

Stationary point

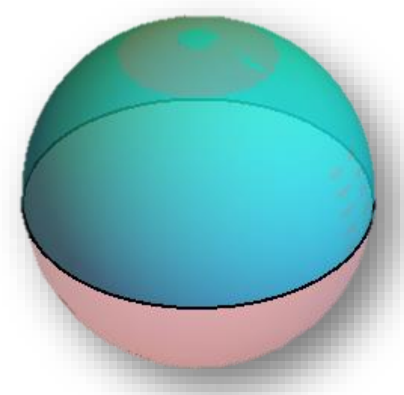
$$\kappa = \pm\omega = \pm\frac{x}{|x|}$$

$$\frac{x}{|x|} \in \mathbb{S}_+^2$$



Proof

Asymptotic expansion of integrals over the upper half sphere



Let $\mu > 0$. Then for any $\varphi \in L^2(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} e^{i\mu x\cdot\omega}\varphi(\omega)\,d\omega \simeq C\frac{e^{i\mu r}}{(\mu r)^{(n-1)/2}}\varphi(\hat{x}) + \overline{C}\frac{e^{-i\mu r}}{(\mu r)^{(n-1)/2}}\varphi(-\hat{x}),$$

where $r = |x|$, $\hat{x} = x/r$ and $C = e^{-(n-1)\pi i/4}(2\pi)^{(n-1)/2}$.

Proof

Asymptotic expansion of integrals over the upper half sphere

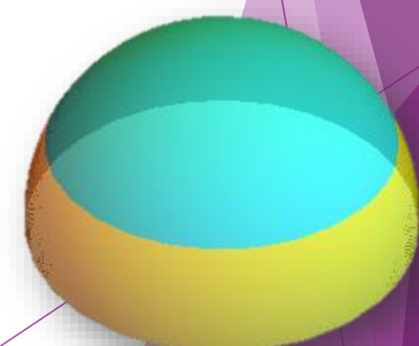
$$\left(J_{SV,1}(\mu)\widehat{\boldsymbol{f}_{SV}}\right)(x) = \int_{\mathbb{S}_{SV}} e^{i\mu x\cdot\widehat{\omega}}\, \widehat{\boldsymbol{f}_{SV}}(\mu\omega)\mathbf{a}_{SV}^{(1)}(\omega)d\omega$$

$$= \int_{\mathbb{S}^2} e^{-i\mu x\cdot\kappa}\, \boldsymbol{\chi}(\mathbb{S}_{SV})\widehat{\boldsymbol{f}_{SV}}(\mu\kappa_-)\mathbf{a}_{SV}^{(1)}(\kappa_-)d\kappa$$

$$\simeq \bar{C}\frac{e^{-i\mu r}}{\mu r}\boldsymbol{\chi}(\mathbb{S}_{SV})\widehat{\boldsymbol{f}_{SV}}(\mu\varphi_-)\mathbf{a}_{SV}^{(1)}(\varphi_-) \qquad x = r\varphi$$

$$(R_0(z)\boldsymbol{f})(x) = B_P(z)\boldsymbol{f} + B_{SV}(z)\boldsymbol{f} + B_{SH}(z)\boldsymbol{f} + B_R(z)\boldsymbol{f}$$

$$B_P(z)\boldsymbol{f} = C_\rho \sum_{l=1}^{3}\int_0^\infty \frac{\mu^2}{c_P^2\mu^2 - z}\left(J_{P,l}(\mu)F_P\boldsymbol{f}\right)(x)d\mu$$



