On the stabilization of the incompressible Navier-Stokes equations in a 2d channel with a normal control

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Outline

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Incompressible Navier-Stokes equations in a 2-d channel:

$$\Omega = \mathbb{T} \times (0,1), \text{ where } \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \text{div } u = 0, & \text{in } (0,\infty) \times \Omega, \\ u(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega. \end{cases}$$

- $u = u(t, x_1, x_2) \in \mathbb{R}^2$ is the velocity.
- $p = p(t, x_1, x_2)$ is the pressure.
- $\nu > 0$ is the viscosity.
- $v = v(t, x_1)$ is the control function, acting on the normal component only.

Choose v to stabilize the state u.

Motivation and related topics

Motivation: Controllability/Stabilization of fluid-structure models with controls acting on the structure.
See Lions Zuazua '95, Osses Puel '99, '09, Lequeurre '13, ...

Related topics:

- Controllability of incompressible Navier-Stokes equations....
  Fursikov Imanuvilov '96, Fernandez-Cara Guerrero Imanuvilov Puel '04, ...
- ... with controls having zero components:
  Coron Guerrero '09, Carreno Guerrero '13, Coron Lissy '15,...
- Coupled parabolic systems with one boundary control:
  Ammar-Khodja Benabdallah Gonzalez-Burgos de Teresa '11, Duprez Lissy '15...
- Stabilization for incompressible Navier-Stokes equations:
  Krstic et al '01, Raymond '06, Barbu '07, Triggiani '07, Vazquez Coron Trélat '08, Munteanu '12,...

To be more precise....

Our goal

Get a local stabilization result around the state $(u, p) = (0, 0)$.

Linearized equations:

$$\begin{cases} \partial_t u - \nu \Delta u + \nabla p = 0, & \text{in } (0, \infty) \times \Omega, \\ \text{div } u = 0, & \text{in } (0, \infty) \times \Omega, \\ u(t, x_1, 0) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

??? $\rightsquigarrow$ *The linearized equations are already stable!* Taking $v = 0$,

$$\frac{d}{dt} \left(\frac{1}{2} \int_\Omega |u(t, x)|^2 \, dx \right) + \nu \int_\Omega |\nabla u(t, x)|^2 \, dx = 0$$

$\rightsquigarrow$ Exponential decay like $t \mapsto \exp(-\nu \pi^2 t)$.

To be more precise.... Part 2

Get a local stabilization result around the state $(u, p) = (0, 0)$
At an exponential rate larger than $\nu\pi^2$.

Linearized equations:

$$\begin{cases} \partial_t u - \nu \Delta u + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \text{div } u = 0, & \text{in } (0,\infty) \times \Omega, \\ u(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

Difficulty:

div $u = 0$ in $(0,\infty) \times \Omega \Rightarrow \int_{\mathbb{T}} v(t, x_1)\, dx_1 = 0$ for all $t > 0$.

To be more precise.... Part 3

The 0-mode:

$$u_0(t, x_2) = \int_{\mathbb{T}} u(t, x_1, x_2)\, dx_1$$

satisfies the uncontrolled heat equation

$$\begin{cases} \partial_t u_{0,1} - \nu \partial_{22} u_{0,1} = 0, & \text{in } (0,\infty) \times (0,1), \\ u_{0,1}(t,0) = u_{0,1}(t,1) = 0, & \text{on } (0,\infty), \\ u_{0,2}(t,x_2) = 0, & \text{in } (0,\infty) \times (0,1). \end{cases}$$

Consequence

The solutions of the linearized equations decay like $\exp(-\nu\pi^2 t)$ and, considering

$$u(t,x) = e^{-\nu\pi^2 t}\Psi_0(x_2) \text{ with } \Psi_0 = \Psi_0(x_2) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin(\pi x_2) \\ 0 \end{pmatrix},$$

this decay estimate is sharp whatever the control v is.

Main result

Theorem (S. Chowdhury, S.E., J.-P. Raymond)

Let $\omega_0 > 0$ be such that $0 < \omega_0 < 4\nu\pi^2$.
There exists $\gamma > 0$ such that for all $u_0 \in \mathbf{V}^1_0(\Omega)$ with $\|u_0\|_{\mathbf{V}^1_0(\Omega)} \leq \gamma$, there exists $v \in L^2((0,\infty)\times\mathbb{T})$ satisfying $\int_{\mathbb{T}} v(t,x_1)\,dx_1 = 0$ for all $t > 0$ such that the solution (u,p) of the incompressible Navier-Stokes equation satisfies, for some constant $C > 0$ independent of t,

$$\forall t \geq 0, \quad \|u(t)\|_{\mathbf{V}^1(\Omega)} \leq Ce^{-\omega_0 t}.$$

$$\mathbf{V}^1(\Omega) = \left\{u = (u_1,u_2) \in H^1(\Omega)\times H^1(\Omega) \,|\, \mathrm{div}\; u = 0\right\},$$
$$\mathbf{V}^1_0(\Omega) = \left\{u \in \mathbf{V}^1(\Omega) \,|\, u(x_1,0) = u(x_1,1) = 0 \text{ for } x_1 \in \mathbb{T}\right\}.$$

