On hierarchic control for coupled parabolic equations

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CIRM
November 2015

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- The leader (first player) does a movement. The follower (second player) reacts trying to win or optimize the response to the leader movement.
- Historical papers due to J. von Neumann and O. Morgenstern (1943) and Nash (1950). [What is now known as Nash equilibria is due to Cournot (1838)].

ON THE THEORY OF GAMES OF STRATEGY[1]

John von Neumann

[A translation by Mrs. Sonya Bargmann of "Zur Theorie der Gesellschaftsspiele," Mathematische Annalen 100 (1928), pp. 295-320.]

INTRODUCTION

1. The present paper is concerned with the following question:

n players $S_1, S_2, \ldots, S_n$ are playing a given game of strategy, $\mathfrak{G}$. How must one of the participants, S_m, play in order to achieve a most advantageous result?

The problem is well known, and there is hardly a situation in daily life into which this problem does not enter. Yet, the meaning of this question is not unambiguous. For, as soon as $n > 1$ (i.e., $\mathfrak{G}$ is a game of strategy in the proper sense), the fate of each player depends not only on his own actions but also on those of the others, and their behavior is motivated by the same selfish interests as the behavior of the first player. We feel that the situation is inherently circular.

Hence we must first endeavor to find a clear formulation of the question. What, exactly, is a game of strategy? A great many different things come under this heading, anything from roulette to chess, from baccarat to bridge. And after all, any event - given the external conditions and the participants in the situation (provided the latter are acting of their own free will) - may be regarded as a game of strategy if one looks at the effect it has on the participants.[2] What element do all these things have in common?

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, **8**, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

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- Two players (controls) one leader, one follower. Called Stackelberg strategy.
- Several players: Nash equilibria, Pareto strategy etc.......
- We concentrate on Stackelberg strategy.

Hierarchic control for the heat equation

Consider

$$
(1)\qquad \begin{cases} y_t - \Delta y = h\chi_\omega + v\chi_{\omega_1} \quad \text{in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y^0(x) \text{ in } \Omega. \end{cases}
$$

Hierarchic control for the heat equation

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h is the leader

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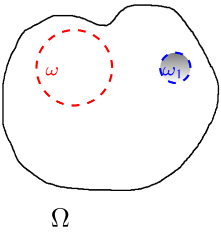
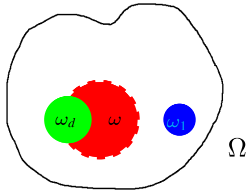


Hierarchic control for the heat equation

We will have the main functional

$$\boxed{J(h) = \frac{1}{2}\iint_{\omega\times(0,T)} |h|^2 dxdt,}$$



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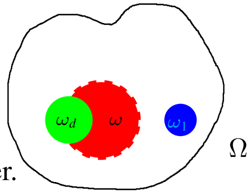

Let $\omega_d \subset \Omega$ be the observation domain of the follower.
Define the *secondary* functional

$$\boxed{J_1(h,v) = \frac{\alpha}{2}\iint_{\omega_d\times(0,T)} |y-y_d|^2 dxdt + \frac{\mu}{2}\iint_{\omega_1\times(0,T)} |v|^2 dxdt,}$$

Hierarchic control

The follower $\bar{v}$ assumes that the leader h has made a choice and intend to be a minimizer for the cost J_1

Main control

Identifying $\overline{v}$ we look for an optimal control $\hat{h}$ such that

$$J\left(\widehat{h}\right) = \min_{h} J\left(h, \overline{v}(h)\right)$$

subject to the restrictions:

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$$y(\cdot, T; h, \overline{v}(h)) \in y^T + \epsilon B.$$

Or null control

$$y(\cdot, T; h, \overline{v}(h)) = 0$$

- Also we may ask controllability to trajectories.
- and two or more followers (Nash equilibrium)

Characterization of the follower: $\mu >> 1$

Characterization of the follower
$(J_1(h,v) = \frac{\alpha}{2} \iint_{\omega_{d\times(0,T)}} |y - y_d|^2 dxdt + \frac{\mu}{2} \iint_{\omega_1\times(0,T)} |v|^2 dxdt)$

Given h, $\bar{v}$ is characterized by

$$\boxed{\bar{v} = -\frac{1}{\mu} p \chi_{\omega_1}}$$

with

$$(1) \qquad \begin{cases} y_t - \Delta y = h\chi_\omega - \frac{1}{\mu} p\chi_{\omega_1} & \text{in } Q, \\ -p_t - \Delta p = \alpha\,(y - y_d)\,\chi_{\omega_d} & \text{in } Q, \\ y(0) = y^0,\ p(T) = 0, \quad y = p = 0 \text{ on } \Sigma. \end{cases}$$

