Control of the movement of a set or particles driven by the stationary Stokes equation

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I. Introduction. Lagrangian controllability

- Lagrangian controllability is a natural notion for control systems modelling a fluid : instead of controlling the state of the system (e.g. the velocity field), one tries to control the flow associated to the system.

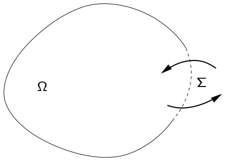
- Let us give an example on the incompressible Navier-Stokes system.
- We consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$.
- Navier-Stokes equation for incompressible Newtonian fluids

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \operatorname{div} u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

- Here, $u : [0, T] \times \Omega \to \mathbb{R}^n$ is the velocity field, $p : [0, T] \times \Omega \to \mathbb{R}$ is the pressure field.

Boundary control

- We consider a non empty open part Σ of the boundary $\partial\Omega$.
- Non-homogeneous boundary conditions can be chosen as follows :
  - on $\partial\Omega \setminus \Sigma$, the fluid sticks to the boundary, $u = 0$.
  - on Σ, we suppose that one can choose the boundary conditions, that is :
$$u(t, x) \ \text{ on } \ [0, T] \times \Sigma.$$
- This boundary condition is a control which we can choose to influence the system, in order to prescribe its behavior.



The standard problem of controllability

- Standard problem of exact/approximate controllability :

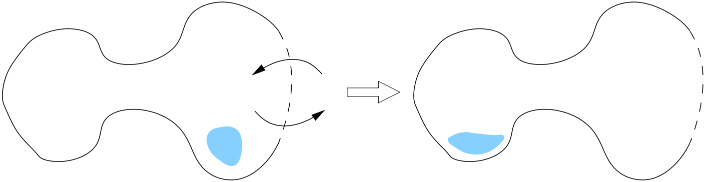
  Given two possible states of the system, say u_0 and u_1, and given a time $T > 0$, can one find a control such that the corresponding solution of the system starting from u_0 at time $t = 0$ reaches the target u_1 at time $t = T$?
  At least such that
  $$\|u(T,\cdot) - u_1\|_X \leq \varepsilon? \tag{AC}$$

- Alternative formulation : given u_0, u_1 and T, can we find a solution of the equation satisfying the constraint on the boundary :
  $$u = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma),$$
  (under-determined system) and driving u_0 to u_1 at time T ? Or to $u(T,\cdot)$ satisfying (AC) ?

- See e.g. Fursikov-Imanuvilov, Fernández-Cara-Guerrero-Imanuvilov-Puel, Coron, etc., for what concerns the boundary controllability of the Navier-Stokes equation.

Another type of controllability

- Another type of controllability is natural for equations from fluid mechanics : is possible to drive a zone in the fluid from a given place to another by using the control ? (Based on a suggestion by J.-P. Puel)



- One can think for instance to a polluted zone in the fluid, which we would like to transfer to a zone where it can be treated.
- It is natural, in order to control the fluid zone during the whole displacement to ask that it remains inside the domain during the whole time interval.

Exact Lagrangian controllability

- Due to the incompressibility of the fluid, the starting zone and the target zone must have the same area.
- We have also to require that there is no topological obstruction to move a zone to the other one.

- In the sequel, we will consider fluids zones given by the interior (supposed to be inside Ω) of smooth (C^∞) Jordan curves/surface.

Definition

We will say that the system satisfies the exact Lagrangian controllability property, if given two smooth Jordan curves/surface γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area/volume, a time $T > 0$ and an initial datum u_0, there exists a control such that the flow given by the velocity fluid drives γ_0 to γ_1, by staying inside the domain.