Proof

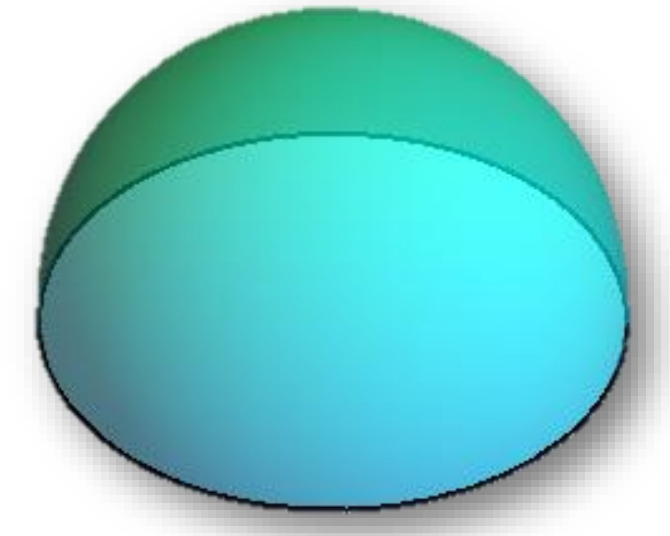
Asymptotic expansion of integrals over the upper half sphere

$$\varphi_- = (-\varphi_1, -\varphi_2, \varphi_3)$$
$$\widehat{\omega} = (\omega_1, \omega_2, -\omega_3)$$
$$\bar{C} = 2\pi e^{-\pi i/2}$$

$$\left(J_{P,1}(\mu)\widehat{\boldsymbol{f}_P}\right)(x) = \int_{\mathbb{S}_+^2} e^{i\mu x\cdot\widehat{\omega}}\,\widehat{\boldsymbol{f}_{\boldsymbol{P}}}(\mu\omega)\mathbf{a}_P^{(1)}(\omega)d\omega \sim \bar{C}\frac{e^{-i\mu r}}{\mu r}\theta(\varphi_-)\mathbf{a}_P^{(1)}(\varphi_-) \quad \text{as } |x| \to \infty$$

as $\varphi_3 \to 0$

?



$$\left(J_{P,1}(\mu)\theta\right)(x_*, 0) \sim \frac{\bar{C}}{2}\frac{e^{-i\mu r}}{\mu r}\theta\left(-\frac{\varphi_*}{|\varphi_*|}, 0\right)\mathbf{a}_P^{(1)}\left(-\frac{\varphi_*}{|\varphi_*|}, 0\right)$$

Let $\omega_3 = \sqrt{1-|\omega_*|^2}$ and $\omega_* = -\nu_*$. Then we find

$$J_{P,1}(\mu)\theta(x_*,0) = \int_{0\le|\nu_*|\le1} e^{-i\mu r\nu_*\cdot\varphi_*}(\theta\boldsymbol{a}_P^{(1)})(-\nu_*,[\nu_*])\,\frac{d\nu_*}{[\nu_*]},$$

where we have used a notation $[x] := \sqrt{1-|x|^2}$. Again letting $\nu_* = \tau\kappa$, $\kappa\in\mathbb{S}^1$,

$$J_{P,1}(\mu)\theta(x_*,0) = \int_0^1 \frac{\tau}{\sqrt{1-\tau^2}}\,d\tau\int_{\mathbb{S}^1} e^{-i\mu r\tau|\varphi_*|\hat{\varphi}_*\cdot\kappa}(\theta\boldsymbol{a}_P^{(1)})(-\tau\kappa,[\tau])\,d\kappa.$$

Applying the stationary phase method to the integral over $\mathbb{S}^1$, we obtain

$$\begin{aligned}J_{P,1}(\mu)\theta(x_*,0) = \frac{\sqrt{2\pi}e^{-3\pi i/4}}{(\mu r)^{1/2}}\int_0^1 \frac{e^{-i\mu r\tau}}{\sqrt{1-\tau}}(\theta\boldsymbol{a}_P^{(1)})(-\tau\hat{\varphi}_*,[\tau])\frac{\sqrt{\tau}}{\sqrt{1+\tau}}\,d\tau \\ + R(x,\mu),\end{aligned}$$

where $R(x,\mu) = O(|x|^{-2/3})$ as $|x|\to\infty$.