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In fact, if $\bar{v}$ is a minimizer, then we have the Euler equation

$$J_1'(h,\bar{v})\hat{v} = 0$$

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$$\text{(E)} \qquad \iint\limits_{\omega_{d\times(0,T)}} (\bar{y} - y_d)\hat{y}\,dxdt + \mu \iint\limits_{\omega_1\times(0,T)} \bar{v}\hat{v}\,dxdt = 0$$

with

$$\begin{cases} \hat{y}_t - \Delta\hat{y} = \hat{v}\chi_{\omega_1} \quad \text{in } Q, \\ \hat{y}(0) = 0, \qquad \hat{y} = 0 \text{ on } \Sigma. \end{cases}$$

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So we recover (E) if we put

$$\begin{cases} -p_t - \Delta p = (\bar{y}-y_d)\chi_{\omega_d} & \text{in } Q, \\ \hat{p}(T) = 0, \text{ in } \Omega, \quad p = 0 \text{ on } \Sigma. \end{cases}$$

and multiply it by $\hat{y}$ so we get $\bar{v} = -\frac{1}{\mu}p\chi_{\omega_1}$

Characterization of the follower: $\mu >> 1$

$$\begin{cases} y_t - \Delta y = h\chi_\omega - \frac{1}{\mu} p\chi_{\omega_1} \quad \text{in } Q, \\ -p_t - \Delta p = \alpha\left(y - y_d\right)\chi_{\omega_d} \quad \text{in } Q, \\ y(0) = y^0,\ p(T) = 0, \quad y = p = 0 \text{ on } \Sigma. \end{cases}$$

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y(0) = y^0, \; p(T) = 0, \quad y = p = 0 \text{ on } \Sigma.
\end{array}
\right.
$$

We get the system of two coupled equations!

Lions result: Stackelberg's equilibrium, approximate control

J. L. Lions proves the existence of an approximate Stackelberg equilibrium. That is, he proves the existence of a control $\hat{h}$ such that the corresponding solution to

$$(1) \qquad \begin{cases} y_t - \Delta y = \widehat{h}\chi_\omega - \frac{1}{\mu} p \chi_{\omega_1} & \text{in } Q, \\ -p_t - \Delta p = \alpha\left(y - y_d\right)\chi_{\omega_d} & \text{in } Q, \\ y(0) = y^0, \; p(T) = 0, \quad y = p = 0 \text{ on } \Sigma, \end{cases}$$

satisfies

$$y(T) \in y^T + \alpha B.$$

and

$$J\left(\widehat{h}\right) = \min_h J(h, p)$$

$$\omega \cap \omega_1 = \emptyset.$$

Stackelberg equilibrium: null controllability

Recall that we want

$$J_1(h,v) = \frac{\alpha}{2} \iint\limits_{\omega_{d \times (0,T)}} |y - y_d|^2 dxdt + \frac{\mu}{2} \iint\limits_{\omega_1 \times (0,T)} |v|^2 dxdt,$$

Araruna, Fernández-Cara, Santos proved for μ large enough, $\omega_d \cap \omega \neq \emptyset$ and for

$$\iint\limits_{\omega_{d \times (0,T)}} \hat{\rho}^2(t)|y_d|^2 < \infty \qquad \boxed{\text{Compatibility condition}}$$

for a (particular) $\hat{\rho}(t)$ blowing up at $t = T$. (Related to Carleman inequality).

There exists h such that the corresponding solution to

$$\begin{cases} y_t - \Delta y = h\chi_\omega - \frac{1}{\mu} p\chi_{\omega_1} & \text{in } Q, \\ -p_t - \Delta p = \alpha\left(y - y_d\right)\chi_{\omega_d} & \text{in } Q, \\ y(0) = y^0,\ p(T) = 0, \quad y = p = 0 \text{ on } \Sigma. \end{cases}$$

satisfies

$$y(T) = 0.$$

That is the pair (v, h) solve the Stackelberg problem:
$v = -\frac{1}{\mu} p$ and h minimizes

$$J\left(h, v(h)\right)$$

subject to $y(T; h, v) = 0$.