Approximate Lagrangian controllability

Definition

We will say that the system satisfies the property of approximate Lagrangian controllability in C^k, if given two smooth Jordan curves/surface γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same volume, a time $T > 0$, an initial datum u_0 and $\varepsilon > 0$, we can find a control such that the flow of the velocity field maintains γ_0 inside Ω for all time $t \in [0, T]$ and satisfies, up to reparameterization :

$$\|\Phi^u(T, 0, \gamma_0) - \gamma_1\|_{C^k} \leq \varepsilon.$$

Here, $(t, s, x) \mapsto \Phi^u(t, s, x)$ is the flow of the vector field u (the position at time t of the particle located at x at time s).

II. Previous results on Lagrangian controllability : Euler equation

- Previous results. Results of Lagrangian controllability have been obtained for :
  - Burgers equation (Horsin),
  - 2D & 3D Euler Equation (G.-Horsin),
  - Other approach for the Euler Equation (Horsin-Kavian).

- The Euler equation corresponds to the high Reynolds number regime ($\nu = 0$ in Navier-Stokes above) :

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \operatorname{div} u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

- Here again $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a smooth domain.
- The control acts on an open part Σ of $\partial\Omega$, that is to say

$$u \cdot n = 0 \text{ on } \partial\Omega \setminus \Sigma.$$

An objection in the case of Euler equation

The exact Lagrangian controllability does not hold in general for the Euler equation, indeed :

- Let us suppose $\omega_0 := \text{curl } u_0 = 0$. In that case if the flow $\Phi(t, x)$ maintains γ_0 inside the domain, then for all t, $\omega(t, \cdot) = \text{curl } u(t, \cdot) = 0$ in the neighborhood of $\Phi(t, \gamma_0)$, since
$$\partial_t \omega + (u \cdot \nabla)\omega = 0.$$
- Since div $u = 0$, locally around γ_0, u is the gradient of a harmonic function ; u is therefore analytic in a neighborhood $\Phi(t, \gamma_0)$.
- Hence if γ_0 is analytic, its analyticity is propagated over time.
- If γ_1 is smooth but non analytic, the exact Lagrangian controllability cannot hold.

Approximate Lagrangian controllability for 2D Euler

Theorem (G.-Horsin)

Consider two smooth smooth Jordan curves γ_0, γ_1 in Ω, homotopic in Ω and surrounding the same area. Let $k \in \mathbb{N}$. We consider $u_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying

$$div\ u_0 = 0 \text{ in } \Omega \text{ and } u_0 \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma).$$

For any $T > 0$, $\varepsilon > 0$, there exists a solution u of the Euler equation in $C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ with

$$u \cdot n = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma) \text{ and } u_{|t=0} = u_0 \text{ in } \Omega,$$

and whose flow satisfies

$$\forall t \in [0, T],\ \Phi^u(t, \gamma_0) \subset \Omega,$$

and up to reparameterization

$$\|\gamma_1 - \Phi^u(T, \gamma_0)\|_{C^k} \leq \varepsilon.$$

Approximate Lagrangian controllability for 3D Euler

Theorem (G.-Horsin)

Let $\alpha \in (0,1)$ *and* $k \in \mathbb{N} \setminus \{0\}$*. Consider* $u_0 \in C^{k,\alpha}(\Omega; \mathbb{R}^3)$ *satisfying*

$$\mathit{div}\, u_0 = 0 \;\; \mathit{in} \;\; \Omega \;\; \mathit{and} \;\; u_0 \cdot n = 0 \;\; \mathit{on} \;\; \partial\Omega \setminus \Gamma,$$

and γ_0 *and* γ_1 *two contractible* C^∞ *embeddings of* $\mathbb{S}^2$ *in* Ω *satisfying*

$$|\mathit{int}(\gamma_0)| = |\mathit{int}(\gamma_1)|.$$

Then for any $\varepsilon > 0$*, there exist a time* $T > 0$ *and a solution* (u, p) *in* $L^\infty(0, T; C^{k,\alpha}(\Omega; \mathbb{R}^4))$ *of the Euler equation on* $[0, T]$ *with*

$$u \cdot n = 0 \; \mathit{on} \; [0, T] \times (\partial\Omega \setminus \Sigma) \; \mathit{and} \; u_{|t=0} = u_0 \; \mathit{in} \; \Omega,$$

such that

$$\forall t \in [0, T], \quad \phi^u(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\phi^u(T, 0, \gamma_0) - \gamma_1\|_{C^k(\mathbb{S}^2)} < \varepsilon.$$

