Rapid stabilization

Jean-Michel Coron

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Energy decay for Timoshenko systems of memory type

F. Ammar-Khodja,[a] A. Benabdallah,[a] J.E. Muñoz Rivera,[b] and R. Racke[c,*]

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NULL-CONTROLLABILITY OF SOME SYSTEMS OF PARABOLIC TYPE BY ONE CONTROL FORCE

Farid Ammar Khodja[1], Assia Benabdallah[2], Cédric Dupaix[1] and Ilya Kostin[3]

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Null-controllability of some reaction–diffusion systems with one control force

Farid Ammar Khodja [a,*], Assia Benabdallah [b], Cédric Dupaix [a]

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RECENT RESULTS ON THE CONTROLLABILITY OF LINEAR COUPLED PARABOLIC PROBLEMS: A SURVEY

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Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem

Assia Benabdallah, Yves Dermenjian, Jérôme Le Rousseau *

Outline

1. Rapid stabilization in finite dimension
2. Linear transformations and rapid exponential stabilization
3. Backstepping and stabilization in finite time of 1-D heat equations
4. Korteweg-de Vries control systems and rapid exponential stabilization

Rapid exponential stabilization in finite dimension

We consider the control system

(1) $$\dot{y} = f(y, u),$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. We assume that $f(0,0) = 0$. We are interested in the following question (rapid exponential stabilization). Is it true that, for every $\nu > 0$, there exist a feedback law $y \in \mathbb{R}^n \mapsto u(y) \in \mathbb{R}^m$, $C > 0$ and $r > 0$ such that, for every solution of the closed loop system $\dot{y} = f(y, u(y))$ such that $|y(0)| \leqslant r$, one has

(2) $$|y(t)| \leqslant Ce^{-\nu t}|y(0)|, \, \forall t \geqslant 0?$$

If the answer is yes, one says that the rapid exponential stabilization property holds for $\dot{y} = f(y, u)$.

Rapid exponential stabilization of controllable linear systems

For a matrix $M \in \mathbb{R}^{n\times n}$, P_M denotes the characteristic polynomial of M: $P_M(z) := \det\,(z\,\mathrm{Id} - M)$. Let us denote by $\mathcal{P}_n$ the set of polynomials of degree n in z such that the coefficients are all real numbers and such that the coefficient of z^n is 1. One has the following theorem.

Theorem (Pole shifting theorem, Wonham (1967))

Let us assume that the linear control system $\dot y = Ay + Bu$ is controllable. Then

$$\left\{P_{A+BK};\, K \in \mathbb{R}^{m\times n}\right\} = \mathcal{P}_n. \tag{1}$$

In particular, if the linear control system $\dot y = Ay + Bu$ is controllable, for every real μ, there exists $K \in \mathbb{R}^{m\times n}$ such that $P_{A+BK} = (z+\mu)^n$. Hence, if the linear control system $\dot y = Ay + Bu$ is controllable, the rapid exponential stabilization property holds for this control system.

Corollary on the rapid exponential stabilization in finite dimension

Theorem (Rapid exponential stabilization when the linearized control system is controllable)

If the linearized control system at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$

$$\dot{y} = \frac{\partial f}{\partial y}(0,0)y + \frac{\partial f}{\partial u}(0,0)u \tag{1}$$

is controllable, then the rapid exponential stabilization property holds for $\dot{y} = f(y,u)$.

Obstruction to the stabilizability

Theorem (R. Brockett (1983))

If the control system $\dot{y} = f(y, u)$ can be locally asymptotically stabilized then