Insensitizing controls

System is close to the system that appears in insensitizing controls:

$$\begin{cases} y_t - \Delta y = h\chi_\omega + y_d \quad \text{in } Q, \\ -p_t - \Delta p = y\chi_{\omega_d} \quad \text{in } Q, \\ y(0) = y^0,\ p(T) = 0, \quad y = p = 0 \text{ on } \Sigma. \end{cases}$$

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- Huge difference: In the first equation we have $\frac{1}{\mu} p \chi_{\omega_1}$
- More complicated coupling but the big parameter μ allows to prove null controllability.

Hierarchic control for systems

What can we say of hierarchic control for systems?
Let us consider a "simple" case $a_{ij}(x,t) \in L^\infty(Q)$:

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h\chi_\omega + v\chi_{\omega_1} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = 0 \quad \text{in } Q, \\ y_j(x,0) = y_j^0(x) \text{ in } \Omega, \quad y_j = 0 \ \text{ on } \Sigma, \ j = 1,2, \end{cases}$$

where for an open set $\omega_0 \subset \omega$ the following holds

$$a_{21} \geq a_0 > 0 \quad \text{or} \quad -a_{21} \geq a_0 > 0 \text{ in} \quad \omega_0 \times (0,T).$$

In this situation we know (González-Burgos, deT [2010]) that this system is null controllable.

What about hierarchic control?

Hierarchic control for systems

Main functional: Minimize

$$J(h) = \frac{1}{2} \iint_{\omega \times (0,T)} |h|^2 dxdt,$$

subject to

$$\boxed{y(\cdot, T; h, v) = 0 \quad \text{in } \Omega.}$$

Secondary functional:

$$\begin{aligned} J_1(h,v) = {} & \frac{\alpha}{2} \iint_{\omega_{d \times (0,T)}} |y_1 - y_{1,d}|^2 + |y_2 - y_{2,d}|^2 dxdt \\ & + \frac{\mu}{2} \iint_{\omega \times (0,T)} |v|^2 dxdt, \end{aligned}$$

Characterization of the follower $\mu >> 1$

$$
\begin{cases}
y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h\chi_\omega - \frac{1}{\mu} p_1 \chi_{\omega_1} & \text{in } Q, \\
y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad \text{in } Q, \\
-p_{1,t} - \Delta p_1 + a_{11} p_1 + a_{21} p_2 = \alpha \left(y_1 - y_{1,d} \right) \chi_{\omega_d} & \text{in } Q, \\
-p_{2,t} - \Delta p_2 + a_{12} p_1 + a_{22} p_2 = \alpha \left(y_2 - y_{2,d} \right) \chi_{\omega_d} & \text{in } Q, \\
y_j(0) = y_j^0,\ p_j(T) = 0, \quad y_j = p_j = 0 \text{ on } \Sigma,\ j = 1, 2.
\end{cases}
$$

So the leader has to control 2 variables of a system of 4 equations!

Characterization by means of the adjoint system

$$
\begin{cases}
-\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \alpha\theta_1\chi_{\omega_d} \quad \text{in } Q, \\
-\varphi_{2,t} - \Delta\varphi_2 + {\color{red}a_{12}}\varphi_1 + a_{22}\varphi_2 = \alpha\theta_2\chi_{\omega_d} \quad \text{in } Q, \\
\theta_{1,t} - \Delta\theta_1 + a_{11}\theta_1 + {\color{red}a_{12}}\theta_2 = -\frac{1}{\mu}\varphi_1\chi_{\omega} \quad \text{in } Q, \\
\theta_{2,t} - \Delta\theta_2 + a_{21}\theta_1 + a_{22}\theta_2 = 0 \quad \text{in } Q, \\
\varphi_j(T) = f_j,\ \theta_j(0) = 0 \text{ in } \Omega, \quad \varphi_i = \theta_j = 0 \text{ on } \Sigma,\ j = 1, 2,
\end{cases}
$$

Carleman inequality $a_{12} = 0$

Remember Poster Session Victor Hernández-Santamaría proved the following Carleman inequality when $\omega \cap \omega_d \neq \emptyset$

$$\boxed{a_{12} = 0}$$

$$I(d_1,\varphi_1) + I(d_2,\varphi_2) + I(m_1,\theta_1) + I(m_2,\theta_2) \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\beta}(s\gamma)^{2m_2-d_2+12}|\varphi_1|^2 dxdt$$

$$I(m,z) := \iint_Q e^{-2s\beta}(s\gamma)^{m-2}|\nabla z|^2 dxdt + \iint_Q e^{-2s\beta}(s\gamma)^{m}|z|^2 dxdt$$

$$\beta_0(x) = e^{2C^*\|\eta^0\|_\infty} - e^{C^*\eta^0(x)}, \quad \beta(x,t) = \frac{\beta_0(x)}{t(T-t)} \quad \gamma(t) := \frac{1}{t(T-t)}$$