III. Lagrangian controllability for the stationary Stokes equation

- The Euler equation corresponds to the high Reynolds number regime ($\nu = 0$ in Navier-Stokes above).
- Here we are concerned with the low Reynolds number regime ($\nu \to +\infty$), which yields the stationary Stokes equation :

$$-\Delta u + \nabla p = 0 \text{ in } \Omega,$$
$$\operatorname{div} u = 0 \text{ in } \Omega,$$

with $u = u(t,x)$ and $p = p(t,x)$.

Stationary Stokes control system

- We consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ and a non empty open part Σ of the boundary $\partial\Omega$.
- We end up with the following control system :

$$\begin{cases} -\Delta u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \text{div } u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

- The boundary conditions are as follows :
  - on $\partial\Omega \setminus \Sigma$, $u = 0$,
  - the value of u on Σ is the control.
- The standard controllability problems would not be very satisfying in this context, but Lagrangian controllability makes sense despite the fact that the equation is stationary.
- See for instance Alouges-Giraldi, Lohéac-Munnier, ... for other controllability results (micro-swimmers) relying on the stationary Stokes equation.

An objection in the case of Stokes

The exact Lagrangian controllability does not hold in general, indeed :

- the properties of Stokes equation yield that u is real-analytic (in the variable x) inside Ω.
- Hence if γ_0 is analytic, its analyticity is propagated over time.
- If γ_1 is smooth but non analytic, the exact Lagrangian controllability cannot hold.
- Consequently, we look for approximate Lagrangian controllability.

The 2-D case

Theorem (G.-Horsin)

Consider two smooth smooth Jordan curves γ_0, γ_1 *in* Ω, *homotopic in* Ω *and surrounding the same area. Let* $k \in \mathbb{N}$. *Then for any* $T > 0$, $\varepsilon > 0$, *there exists a solution* (u, p) *of the Stokes equation in* $C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^2)$ *with*

$$u = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma),$$

and whose flow satisfies

$$\forall t \in [0, T], \ \Phi^u(t, 0, \gamma_0) \subset \Omega,$$

and up to reparameterization

$$\|\gamma_1 - \Phi^u(T, 0, \gamma_0)\|_{C^k} \leq \varepsilon.$$

The 3-D case

Theorem (G.-Horsin)

Let $k \in \mathbb{N} \setminus \{0\}$. Let γ_0 and γ_1 two C^∞ Jordan surfaces in Ω such that

$$\gamma_0 \textit{ and } \gamma_1 \textit{ are contractible in } \Omega \quad \textit{and} \quad |Int(\gamma_0)| = |Int(\gamma_1)|.$$

Then for any $\varepsilon > 0$, for all $T > 0$, there is a solution (u, p) in $C^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^4)$ of the Stokes equation on $[0, T]$ with $u = 0$ on $\partial\Omega \setminus \Sigma$ such that

$$\forall t \in [0, T], \ \Phi^u(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\Phi^u(T, 0, \gamma_0) - \gamma_1\|_{C^k(\mathbb{S}^2)} < \varepsilon.$$

IV. A related question of approximation

One of the main ideas to get to obtain the Lagrangian controllability in the case of the Euler equation was to use results from holomorphic/harmonic approximation, such as Runge's theorem, Mergelyan's theorem or Walsh's theorem :

Theorem (Runge)

Let K a compact set in $\mathbb{C}$, Ω an open set such that $K \subset \Omega$. Let A a set such that any connected component of $\overline{\mathbb{C}} \setminus \Omega$ contains at least a point of A. Then, for each holomorphic function u on Ω and each $\varepsilon > 0$, there is a rational function v whose poles are in A, and such that $\|v - u\|_\infty < \varepsilon$ on K.