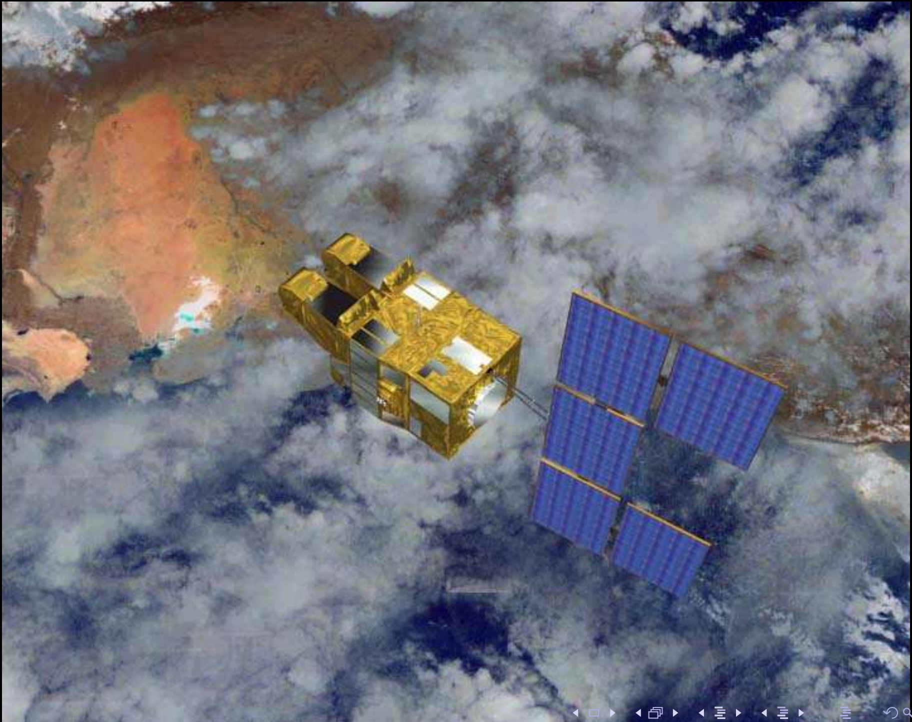
(N) *the image by f of every neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^n$.*

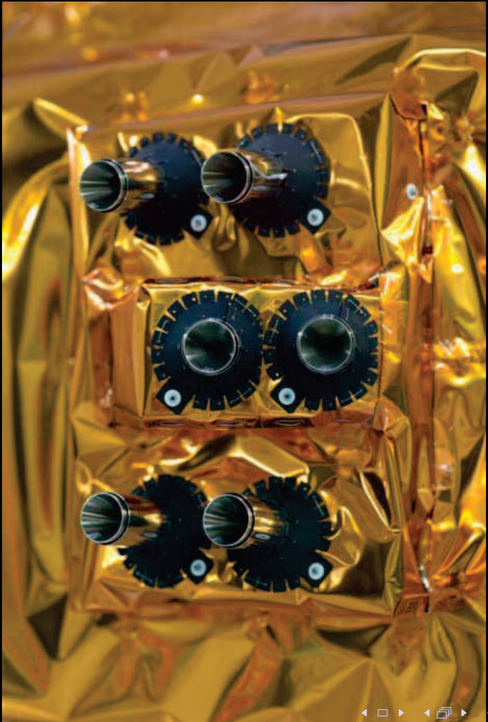
Sketch of proof. By a theorem due to Krasnosel′skiĭ, the fact that 0 is locally asymptotically stable for $\dot{y} = X(y) = f(y, u(y))$ implies de existence of $\varepsilon > 0$ such that,

(2) $$X(y) \neq 0, \ \forall y \in \mathbb{R}^n \text{ such that } 0 < |y| < \varepsilon,$$

(3) $$\text{degree}\,(X, B_\varepsilon, 0) = (-1)^n,$$

with $B_\varepsilon := \{(y \in \mathbb{R}^n;\ |y| < \varepsilon\}$. Properties (2) and (3) imply that, for every $\eta \in (0, \varepsilon]$, $X(B_\eta)$ is a neighborhood of 0.





The under-actuated satellite

(1) $$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i,\ \dot{\eta} = A(\eta)\omega,$$

with $S(\omega)y := y \wedge \omega$. One has $A(0) = \mathrm{Id}$. The vectors $b_1, \ldots, b_m$ are independent. If $m = 3$, the linearized control system around the equilibrium $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^3$ is controllable and the control system is locally asymptotically stabilizable. We now turn to the case where $m = 2$. One easily sees that (B) never holds. However, if

(2) $$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3,$$

then the control system (1) is small-time locally controllable at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$. (This follows from a sufficient condition for local controllability proved by H. Sussmann in 1987.) However, if $m < 3$, (1) does not satisfy the Brockett condition (B).