Comments

- Straightforward when $\omega < \nu\pi^2$
  $\rightsquigarrow$ Interesting case $\omega \in (\nu\pi^2, 4\nu\pi^2)$.
- $4\nu\pi^2$ is the second eigenvalue of the elliptic operator generating the heat equation satisfied by the 0-mode:

$$\begin{cases} \partial_t u_{0,1} - \nu \partial_{22} u_{0,1} = 0, & \text{in } (0,\infty) \times (0,1), \\ u_{0,1}(t,0) = u_{0,1}(t,1) = 0, & \text{on } (0,\infty), \\ u_{0,2}(t,x_2) = 0, & \text{in } (0,\infty) \times (0,1). \end{cases}$$

- The stabilization result cannot be true for the linearized model
  $\Rightarrow$ We have to use the non-linearity to improve the exponential decay.
  Strategy based on the so-called Power Series Expansion:
  see Coron Crépeau '04, Cerpa '07, Cerpa Crépeau '09, Coron Rivas '15.

Outline

Strategy Part 1

Write $u = \varepsilon\alpha + \varepsilon^2\beta$, $v = \varepsilon v_1 + \varepsilon^2 v_2$, with

$$\begin{cases} \partial_t\alpha - \nu\Delta\alpha + \nabla p_1 = 0, & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div}\,\alpha = 0, & \text{in } (0,\infty)\times\Omega, \\ \alpha(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(t,x_1,1) = (0, v_1(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(0,x_1,x_2) = \alpha^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t\beta - \nu\Delta\beta + \nabla p_2 = -(\alpha+\varepsilon\beta)\cdot\nabla(\alpha+\varepsilon\beta), & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div}\,\beta = 0, & \text{in } (0,\infty)\times\Omega, \\ \beta(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(t,x_1,1) = (0, v_2(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(0,x_1,x_2) = \beta^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

Strategy Part 2

- α satisfies the linearized incompressible Navier-Stokes equations.

$\Rightarrow$ If α contains 0-modes decaying slower than $\exp(-\omega_0 t)$, one cannot achieve an exponential decay rate ω_0.

$\Rightarrow$ The component of the solution u on the eigenfunction

$$\Psi_0 = \Psi_0(x_2) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin(\pi x_2) \\ 0 \end{pmatrix}$$

- Is in β.
- Should be handled by constructing a suitable α.

Preliminaries

- The Stokes operator A is self-adjoint, positive definite, with compact resolvent on the space $\mathbf{V}_n^0(\Omega) = \left\{ u \in (L^2(\Omega))^2 \mid \text{div}\,(u) = 0 \text{ on } \Omega \text{ and } u \cdot n = 0 \text{ on } \Gamma \right\}$
$\Rightarrow$ Sequences of positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and corresponding orthonormal basis of eigenvectors (Ψ_j).

$$A\Psi = \lambda\Psi \Leftrightarrow \begin{cases} -\nu\Delta\Psi + \nabla q = \lambda\Psi, & \text{in } \Omega, \\ \text{div } \Psi = 0, & \text{in } \Omega, \\ \Psi = 0, & \text{on } \Gamma, \end{cases}$$

Adjoint of the control operator: $B^*\Psi = q(x_1, 1) - \frac{1}{2\pi} \int_{\mathbb{T}} q(x_1, 1) dx_1.$

Lemma

$A\Psi = \lambda\Psi$ and $B^*\Psi = 0$ imply $\Psi(x) = \Psi(x_2)$.

Decomposition of the space $\mathbf{V}_n^0(\Omega)$:

- A stable space: $\mathbf{Z}_s = \text{Span}\,\{\Phi \,|\, A\Phi = \lambda\Phi, \text{ with } \lambda > \omega\}$.
- An unstable space: $\mathbf{Z}_u = \mathbf{Z}_s^\perp$, itself decomposed as
  - An unstable uncontrollable space $\mathbf{Z}_{uu} = \text{Span}\,\Psi_0$.
  - An unstable detectable space $\mathbf{Z}_{ud} = \mathbf{Z}_{uu}^\perp \cap \mathbf{Z}_u$.