Observability inequality

Introduce:

$$l(t) = \begin{cases} T^2/4 & \text{for} \quad 0 \le t \le T/2, \\ t(T-t) & \text{for} \quad T/2 \le t \le T, \end{cases}$$

and the following associated weight functions

$$\tilde{\alpha}(x,t) = \frac{\beta_0(x)}{l(t)}, \quad \tilde{\gamma}(t) = \frac{1}{l(t)}, \quad \tilde{\alpha}^*(t) = \max_{\tilde{\Omega}} \tilde{\alpha}(x,t), \quad \tilde{\rho}(t) := e^{\tilde{s}\tilde{\alpha}^*}.$$

$$\int_\Omega |\varphi_1(0)|^2 dx + \int_\Omega |\varphi_2(0)|^2 dx + \iint_Q \tilde{\rho}^{-2}|\theta_1|^2 dxdt + \iint_Q \tilde{\rho}^{-2}|\theta_2|^2 dxdt$$
$$+ \iint_Q \tilde{\rho}^{-2}|\varphi_1|^2 dxdt + \iint_Q \tilde{\rho}^{-2}|\varphi_2|^2 dxdt \le C \iint_{\omega\times(0,T)} |\varphi_1|^2 dxdt$$

Key points: $\theta_i(x,0) = 0$ and $\mu >> 1$.

Controllability result

That means that there exists a control $h \in L^2((0,T)\times\omega)$ such that the solution to

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + = h\chi_\omega - \frac{1}{\mu}p_1\chi_{\omega_1} \quad \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = 0 \quad \text{in } Q, \\ -p_{1,t} - \Delta p_1 + a_{11}p_1 + a_{21}p_2 = \alpha\left(y_1 - y_{1,d}\right)\chi_{\omega_d} \quad \text{in } Q, \\ -p_{2,t} - \Delta p_2 + a_{22}p_2 = \alpha\left(y_2 - y_{2,d}\right)\chi_{\omega_d} \quad \text{in } Q, \\ y_j(0) = y_j^0,\ p_j(T) = 0, \quad y_j = p_j = 0 \text{ on } \Sigma,\ j = 1,2. \end{cases}$$

satisfies

$$\boxed{y_1(T) = y_2(T) = 0}$$

as soon as

$$\int_0^T\int_{\omega_d} y_{i,d}^2\tilde{\rho}^2 dxdt < \infty,\ i = 1,2 \qquad \boxed{\text{Compatibility condition!}}$$

Theorem

Suppose $\boxed{a_{12} = 0,}$ a_{21} *satisfies "coupling" condition,* $\omega \cap \omega_d \neq \emptyset$, $\mu >> 1$ *and* $y_{i,d}$ *satisfy the compatibility condition, then there exists a Stackelberg* (h, v) *optimal strategy for* $J(h)$ *subject to* $y_i(\cdot, T; h, v) = 0 \quad in\ \Omega$ *and* $J_1(h, v)$ *and* (y_1, y_2) *solution to*

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 = h\chi_\omega + v\chi_{\omega_1} \quad in\ Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad in\ Q, \\ y_j(x, 0) = y_j^0(x)\ in\ \Omega, \quad y_j = 0\ \ on\ \Sigma,\ j = 1, 2, \end{cases}$$

$$J(h) = \frac{1}{2} \iint_{\omega \times (0,T)} |h|^2 dxdt,$$

$$J_1(h, v) = \frac{\alpha}{2} \iint_{\omega_{d \times (0,T)}} |y_1 - y_{1,d}|^2 + |y_2 - y_{2,d}|^2 dxdt$$
$$+ \frac{\mu}{2} \iint_{\omega \times (0,T)} |v|^2 dxdt,$$

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Furthermore, there exists a constant $C > 0$ such that

$$\|h\|^2_{L^2} + \|v\|^2_{L^2} \leq C(\|y^0\|^2_{L^2} + \sum \int_0^T \int_{\omega_d} y^2_{i,d} \tilde{\rho}^2 dxdt)$$

Series solution

- What can we do if $a_{12} \neq 0$?