A solution: Time-varying feedback laws

Instead of $u(y)$, use $u(t, y)$. Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function). First use of time-varying feedback laws:

- $n = 1$: E. Sontag and H. Sussmann (1980).
- For a driftless control system with $n = 3$ and $m = 2$: C. Samson (1992).

Continuous reachability

In order to deal with systems for which the linearized system is not controllable, we use the following definition.

Definition

The origin (of $\mathbb{R}^n$) is *locally continuously reachable in small time* for the control system $\dot{y} = f(y, u)$ if, for every positive real number T, there exist a positive real number ε and $u : \bar{B}_\varepsilon \to L^1((0, T); \mathbb{R}^m)$ such that

(1) $$u \in C^0\left(\bar{B}_\varepsilon; L^1((0, T); \mathbb{R}^m)\right)$$

(2) $$\text{Sup}\{|u(a)(t)|; t \in (0, T)\} \to 0 \text{ as } a \to 0,$$

(3) $$((\dot{y} = f(y, u(a)(t)),\ y(0) = a) \Rightarrow (y(T) = 0)), \forall a \in \bar{B}_\varepsilon.$$

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Open problem

Assume that f is analytic and that $\dot{y} = f(y, u)$ is small-time locally controllable at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$. Is the origin (of $\mathbb{R}^n$) locally continuously reachable in small time for the control system $\dot{y} = f(y, u)$?

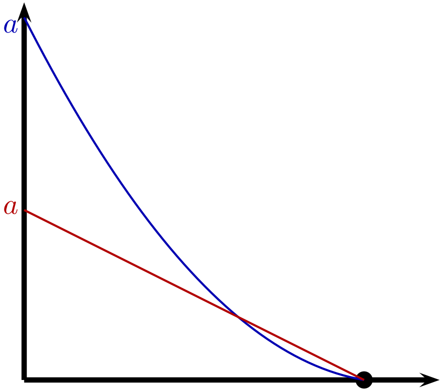
Theorem (JMC (1995))

Assume f is analytic, that $0 \in \mathbb{R}^n$ is locally continuously reachable in small time for the control system $\dot{y} = f(y,u)$, and that $n \notin \{2,3\}$. Then, for every positive real number T, there exist ε in $(0,+\infty)$ and u in $C^0(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^m)$, of class C^∞ on $\mathbb{R}\times(\mathbb{R}^n\setminus\{0\})$, T-periodic with respect to time, vanishing on $\mathbb{R}\times\{0\}$ and such that, for every $s\in\mathbb{R}$,

(1) $\quad ((\dot{y} = f(y,u(t,y)) \text{ and } y(s)=0) \Rightarrow (y(\tau)=0,\ \forall \tau \geqslant s))\,,$

(2) $\quad (\dot{y} = f(y,u(t,y)) \text{ and } |y(s)| \leqslant \varepsilon) \Rightarrow (y(\tau)=0,\ \forall \tau \geqslant s+T))\,.$

Step 1: Excite the system in order to have controllable linearized systems around the trajectories



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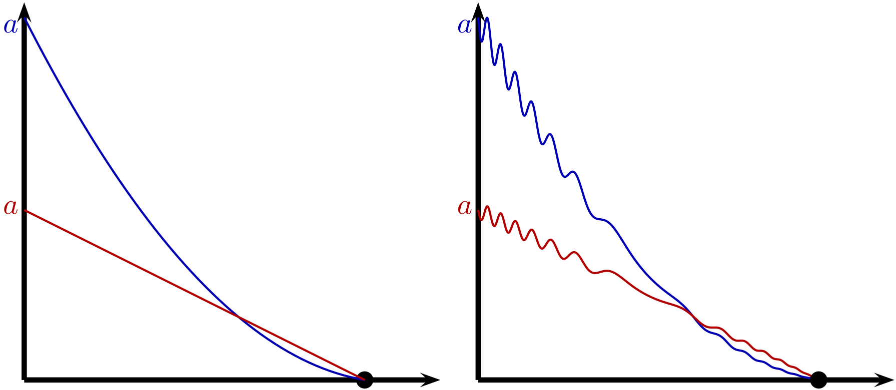