Theorem (Mergelyan)

Let K a compact set in $\mathbb{C}$ whose complement is connected. If u is a continuous complex function on K which is holomorphic in the interior of K, then for any $\varepsilon > 0$ there is a polynomial function v such that $\|v - u\|_\infty < \varepsilon$ on K.

Walsh's theorem

Walsh's theorem is the equivalent of Runge's theorem for harmonic functions in dimension n :

Theorem (Walsh, Gardiner)

Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ and let K be a compact set in $\mathbb{R}^N$ such that that $\mathcal{O}^* \setminus K$ is connected, where $\mathcal{O}^*$ is the Alexandroff compactification of $\mathcal{O}$. Then, for each function u which is harmonic on an open set containing K and each $\varepsilon > 0$, there is a harmonic function v in $\mathcal{O}$ such that $\|v - u\|_\infty < \varepsilon$ on K.

A weak Runge/Walsh theorem for stationary Stokes equation

Theorem (G.-Horsin)

Let K a compact set in $\mathbb{R}^N$, $N = 2$ or 3. Let $\mathcal{V}$ and Ω two bounded open sets such that $K \subset \mathcal{V}$, $\overline{\mathcal{V}} \subset \Omega$ and each connected component of $\mathbb{R}^N \setminus K$ contains an interior point of $\mathbb{R}^N \setminus \Omega$. Then for any solution $(u, p) \in C^\infty(\mathcal{V}; \mathbb{R}^{N+1})$ of the Stokes equation in $\mathcal{V}$:

$$\begin{cases} -\Delta u + \nabla p = 0, \\ div\ u = 0 \end{cases} \quad \text{in } \mathcal{V},$$

for any $k \in \mathbb{N}$ and any $\varepsilon > 0$ there exists $(\overline{u}, \overline{p}) \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^{N+1})$ a solution of the Stokes equation in Ω :

$$\begin{cases} -\Delta \overline{u} + \nabla \overline{p} = 0, \\ div\ \overline{u} = 0 \end{cases} \quad \text{in } \Omega,$$

and

$$\|\overline{u} - u\|_{C^k(K)} \leq \varepsilon.$$

V. Ideas of proof of the Runge-Walsh-type result

- We start from a solution (u, p) defined on $\mathcal{V}$ an open neighborhood of K, and want to construct $(\overline{u}, \overline{p})$ on Ω approximating well (u, p) on K.

- First, we can assume that K, $\mathcal{V}$ and Ω have smooth boundaries and even that there is a regular intermediate domain $\mathcal{V}_0$:

$$K \subset \mathcal{V}_0 \subset \overline{\mathcal{V}_0} \subset \mathcal{V} \subset \overline{\mathcal{V}} \subset \Omega.$$

Idea : introduce a function $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi \geq 0$ such that

$K = \varphi^{-1}(\{0\})$ (Whitney) and set

$$\tilde{K} = \varphi^{-1}([0, \varepsilon_K]), \quad \tilde{V} = \varphi^{-1}([0, \varepsilon_{\mathcal{V}})),$$

for $\varepsilon_K < \varepsilon_{\mathcal{V}}$ regular values of φ (Sard).

- Step 1. We first set

$$\hat{u} := u - \nabla \Delta^{-1} \tilde{p} \quad \text{in} \quad \mathcal{V},$$

with $\Delta^{-1} := \cdot * G$, $G(x) = \frac{1}{2\pi} \ln |x|$ for $n = 2$, $G(x) = -\frac{1}{4\pi |x|}$ for $n = 3$, and $\tilde{p}$ a smooth extension of p. Then

$$\operatorname{curl} \hat{u} = \operatorname{curl} u \quad \text{in } \mathcal{V}, \qquad \Delta \hat{u} = 0 \quad \text{in } \mathcal{V}.$$