In order to evaluate the integral, we consider

$$F(s) = \int_{-i\infty}^{s} \frac{e^{-i\mu r\tau}}{\sqrt{1-\tau}}\, d\tau = -i \int_0^{+\infty} \frac{e^{-i\mu r\tau - \mu r\zeta}}{\sqrt{1-(\tau+i\zeta)}}\, d\zeta.$$

Note that $F(1)$ becomes

$$\begin{aligned} F(1) &= -ie^{-i\mu r} \int_0^{+\infty} \frac{e^{-\mu r\zeta}}{\sqrt{-i\zeta}}\, d\zeta = e^{-i\mu r} e^{-3\pi i/4} \int_0^{+\infty} e^{-\mu r\zeta} \zeta^{-1/2}\, d\zeta \\ &= \frac{e^{-i\mu r}}{\sqrt{\mu r}} \Gamma\left(\frac{1}{2}\right), \end{aligned}$$

where $\Gamma(p)$ is the Gamma function. To obtain a leading term of the integral, we may assume that the support of $\theta \boldsymbol{a}$ is contained in a neighborhood of $\tau = 1$. Integrating by parts, we find

$$\begin{aligned} J_{P,1}(\mu)\theta(x_*, 0) = \frac{\sqrt{2\pi} e^{-3\pi i/4}}{(\mu r)^{1/2}} \Big\{ & F(1)(\theta \boldsymbol{a}_P^{(1)})(-\hat{\varphi}_*, 0) \\ & - \int_0^1 F(\tau) \frac{d}{d\tau} \left((\theta \boldsymbol{a}_P^{(1)})(-\tau\hat{\varphi}_*, [\tau]) \frac{\sqrt{\tau}}{\sqrt{1+\tau}} \right) d\tau \Big\} \\ & + R(x, \mu) \\ = \pi e^{\pi i/2} \frac{e^{-i\mu r}}{\mu r} & (\theta \boldsymbol{a}_P^{(1)})(-\hat{\varphi}_*, 0) + R_1(x, \mu). \end{aligned}$$

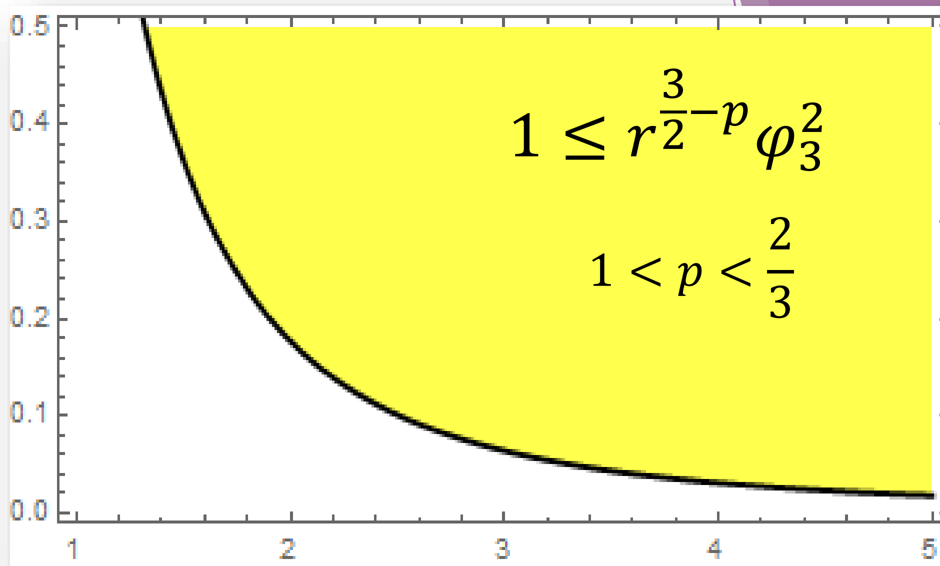
Lemma

$$A \in C^{\infty}(\mathbb{S}^2_+) \quad \mu > 0$$

$$\beta(s) = \sqrt{1-s}$$

$$\mathrm{Fr}(x) = \int_x^{\infty} e^{is^2}\, ds.$$

$$x = r\varphi$$



$$\big(J_{P,1}(\mu)A\big)(x) = C_+ \frac{e^{-i\mu r}}{\mu r} \mathrm{Fr}\big(-\sqrt{\mu r}\,\beta(|\varphi_*|)\big) \left(A a_P^{(1)}\right)(\varphi_-)$$

$$+ \frac{R(x,\mu)}{\mu^{3/2} r^p} \qquad |R(x,\mu)| \leq D \sum_{|l|\leq 2} \sup_{\theta \in \mathbb{S}^2_+} \left|\nabla^{(l)} A(\varphi)\right|$$