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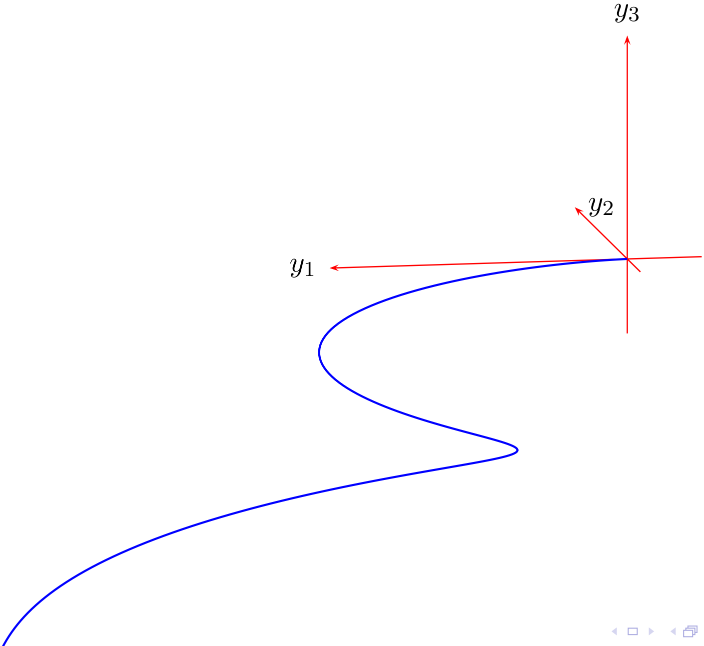

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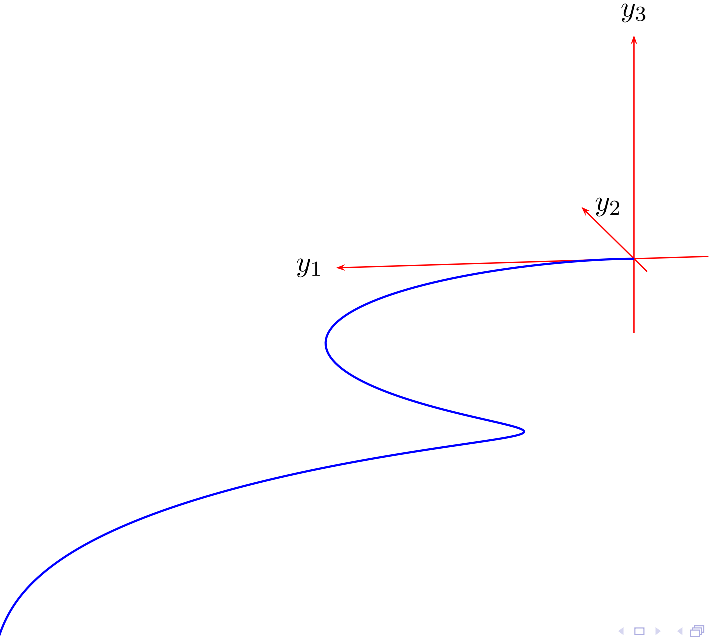

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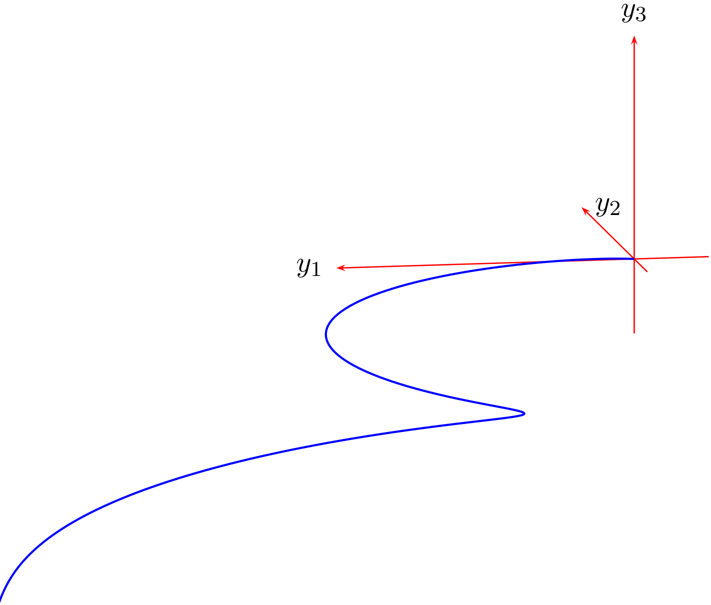