- Step 2. We use Walsh's theorem on $\hat{u}$. We obtain some $\tilde{u}$ defined on Ω such that

$$\Delta \tilde{u} = 0 \text{ in } \mathbb{R}^n \setminus \{A_1, \ldots, A_N\}, \ \|\hat{u} - \tilde{u}\|_{C^{\boldsymbol{k+1},\alpha}(\overline{\mathcal{V}_{\mathbf{0}}})} \leq \varepsilon,$$

In particular we have

$$\left| \oint_\gamma (u - \tilde{u}) \cdot \tau \right| \leq C\varepsilon \quad \text{and} \quad \| \operatorname{curl} u - \operatorname{curl} \tilde{u} \|_{C^{\boldsymbol{k},\alpha}(\overline{\mathcal{V}_{\mathbf{0}}})} \leq C\varepsilon.$$

for all smooth loop $\gamma \subset \mathcal{V}_0$

- Step 3. Now $\operatorname{curl} u - \operatorname{curl} \tilde{u} = \mathcal{O}(\varepsilon)$ gives

$$u = \tilde{u} + \nabla\theta + H + \mathcal{O}(\varepsilon),$$

for some regular θ on $\mathcal{V}$ and H in the first de Rham cohomology space, but small since

$$\oint_\gamma (u - \tilde{u}) \cdot \tau = \mathcal{O}(\varepsilon).$$

We extend θ to Ω arbitrarily. Now $\tilde{u} + \nabla\theta$ is a good candidate since

$$\Delta(\tilde{u} + \nabla\theta) = \nabla(\Delta\theta),$$

but the divergence is not 0! It is, however, of order $\mathcal{O}(\varepsilon)$ on $\mathcal{V}_0$.

- Step 4. For that we introduce q and $\tilde{q}$ solutions of

$$-\Delta q = \operatorname{div}(\tilde{u} + \nabla\theta) \text{ in } \Omega, \quad \tilde{q} = 0 \text{ on } \partial\Omega.$$

$$-\Delta \tilde{q} = \operatorname{div} \hat{U} \text{ in } \Omega, \quad \tilde{q} = 0 \text{ on } \partial\Omega,$$

  where $\hat{U}$ is constructed in order to have the same divergence as $\tilde{u} + \nabla\theta$ in $\mathcal{V}_0$ and is of order $\mathcal{O}(\varepsilon)$.
- Then $q - \tilde{q}$ is harmonic on $\mathcal{V}_0$, and can be approximated by a harmonic function ψ defined on Ω.
- We set

$$\beta = q - \psi,$$

  and one checks that

$$\overline{u} := \tilde{u} + \nabla\theta - \nabla\beta \quad \text{and} \quad \overline{p} := \Delta(-\theta + \beta),$$

  is a solution to our problem.

- Step 4'. How to obtain $\hat{U}$?
- We start from $\operatorname{div}(\tilde{u} + \nabla\theta) = \mathcal{O}(\varepsilon)$ in $\mathcal{V}_0$ and want to obtain $\hat{U}$ of size $\mathcal{O}(\varepsilon)$ such that

$$\operatorname{div}(\tilde{u} + \nabla\theta) = \operatorname{div}\ \hat{U}.$$

- But from $\operatorname{div}(\tilde{u} - \nabla\theta) = \mathcal{O}(\varepsilon)$ one can obtain easily that

$$\tilde{u} - \nabla\theta = \operatorname{curl}(A) + h + R,$$

  where
  - A is some vector field,
  - h belong to the *second* de Rham cohomology space of $\mathcal{V}_1$
  - the remaining term satisfies $R = \mathcal{O}(\varepsilon)$.
- Extend R in Ω keeping it size of $\mathcal{O}(\varepsilon)$ and you are done.

VI. Ideas of proof. From the Runge-Walsh-type result to Lagrangian controllability

- One seeks a vector field satisfying Stokes' equation, fulfilling the boundary condition on $\partial\Omega \setminus \Sigma$ and whose flow drives γ_0 to γ_1 (approximately in C^k).