Proof

of Lemma

surface integral → **single integral**

$$\omega_* = -\tau\kappa \qquad \kappa \in \mathbb{S}^1$$

$$\omega_3 = \sqrt{1 - |\omega_*|^2} =: [\omega_*]$$

$$\big(J_{P,1}(\mu)A\big)(x) = \int_{\mathbb{S}^2_+} e^{i\mu x\cdot\hat{\omega}} A(\omega)\mathbf{a}_P^{(1)}(\omega)d\omega$$

$$\int_0^1 e^{-i\mu r\varphi_3[\tau]} \frac{\tau}{[\tau]} \left(\underline{\int_{\mathbb{S}^1} e^{-i\mu r\tau|\varphi_*|\kappa\cdot\widehat{\varphi_*}} \left(A\mathbf{a}_P^{(1)}\right)(-\tau\kappa,[\tau])d\kappa} \right) d\tau$$

Stationary Phase Method

$$\left(J_{P,1}(\mu)A\right)(r\varphi) = \frac{\sqrt{2\pi}}{(\mu r|\varphi_*|)^{\frac{1}{2}}}\left(e^{\pi i/4}I_1(\mu, r\varphi) + e^{-\pi i/4}I_2(\mu, r\varphi)\right) + \frac{R(\mu x)}{(\mu r)^{3/2}}$$

$$r \to \infty$$

$$I_1(\mu, r\varphi) = \int_0^1 e^{-i\mu r\psi^+(\tau)}B^+(\tau)d\tau$$

$|R(\mu x)| \leq C$, independent of μ

$$I_2(\mu, r\varphi) = \int_0^1 e^{i\mu r\psi^-(\tau)}B^-(\tau)d\tau$$

$$\psi^\pm(\tau) = |\varphi_*|\tau \pm \varphi_3[\tau]$$

$$B^\pm(\tau) = A\left(\mp\tau\frac{\varphi_*}{|\varphi_*|}, [\tau]\right)\frac{\sqrt{\tau}}{[\tau]}$$

Stationary Point

$$\tau^\dagger = |\varphi_*| \in [0,1]$$

$$I_1(\mu, r\varphi) = \int_0^1 e^{-i\mu r\psi^+(\tau)} B^+(\tau) d\tau$$

$$I_2(\mu, r\varphi) = \int_0^1 e^{i\mu r\psi^-(\tau)} B^-(\tau) d\tau$$

$$\psi^+(\tau) = |\varphi_*|\tau + \varphi_3[\tau]$$

$$(\psi^+)''(\tau) = -\frac{\varphi_3}{[\tau]^3} < 0$$

$$\psi^-(\tau) = |\varphi_*|\tau - \varphi_3[\tau]$$

$$(\psi^+)''(\tau) = \frac{\varphi_3}{[\tau]^3} > 0$$

Stationary Point

$$\tau^\dagger = |\varphi_*| \in [0,1]$$

No Stationary Point

$$B^\pm(\tau) = A\left(\mp\tau\frac{\varphi_*}{|\varphi_*|}, [\tau]\right)\frac{\sqrt{\tau}}{[\tau]}$$