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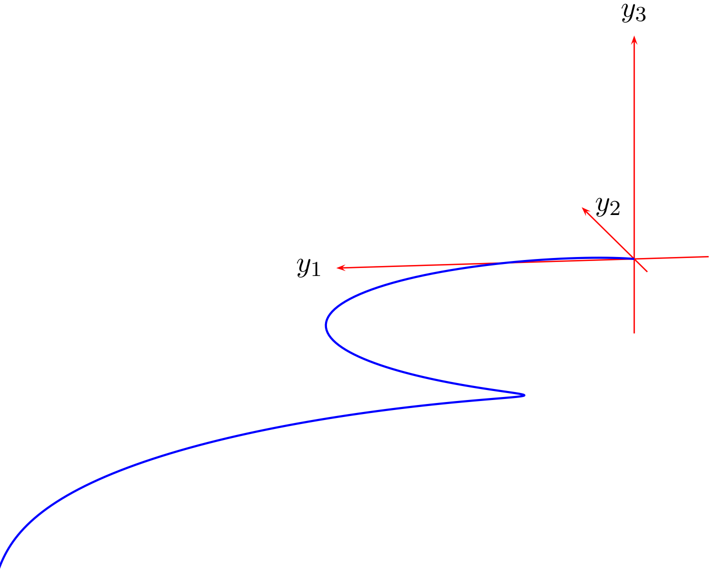

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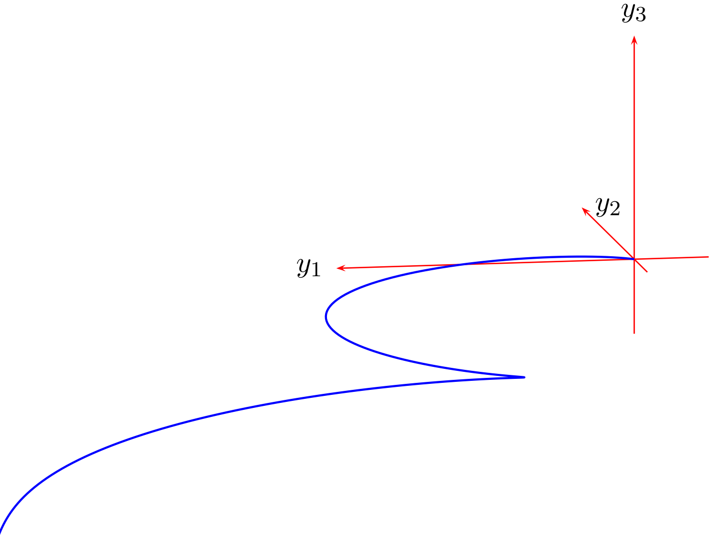

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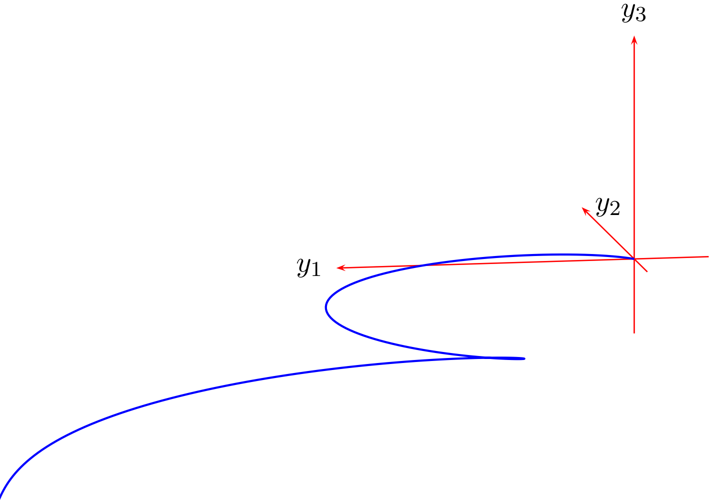