Corresponding (orthogonal) projections: $\mathbb{P}_s$, $\mathbb{P}_u$, $\mathbb{P}_{ud}$ and $\mathbb{P}_{uu}$.

Strategy

Iterative strategy: $(0, \infty) = \cup_{n \in \mathbb{N}} [nT, (n+1)T]$ for some $T > 0$. $(T = 1)$.

Starting point: $u^0 = \varepsilon \alpha^0 + \varepsilon^2 \beta^0$ with

$$\|\alpha^0\|^2_{\mathbf{V}^1_0(\Omega)} + \|\beta^0\|_{\mathbf{V}^1_0(\Omega)} \leq 1, \quad \text{and} \quad \mathbb{P}_{uu}\alpha^0 = 0.$$

Initialization Step: On $[0, T]$, choose

- v_1 such that $\mathbb{P}_u \alpha(T) = 0$.
- $v_2 = 0$.

Strategy

Part 4

Iteration step: In each time interval $[nT, (n+1)T]$, we design controls v_1 and v_2 such that

$$\mathbb{P}_u \alpha((n+1)T) = 0, \quad \text{and} \quad \mathbb{P}_u \beta^*((n+1)T) = 0,$$

where β^* is the solution of

$$\begin{cases} \partial_t \beta^* - \nu \Delta \beta^* + \nabla p^* = -\alpha \cdot \nabla \alpha, & \text{in } (nT, (n+1)T) \times \Omega, \\ \text{div } \beta^* = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \beta^*(t, x_1, 0) = (0, 0), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \beta^*(t, x_1, 1) = (0, v_2(t, x_1)), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \beta^*(nT^+, x) = \beta(nT^-, x), & \text{in } \Omega. \end{cases}$$

Remark: $\beta^* \neq \beta$, but $\|\beta - \beta^*\| \lessapprox \varepsilon$

Lemma for the initialization step

Given $\alpha^0 \in \mathbf{V}^1_0(\Omega)$ and $\alpha^f \in \mathbf{Z}_{ud}$, there exists a control function

$$v_1 \in H^1_0(0,T;L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T})) \text{ with } \int_{\mathbb{T}} v_1(t,x_1)\,dx_1 = 0$$

such that the solution α of

$$\begin{cases} \partial_t \alpha - \nu\Delta\alpha + \nabla p_1 = 0, & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div}\alpha = 0, & \text{in } (0,\infty)\times\Omega, \\ \alpha(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(t,x_1,1) = (0, v_1(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(0,x_1,x_2) = \alpha^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{ud}\alpha(T) = \alpha^f$.
The projection $\mathbb{P}_{uu}\alpha(T)$ cannot be controlled, but satisfies $\mathbb{P}_{uu}\alpha(T) = \exp(-\nu\pi^2 T)\mathbb{P}_{uu}\alpha^0$.

Key Lemma for the iteration process

Let $\tilde\beta^0 \in \mathbf{Z}_{uu}$ and $f \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{V}^0(\Omega))$. Then there exists a control function $\tilde v_1 \in H^1_0(0,T;H^2(\mathbb{T}) \cap L^2_0(\mathbb{T}))$ such that the solution $\tilde\alpha$ of

$$\begin{cases} \partial_t \tilde\alpha - \nu\Delta\tilde\alpha + \nabla \tilde p_1 = 0, & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\,\tilde\alpha = 0, & \text{in } (0,T)\times\Omega, \\ \tilde\alpha(t,x_1,0) = (0,0), & \text{on } (0,T)\times\mathbb{T}, \\ \tilde\alpha(t,x_1,1) = (0,\tilde v_1(t,x_1)), & \text{on } (0,T)\times\mathbb{T}, \\ \tilde\alpha(0,x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\tilde\alpha(T) = 0$ in Ω, and such that the solution $\tilde\beta$ of

$$\begin{cases} \partial_t \tilde\beta - \nu\Delta\tilde\beta + \nabla \tilde p_2 = -(f+\tilde\alpha)\cdot\nabla(f+\tilde\alpha), & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\,\tilde\beta = 0, & \text{in } (0,T)\times\Omega, \\ \tilde\beta(t,x_1,0) = \tilde\beta(t,x_1,1) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \tilde\beta(0,x_1,x_2) = \tilde\beta^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{uu}\tilde\beta(T) = 0$.