Lewis [9] (1967)

$$t^2 = \psi^+\left(\tau^\dagger\right) - \psi^+(\tau)$$

Second Mean Value Theorem for integration

$$I_1(\mu, r\varphi) = \int_0^1 e^{-i\mu r\psi^+(\tau)} B^+(\tau) d\tau$$

$$I_2(\mu, r\varphi) = \int_0^1 e^{i\mu r\psi^-(\tau)} B^-(\tau) d\tau$$

$$t^2 = 1 - \psi^+(\tau)$$

$$= e^{-i\mu r} \int_{-\infty}^{\sqrt{1-|\varphi_*|}} e^{i\mu r t^2} \underbrace{B^+\big(\tau(t)\big) \frac{d\tau}{dt}}_{H(t)} dt$$

$$s = \psi^-(\tau)$$

$$\int_{\psi^-(\delta_2)}^{|\varphi_*|} e^{i\mu r s} G(s) d\tau$$

$$G(s) = A\left(\tau \frac{\varphi_*}{|\varphi_*|}, [\tau]\right) \frac{\sqrt{\tau}}{[\tau]|\varphi_*| + \varphi_3 \tau}\Bigg|_{\tau=\tau(s)}$$

$$= \frac{e^{-i\mu r} H(0)}{\sqrt{\mu r}} \int_{-\infty}^{\sqrt{\mu r(1-|\varphi_*|)}} e^{is^2} ds$$

$$\boxed{+ \frac{1}{2i\mu r} \int_{-\infty}^{\sqrt{1-|\varphi_*|}} \frac{H(t) - H(0)}{t} \frac{d}{dt}\left(e^{i\mu r t^2}\right) dt}$$

Second Mean Value Theorem for integration

$$I_2(\mu, r\varphi) = \frac{R(\mu x)}{\mu r}, |R(\mu x)| \leq \frac{1}{\varphi_3}$$

$$I_1(\mu, r\varphi) = \int_0^1 e^{-i\mu r\psi^+(\tau)} B^+(\tau)d\tau$$

$$I_2(\mu, r\varphi) = \frac{R(\mu x)}{\mu r}, |R(\mu x)| \leq \frac{1}{\varphi_3}$$

$$\left|\frac{1}{2i\mu r}\int_{-\infty}^{\sqrt{1-|\varphi_*|}} \frac{H(t)-H(0)}{t}\frac{d}{dt}\left(e^{i\mu r t^2}\right)dt\right|$$

$$\boxed{\leq \frac{C}{2\mu r}\frac{1}{\varphi_3^2}}$$

$$h(t) = \frac{H(t)-H(0)}{t}$$

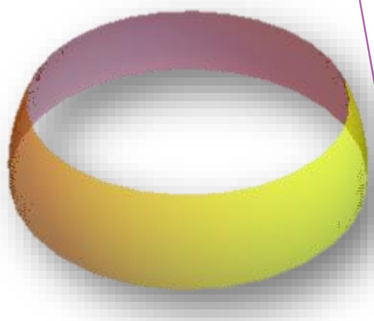
$$|h(\beta)| \leq \frac{C}{\varphi_3} \quad |h'(t)| \leq \frac{C}{\varphi_3^2}$$

$$I_1(\mu, r\varphi) = \int_0^1 e^{-i\mu r\psi^+(\tau)} B^+(\tau) d\tau$$

$$I_2(\mu, r\varphi) = \frac{R(\mu x)}{\mu r}, |R(\mu x)| \leq C$$

$$= \frac{e^{-i\mu r} H(0)}{\sqrt{\mu r}} \int_{-\infty}^{\sqrt{\mu r(1-|\varphi_*|)}} e^{is^2} ds$$

$$+ \frac{R(\mu x)}{\mu r} \quad |R(\mu x)| \leq \frac{C}{\varphi_3^2}$$



$$\left(J_{P,1}(\mu)A\right)(r\varphi) = \frac{\sqrt{2\pi}}{(\mu r|\varphi_*|)^{\frac{1}{2}}} \left(e^{\pi i/4} I_1(\mu, r\varphi) + e^{-\pi i/4} I_2(\mu, r\varphi)\right) + \frac{R(\mu x)}{(\mu r)^{3/2}}$$

$$\frac{1}{r^{3/2}} \frac{1}{\varphi_3^2} \leq \frac{C}{r^p} \quad 1 < p < \frac{3}{2}$$

$$= C_+ \frac{e^{-i\mu r}}{\mu r} \mathrm{Fr}\left(-\sqrt{\mu r}\beta(|\varphi_*|)\right) \left(A a_P^{(1)}\right)(\varphi_-) + \frac{R(x, \mu)}{\mu^{3/2} r^p}$$