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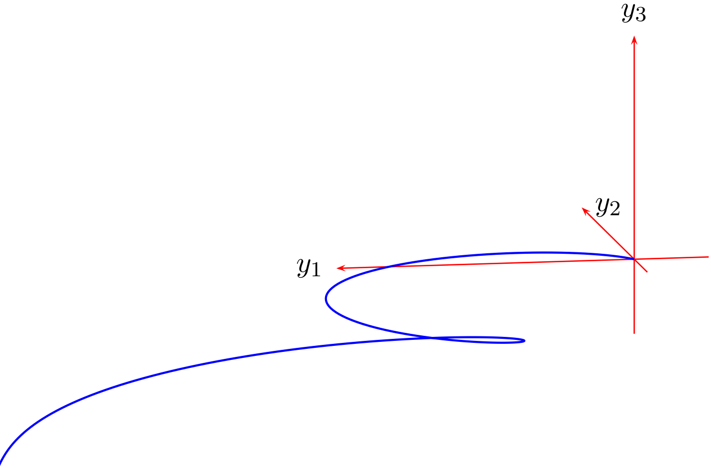

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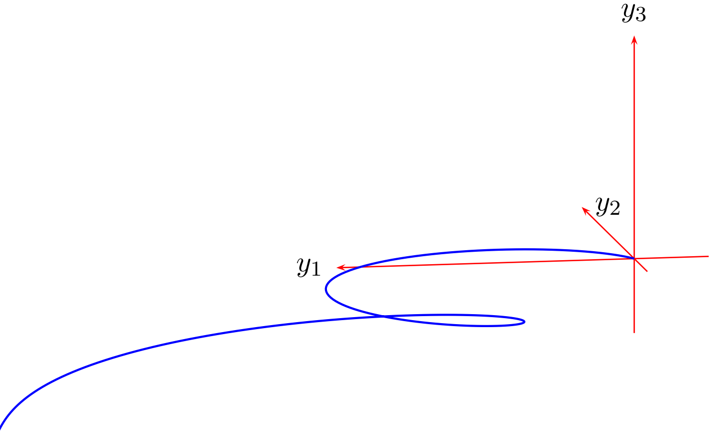

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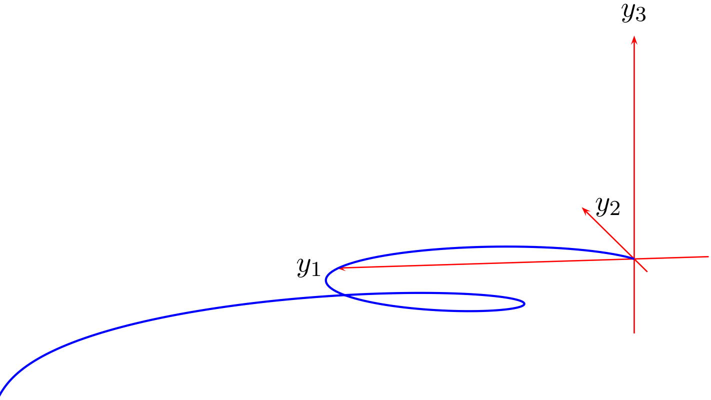

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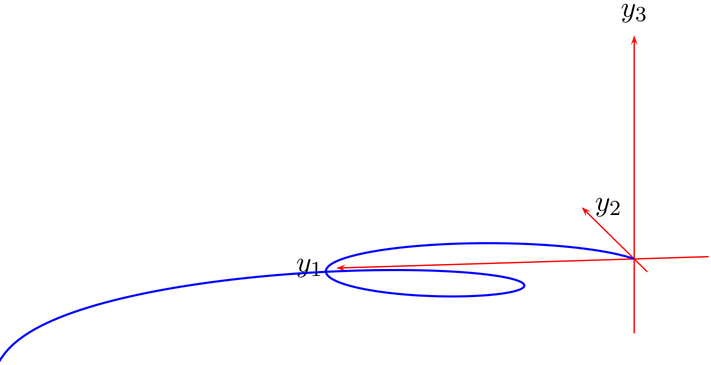