$\rightsquigarrow$ These lemmata come with estimates which are needed to conclude. For instance, we get

$$\|\tilde{\alpha}\|^2_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} + \left\|\tilde{\beta}\right\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))}$$
$$\leq C\left(\|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{V}^0(\Omega))} + \left\|\tilde{\beta}^0\right\|_{\mathbf{V}^1_0(\Omega)}\right).$$

After tedious estimates, at each step,

$$\|\alpha((n+1)T)\|_{\mathbf{V}^1_0(\Omega)} \leq e^{-\omega T}\,\|\alpha(nT)\|_{\mathbf{V}^1_0(\Omega)},$$

$$\begin{pmatrix}\|\mathbb{P}_s\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)}\\ \|\mathbb{P}_u\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)}\end{pmatrix} \leq K_\varepsilon \begin{pmatrix}\|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}^1_0(\Omega)}\\ \|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}^1_0(\Omega)}\end{pmatrix} + \|\alpha(nT)\|^2_{\mathbf{V}^1_0(\Omega)}\begin{pmatrix}C\\ C\varepsilon\end{pmatrix}$$

$$\text{where } K_\varepsilon = \begin{pmatrix} e^{-\omega T} + C\sqrt{\varepsilon} & C\\ C\sqrt{\varepsilon} & C\sqrt{\varepsilon}\end{pmatrix}.$$

$\Rightarrow$ Decay as $\exp(-\omega_0 t)$.

Outline

Key Lemma

Let $\tilde{\beta}^0 \in \mathbf{Z}_{uu}$ and $f \in L^2(0, T; \mathbf{H}^2(\Omega)) \cap H^1(0, T; \mathbf{V}^0(\Omega))$. Then there exists a control function $\tilde{v}_1 \in H^1_0(0, T; H^2(\mathbb{T}) \cap L^2_0(\mathbb{T}))$ such that the solution $\tilde{\alpha}$ of

$$\begin{cases} \partial_t \tilde{\alpha} - \nu \Delta \tilde{\alpha} + \nabla \tilde{p}_1 = 0, & \text{in } (0, T) \times \Omega, \\ \text{div } \tilde{\alpha} = 0, & \text{in } (0, T) \times \Omega, \\ \tilde{\alpha}(t, x_1, 0) = (0, 0), & \text{on } (0, T) \times \mathbb{T}, \\ \tilde{\alpha}(t, x_1, 1) = (0, \tilde{v}_1(t, x_1)), & \text{on } (0, T) \times \mathbb{T}, \\ \tilde{\alpha}(0, x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\tilde{\alpha}(T) = 0$ in Ω, and such that the solution $\tilde{\beta}$ of

$$\begin{cases} \partial_t \tilde{\beta} - \nu \Delta \tilde{\beta} + \nabla \tilde{p}_2 = -(f + \tilde{\alpha}) \cdot \nabla (f + \tilde{\alpha}), & \text{in } (0, T) \times \Omega, \\ \text{div } \tilde{\beta} = 0, & \text{in } (0, T) \times \Omega, \\ \tilde{\beta}(t, x_1, 0) = \tilde{\beta}(t, x_1, 1) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\ \tilde{\beta}(0, x_1, x_2) = \tilde{\beta}^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{uu} \tilde{\beta}(T) = 0$.

A slightly stronger Result

There exist control functions $v^a, v^b \in H^1_0(0,T;H^2(\mathbb{T})\cap L^2_0(\mathbb{T}))$ such that, for all $a, b \in \mathbb{R}$, the solution α of

$$\begin{cases} \partial_t \alpha - \nu\Delta\alpha + \nabla p_1 = 0, & \text{in } (0,T)\times\Omega, \\ \text{div } \alpha = 0, & \text{in } (0,T)\times\Omega, \\ \alpha(t,x_1,0) = (0,0), & \text{on } (0,T)\times\mathbb{T}, \\ \alpha(t,x_1,1) = (0,(av^a + bv^b)(t,x_1)), & \text{on } (0,T)\times\mathbb{T}, \\ \alpha(0,x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\alpha(T) = 0$ in Ω, and such that the solution β of

$$\begin{cases} \partial_t \beta - \nu\Delta\beta + \nabla p_2 = -\alpha\cdot\nabla\alpha, & \text{in } (0,T)\times\Omega, \\ \text{div } \beta = 0, & \text{in } (0,T)\times\Omega, \\ \beta(t,x_1,0) = \beta(t,x_1,1) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(0,x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{uu}\beta(T) = ab\,\Psi_0$.