Proof

Asymptotic expansion of resolvent

$$(R_0(z)\boldsymbol{f})(x) = B_P(z)\boldsymbol{f} + B_{SV}(z)\boldsymbol{f} + B_{SH}(z)\boldsymbol{f} + B_R(z)\boldsymbol{f}$$

$$(B_P(z)\boldsymbol{f})(x) = C_\rho \sum_{l=1}^{3} \int_0^\infty \frac{\mu^2}{c_P^2\mu^2 - z} \big(J_{P,l}(\mu) F_P \boldsymbol{f}\big)(x) d\mu$$

$$B_P^{(1)}(\lambda - i0)\boldsymbol{f}(x) =$$

$$A_P^{(1)}(\lambda, x) + \boxed{\frac{C_\rho}{r^p} \int_0^\infty \frac{\sqrt{\mu}}{c_P^2\mu^2 - (\lambda - i0)} R(\mu, x)\, d\mu}$$

Absolute Value of $\boxed{} \leq Cr^{-(p-1+\gamma)}, 0 < \gamma < 1$

$$(B_P(z)\boldsymbol{f})(x) = C_\rho \sum_{l=1}^{3} \int_0^\infty \frac{\mu^2}{c_P^2\mu^2 - z} \big(J_{P,l}(\mu)F_P\boldsymbol{f}\big)(x)d\mu$$

$$1 \le r^{\frac{3}{2}-p}\varphi_3^2$$

$$B_P^{(1)}(\lambda - i0)\boldsymbol{f}(x) = A_P^{(1)}(\lambda, x) + \frac{C_\rho}{r^p}\int_0^\infty \frac{\sqrt{\mu}}{c_P^2\mu^2 - (\lambda - i0)} R(\mu, x)\, d\mu$$

$$|\cdots| \le C_1 r^{-(p-1+\gamma)},\ 0 < \gamma < 1$$

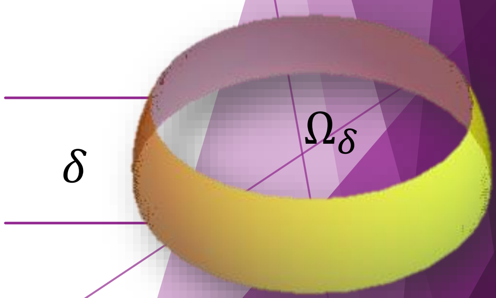
$$\frac{1}{R}\int_{B_R} \rho\left(\frac{|x|}{R}\right)\left|B_P^{(1)}(\lambda - i0)\boldsymbol{f} - A_P^{(1)}(\lambda, x)\right|^2 dx$$

$$\le \frac{1}{R}\int_{B_R\setminus\Omega_\delta} \rho\left(\frac{|x|}{R}\right)\left|B_P^{(1)}(\lambda - i0)\boldsymbol{f} - A_P^{(1)}(\lambda, x)\right|^2 dx$$

$$+ \left\|B_P^{(1)}(\lambda - i0)\boldsymbol{f}\right\|_{\mathcal{B}^*(\Omega_\delta)} + \left\|A_P^{(1)}\right\|_{\mathcal{B}^*(\Omega_\delta)}$$

$$\le \varepsilon$$

$$\forall R > R_0 = \max\left\{\delta^{-\frac{4}{3-2p}}, 3C_1\varepsilon^{-1}\right\}$$



Future Work

Perturbation

Scattering Matrix

Inverse Scattering

Summary

$\mathcal{B}^*$-space key in asymptotics of Body wave and surface wave

Asymptotic analysis of integrals near critical angle and boundary important

Integration by parts

Stationary Phase Method

Saddle Point Method

Uniform expansion of integral
with a stationary point near an endpoint

Asymptotic expansion of resolvent

Characterization of solutions

Lorsque J'apprenais Francais, on m'apris que tout les femme ont 25 ans

Bon Anniversaire
Assia

Toujours, Toujours
25 ans