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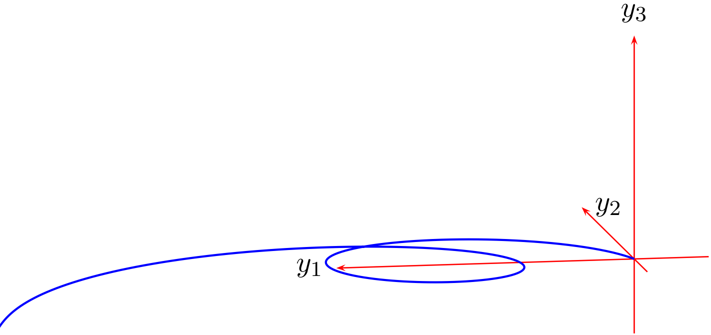

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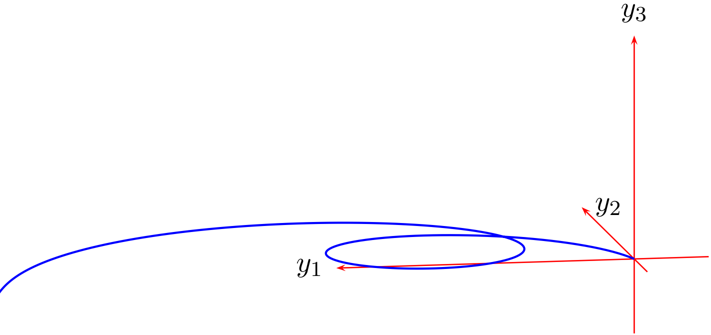

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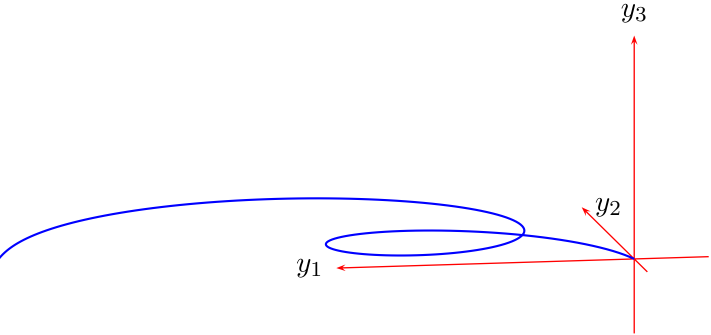

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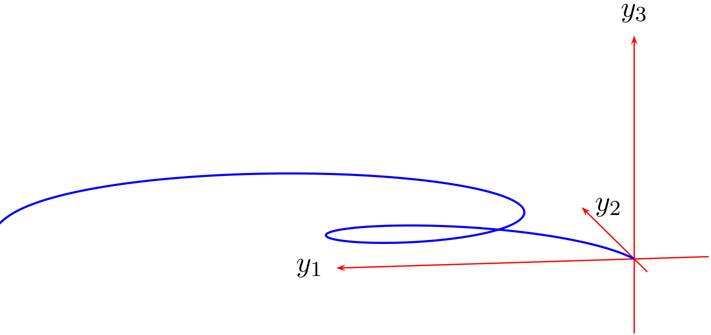

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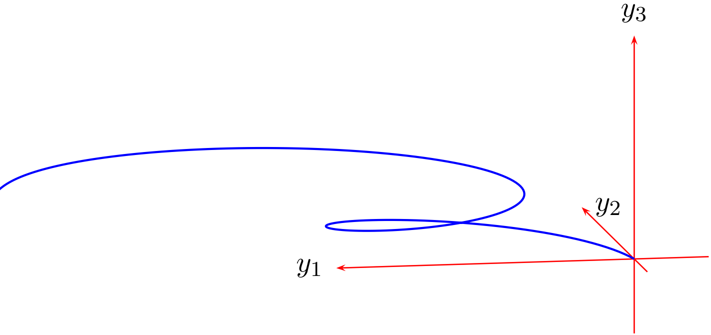