Series solution

- What can we do if $a_{12} \neq 0$?
- Following Castro-deT, we propose

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + \varepsilon a_{12} y_2 = h\chi_\omega - \frac{1}{\mu} p_1 \chi_{\omega_1} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad \text{in } Q, \\ -p_{1,t} - \Delta p_1 + a_{11} p_1 + a_{21} p_2 = \alpha \left(y_1 - y_{1,d} \right) \chi_{\omega_d} & \text{in } Q, \\ -p_{2,t} - \Delta p_2 + \varepsilon a_{12} p_1 + a_{22} p_2 = \alpha \left(y_2 - y_{2,d} \right) \chi_{\omega_d} & \text{in } Q, \\ y_j(0) = y_j^0, \ p_j(T) = 0, \quad y_j = p_j = 0 \text{ on } \Sigma, \ j = 1, 2. \end{cases}$$

Series solution

- What can we do if $a_{12} \neq 0$?
- Following Castro-deT, we propose

$$
\begin{cases}
y_{1,t} - \Delta y_1 + a_{11} y_1 + \varepsilon a_{12} y_2 = h\chi_\omega - \frac{1}{\mu} p_1 \chi_{\omega_1} & \text{in } Q, \\
y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = 0 \quad \text{in } Q, \\
-p_{1,t} - \Delta p_1 + a_{11} p_1 + a_{21} p_2 = \alpha \left(y_1 - y_{1,d} \right) \chi_{\omega_d} & \text{in } Q, \\
-p_{2,t} - \Delta p_2 + \varepsilon a_{12} p_1 + a_{22} p_2 = \alpha \left(y_2 - y_{2,d} \right) \chi_{\omega_d} & \text{in } Q, \\
y_j(0) = y_j^0, \; p_j(T) = 0, \quad y_j = p_j = 0 \text{ on } \Sigma, \; j = 1, 2.
\end{cases}
$$

- Suppose that

$$
y_i = \sum_k \varepsilon^k y_i^k; \quad p_i = \sum \varepsilon^k p_i^k; \quad h = \sum_k \varepsilon^k h^k.
$$

Controlled family $k = 0$

$$\begin{cases} y^0_{1,t} - \Delta y^0_1 + a_{11} y^0_1 = h^0 \chi_\omega - \frac{1}{\mu} p^0_1 \chi_{\omega_1} & \text{in } Q, \\ y^0_{2,t} - \Delta y^0_2 + a_{21} y^0_1 + a_{22} y^0_2 = 0 & \text{in } Q, \\ -p^0_{1,t} - \Delta p^0_1 + a_{11} p^0_1 + a^0_{21} p_2 = \alpha \left(y^0_1 - y_{1,d} \right) \chi_{\omega_d} & \text{in } Q, \\ -p^0_{2,t} - \Delta p^0_2 + a_{22} p^0_2 = \alpha \left(y^0_2 - y_{2,d} \right) \chi_{\omega_d} & \text{in } Q, \\ y^0_j(0) = y_{0,j},\ p^0_j(T) = 0, \quad y^0_j = p^0_j = 0 \text{ on } \Sigma,\ j = 1, 2. \end{cases}$$

k=0

Under previous assumptions on ω, ω_d, y_i, d and μ, There exists $h^0 \in L^2(\omega \times (0,T))$ such that $y^0_1(T) = y^0_2(T) = 0$ with

$$\|y^0_1\|^2 + \|y^0_2\|^2 + \|p^0_1\|^2 + \|p^0_2\|^2 + \|h^0\|^2 \leq C\Big(\sum_i \|y_{0,i}\|^2 + \int_0^T \int_{\omega_d} y^2_{i,d} \tilde{\rho}^2 \, dxdt\Big).$$

Controlled family $k > 0$

$$\begin{cases} y^k_{1,t} - \Delta y^k_1 + a_{11} y^k_1 + a_{12} y^{k-1}_2 = h^k \chi_\omega - \frac{1}{\mu} p^k_1 \chi_{\omega_1} \quad \text{in } Q, \\ y^k_{2,t} - \Delta y^k_2 + a_{21} y^k_1 + a_{22} y^k_2 = 0 \quad \text{in } Q, \\ -p^k_{1,t} - \Delta p^k_1 + a_{11} p^k_1 + a^k_{21} p^k_2 = \alpha y^k_1 \quad \text{in } Q, \\ -p^k_{2,t} - \Delta p^k_2 + a_{22} p^k_2 = \alpha \left(y^k_2 \chi_{\omega_d} - \frac{1}{\alpha} a_{12} p^{k-1}_1 \right) \quad \text{in } Q, \\ y^k_j(0) = 0,\ p^k_j(T) = 0, \quad y^k_j = p^k_j = 0 \text{ on } \Sigma,\ j = 1, 2. \end{cases}$$

Question

Is it possible to construct h^k such that $y^k_j(T) = 0$ and such that for $C > 0$ independent of k we get

$$\|y^k_1\|^2 + \|y^k_2\|^2 + \|p^k_1\|^2 + \|p^k_2\|^2 \le C \|h^k\|^2?$$

Furthermore, can $\|h^k\|^2$ be uniformly bounded?