Ideas of the proof

- Take

$$v^a(t,x_1) = v^c(t)\cos(x_1), \quad v^b(t,x_1) = v^s(t)\sin(x_1).$$

$\rightsquigarrow$ α^a and α^b are supported on the first mode of the equations:

$$\alpha^a(t,x_1,x_2) = \begin{pmatrix} \sin(x_1)\alpha_1^c(t,x_2) \\ \cos(x_1)\alpha_2^c(t,x_2) \end{pmatrix}$$

- The equation satisfied by the first modes of the linear incompressible Stokes equations is null-controllable.
$\rightsquigarrow$ Proof by spectral estimates and Müntz Theorem.

- Writing $\langle \beta(T), \Psi_0 \rangle$ in a suitable way.

$$e^{\nu\pi^2 T}\langle \beta(T), \Psi_0\rangle = \pi^{5/2}\int_0^T v^s(t) q(t,1)\, dt,$$

where q is obtained by solving

$$\begin{cases} -\partial_t Z + \nu Z - \nu \partial_{22} Z + \begin{pmatrix} q \\ \partial_2 q \end{pmatrix} = F(t, x_2), & \text{in } (0,T)\times(0,1), \\ -Z_1 + \partial_2 Z_2 = 0, & \text{in } (0,T)\times(0,1), \\ Z(t,0) = Z(t,1) = (0,0), & \text{in } (0,T), \\ Z(T, x_2) = 0, & \text{in } (0,1). \end{cases}$$

with $F(t,x_2) = \cos(\pi x_2) e^{\nu\pi^2 t}\begin{pmatrix} \alpha_2^c(t,x_2) \\ \alpha_1^s(t,x_2)\end{pmatrix}$, depending only on $v^a(t,x_1) = v^c(t)\cos(x_1)$.

$\rightsquigarrow$ Show the existence of v^a/v^c such that $\|q(t,1)\|_{L^2(0,T)} \neq 0$.

Construction of v^c, generation of a suitable trajectory

For $\mu \in \mathbb{R}$, introduce $(\alpha^*(x_2), p^*(x_2))$ solving

$$\begin{cases} \mu\alpha_1^* + \nu\alpha_1^* - \nu\partial_{22}\alpha_1^* - p^* = 0, & \text{in } (0,1), \\ \mu\alpha_2^* + \nu\alpha_2^* - \nu\partial_{22}\alpha_2^* + \partial_2 p^* = 0, & \text{in } (0,1), \\ \alpha_1^* + \partial_2\alpha_2^* = 0, & \text{in } (0,1), \\ \alpha_1^*(0) = \alpha_1^*(1) = \alpha_2^*(0) = 0, \quad \alpha_2^*(1) = 1. \end{cases}$$

Then $\overline{\alpha}(t, x_1, x_2) = e^{\mu t}(\sin(x_1)\alpha_1^*(x_2), \cos(x_1)\alpha_2^*(x_2))$, $\overline{v}(t) = e^{\mu t}$, solves the linear Stokes equations.

Lemma

There exists a suitable $\mu \in \mathbb{R}$ such that if $\alpha(t) = \overline{\alpha}(t)$ on some time interval then the boundary pressure $q(t,1)$ given by the aforementioned process cannot be identically 0 on that time interval.

Reduction to the stationary case and numerically checked.

Construction of v^a/v^c and v^b/v^s

Construction of v^a/v^c in 4 steps:

- On $(0, T/4)$, control α^a to go from 0 to $\overline{\alpha}(T/4)$.
- On $(T/4, T/2)$, take $v^a(t) = e^{\mu t}$ and $\alpha^a(t) = \overline{\alpha}(t)$.
- On $(T/2, 3T/4)$, control α^a goes from $\overline{\alpha}(T/2)$ to 0.
- On $(3T/4, T)$, take $v^a(t) = 0$, and $\alpha^a(t) = 0$, hence $q(t) = 0$.

Construction of v^b/v^s:

- On $(0, 3T/4)$, take v^s such that $\int_0^{3T/4} v^s(t) q(t, 1)\, dt = 1$.
- On $(3T/4, T)$, control α^b to go from $\alpha^b(3T/4)$ to 0.

Outline

Open Question

- Exponential stabilization result at a rate higher than $4\nu\pi^2$?

Difficulty: One has to guarantee that we can enter the space of missing directions

$$\text{Span}\left\{ \begin{pmatrix} \sin(\pi x_2) \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(2\pi x_2) \\ 0 \end{pmatrix} \right\}$$

in both directions independently.

Exponential stabilization at any given rate is open, so is the controllability of the system.

Thank you for your attention!

Comments Welcome

Reference:

Open loop stabilization of incompressible Navier-Stokes equations in a 2d channel using power series expansion.
S. Chowdhury, S. Ervedoza, and J.-P. Raymond, in preparation.