- This is proven in two parts :

  - Part 1 : find a solenoidal (divergence-free) vector field driving γ_0 to γ_1.

  - Part 2 : approximate (at each time) the above vector field on the curve (or to be more precise, its normal part), by a solution of Stokes' system defined on Ω and satisfying the constraint.

Part 1

Proposition

Consider γ_0 and γ_1 two smooth (C^∞) Jordan curves/surface isotopoic in Ω. If γ_0 and γ_1 satisfy

$$|Int(\gamma_0)| = |Int(\gamma_1)|,$$

then there exists $v \in C_0^\infty((0,1)\times\Omega;\mathbb{R}^2)$ such that

$$div\ v = 0 \ \ in\ (0,1)\times\Omega,$$

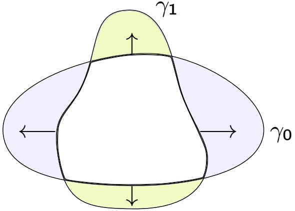
$$\Phi^v(1,0,\gamma_0) = \gamma_1.$$

This was also used in the case of the Euler equation.

(Isotopic : one can deform γ_0 to γ_1 by a continuous family of smooth embeddings)

Idea of proof for Part 1

- In 2-D, one can make moves like the ones described below.



- But it turns out that a very general result due to A. B. Krygin.

Krygin's theorem

Theorem

Let W a contractible (to simplify) Jordan surface in an orientable manifold M. Consider a smooth family of embeddings $f_t : W \to M$, $t \in [0, 1]$, satisfying the properties : $f_0 = id$ and $|Int(f_1(W))| = |Int(W)|$. Then, there exists a family of volume-preserving diffeomorphisms $F_t : M \to M$ such that $F_0 = id$ and $F_{1|\partial W} = f_1$.

- The proof relies on J. Moser's celebrated result on deformation of volume forms : on a manifold, one can deform a smooth volume form onto another one via a smooth diffeomorphism, provided they have same total mass.

Part 2

Proposition

Let γ_0 a smooth (C^∞) Jordan curve/surface ; let $X \in C^0([0,1]; C^\infty(\overline{\Omega}))$ a smooth solenoidal vector field, with $X = 0$ on $[0,1] \times \partial\Omega$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $(u, p) \in C^\infty([0,1] \times \overline{\Omega}; \mathbb{R}^{3,4})$ such that

$$-\Delta u + \nabla p = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$div\ u = 0 \text{ in } \Omega, \text{ for all } t \in [0,1],$$
$$u = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Sigma),$$

and whose flow satisfies

$$\forall t \in [0,1],\ \Phi^u(t, 0, \gamma_0) \subset \Omega,$$

and, up to reparameterization,

$$\|\Phi^X(t, 0, \gamma_0) - \Phi^u(t, 0, \gamma_0)\|_{C^k} \leq \varepsilon, \quad \forall t \in [0,1].$$

Ideas of proof for Part 2

- To construct a Stokes flow u transporting the fluid zone in the same way as X, we introduce for each time the solution of the following Stokes problem :

$$\begin{cases} -\Delta u(t,\cdot) + \nabla p = 0 \text{ in } \mathrm{Int}(\gamma(t)), \\ \mathrm{div}\; u(t,\cdot) = 0 \text{ in } \mathrm{Int}(\gamma(t)), \\ u(t,\cdot) = X(t,\cdot) \text{ on } \gamma(t). \end{cases}$$

- The main issue here is that in general, u cannot be extended to Ω, not to mention in a way that fulfills Stokes equation and

$$u = 0 \text{ on } \partial\Omega \setminus \Sigma.$$

Ideas of proof for Part 2

- The idea is to use our Runge-Walsh-type theorem to obtain a suitable approximation.
- the problem being that ψ is not defined in a neighborhood of $\gamma(t)$ and to obtain a approximation uniformly in time (γ is moving !)