Theorem

Under previous assumptions: Suppose that $a_{12} \in L^\infty(Q)$ *is such that*

$$\int_Q \tilde{\rho}^2 a_{12}^2(x,t)dxdt < \infty.$$

then the problem can be solved if $\varepsilon << 1$.

Obstruction from Carleman inequality

- Relation between d_1, d_2, m_1, m_2 in

$$I(d_1,\varphi_1)+I(d_2,\varphi_2)+I(m_1,\theta_1)+I(m_2,\theta_2)$$
$$\leq C\iint_{\omega\times(0,T)} e^{-2s\beta}(s\gamma)^{2m_2-d_2+12}|\varphi_1|^2 dxdt$$

do not allow to construct a solution $(y_1^1, y_2^1, p_1^1, p_2^2)$ decaying to zero exponentially as $t\to T$.

Theorem

Under previous assumptions: Suppose that $a_{12} \in L^\infty(Q)$ is such that for some $\delta > 0$

$$\int_Q e^{\frac{\delta}{T-t}} a_{12}^2(x,t)\,dxdt < \infty.$$

then the problem can be solved if $\varepsilon << 1$.

The proof requires to prove Carleman inequalities for the system

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \alpha\theta_1\chi_{\omega_d} + F_1 \quad \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{22}\varphi_2 = \alpha\theta_2\chi_{\omega_d} + F_2 \quad \text{in } Q, \\ \theta_{1,t} - \Delta\theta_1 + a_{11}\theta_1 = -\frac{1}{\mu}\varphi_1\chi_\omega + F_3 \quad \text{in } Q, \\ \theta_{2,t} - \Delta\theta_2 + a_{21}\theta_1 + a_{22}\theta_2 = F_1 \quad \text{in } Q, \\ \varphi_j(T) = f_j,\ \theta_j(0) = 0 \text{ in } \Omega, \quad \varphi_i = \theta_j = 0 \text{ on } \Sigma,\ j = 1,2, \end{cases}$$

Decaying-null control

$$
\begin{cases}
z_{1,t} - \Delta z_1 + a_{11} z_1 = \hat{h}\chi_\omega - \frac{1}{\mu} z_3 \chi_{\omega_1} + g_1 & \text{in } Q, \\
z_{2,t} - \Delta z_2 + a_{21} z_1 + a_{22} z_2 = + g_2 & \text{in } Q, \\
-z_{3,t} - \Delta z_3 + a_{11} z_3 + a_{21} z_4 = \alpha y_{1,d} \chi_{\omega_d} + g_3 & \text{in } Q, \\
-z_{4,t} - \Delta z_4 + a_{22} z_4 = \alpha y_{2,d} \chi_{\omega_d} + g_4 & \text{in } Q, \\
z_j(0) = 0, j = 1, 2 \; z_k(T) = 0, k = 3, 4 & z_j = 0 \text{ on } \Sigma, \; j = 1, 2, 3, 4.
\end{cases}
$$

Does there exists $\hat{h}$ such that all the variables z_1, z_2, z_3, z_4 decay exponentially to zero as $t \to T$?

- The result can be extended to $a_{12}(x,t) = b_{12}(x,t) + \varepsilon d_{12}(x,t)$ with

$$b_{12}(x,t) \text{ decaying exponentially to zero at } t = 0, t = T$$

$$d_{12}(x,t) \text{ decaying exponentially to zero at } t = T$$

- The result can be extended to $a_{12}(x,t) = b_{12}(x,t) + \varepsilon d_{12}(x,t)$ with

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- For $a_{12} = 0$ we can introduce more followers and get a Nash equilibrium.

- The result can be extended to $a_{12}(x,t) = b_{12}(x,t) + \varepsilon d_{12}(x,t)$ with

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$$d_{12}(x,t) \text{ decaying exponentially to zero at } t = T$$

- For $a_{12} = 0$ we can introduce more followers and get a Nash equilibrium.
- Work in progress for constant coefficients.

¡Gracias!

Merci!

Thank you!

¡Felicidades Assia!