Remark. Here instead of relying on the Runge-Walsh type theorem, we could use the following statement.

Theorem. Assume that γ is a C^∞ Jordan surface included in Ω, then the set $\left\{u_{|\gamma},\ (u,p) \text{ is a solution of Stokes in } \Omega \text{ such that } u_{|\partial\Omega} \in H_m^{1/2}(\Sigma)\right\}$, is dense in $H_m^{1/2}(\gamma)$.

This result can be proved by a duality argument and using the results of Fabre-Lebeau on the unique continuation for the Stokes operator.

Three steps

- The proof follows three successive steps of growing generality.
  - The case where X and γ_0 are analytic,
  - The case where γ_0 is analytic but X is merely C^∞,
  - The case where X and γ_0 are C^∞.
- First step : when the data are real analytic : $\gamma_0 \in C^\omega(\mathbb{S}^{n-1}; \mathbb{R}^2)$ and $X \in C^0([0,1]; C^\omega(\overline{\Omega}))$.
- In that case, the curve $\gamma(t)$ is an analytic curve and so is the Dirichlet boundary data $u(t,\cdot) = X(t,\cdot)$ on $\gamma(t)$!

First step, sequel

- As $\gamma(t)$ and X on $\gamma(t)$ are analytic, we can extend the solution u across the boundary $\gamma(t)$ (this is a “classical” analyticity result for the Stokes equation).
- Using the continuity in time of X and γ with values in C^ω, we see that the size of the neighborhood of $\gamma(t)$ where this solution can be extended can be estimated from below.
- With the Runge-Walsh-type theorem we can obtain approximations defined on $\overline{\Omega}$, and which satisfy

$$u = 0 \text{ on } \partial\Omega \setminus \Sigma.$$

- We obtain the function the final u as :

$$u(t,x) = \sum_{k=1}^{N} \rho_i(t) u(t_i, \cdot),$$

with ρ_i a certain partition of unity of $[0,1]$.

Second step

- Second step : when only the vector field is real analytic : $X \in C^0([0,1]; C^\omega(\overline{\Omega}))$ but $\gamma_0 \in C^\infty(\mathbb{S}^{n-1}; \mathbb{R}^2)$.

- We can approach γ_0 (and γ_1) by real analytic curves, from the outside. This comes from a general result by H. Whitney.
- Next, we apply the process of Part 1 on the ν-approximations γ_0^ν and γ_1^ν of γ_0 and γ_1. We obtain a function u^ν.

- The central point is to show that, on $\Phi^{u^\nu}(t, \gamma_0)$, we have uniform estimates on u^ν as $\nu \to 0^+$.

- This comes from the construction and the fact that the constants in elliptic estimates in $\mathrm{Int}(\gamma^\nu)$ are bounded independently from ν.

- We conclude then by Gronwall's lemma.

Third step

- Third step : when both data are merely C^∞ : $\gamma_0 \in C^\infty(\mathbb{S}^{n-1}; \mathbb{R}^2)$ and $X \in C^0([0,1]; C^\infty(\overline{\Omega}))$.
- We use Whitney's analytic approximation theorem : X can be approached arbitrarily for the $C^0([0,1]; C^\infty(\overline{\Omega}))$-topology by $X^\nu \in C^0([0,1]; C^\omega(\overline{\Omega}))$.
- We conclude by using the previous step and Gronwall's lemma.

Open problems

- Navier-Stokes equations. Can we obtain a similar result for incompressible Navier-Stokes equations ?

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \text{ in } [0, T] \times \Omega, \\ \text{div } u = 0 \text{ in } [0, T] \times \Omega. \end{cases}$$

  With Dirichlet's boundary conditions ? With Navier's (cf. Coron, Chapouly) ?

- This is also open for the evolutionary Stokes equation !

- Stabilization. Can we find a feedback control :

$$\text{control(t)} = f(\gamma(t), u(t)),$$

  stabilizing a fluid zone at a fixed place ?