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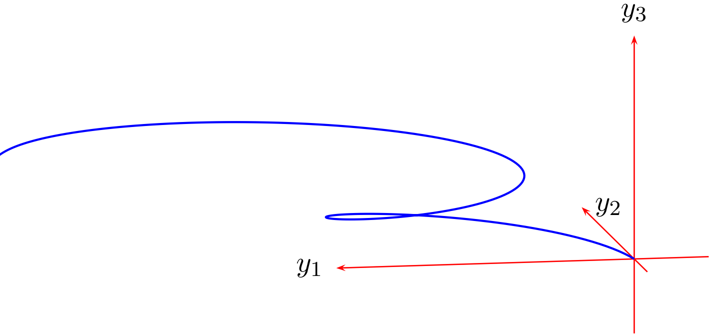

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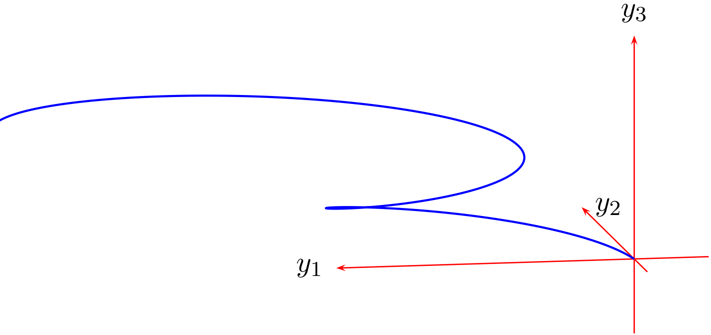

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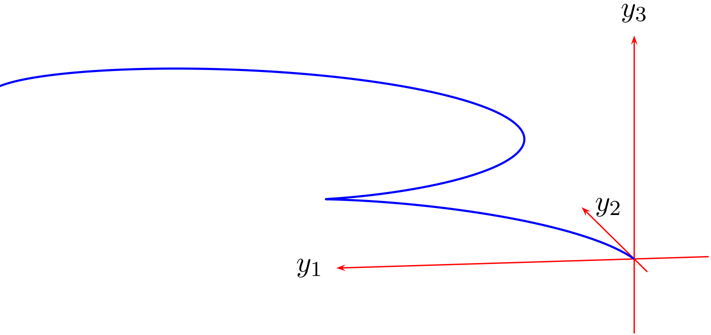

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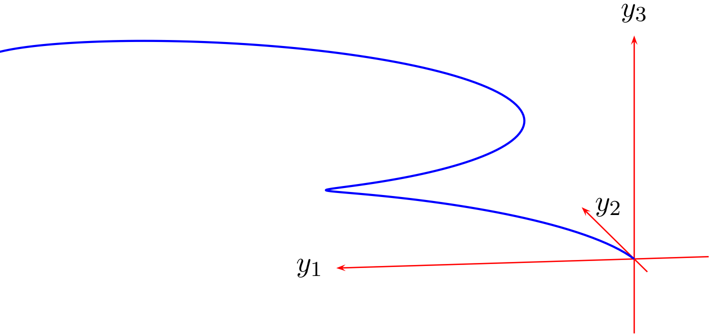

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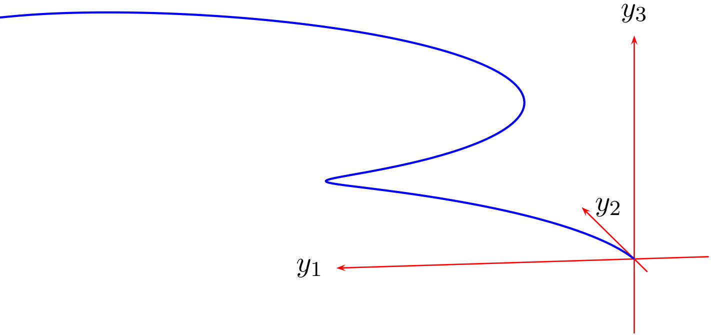

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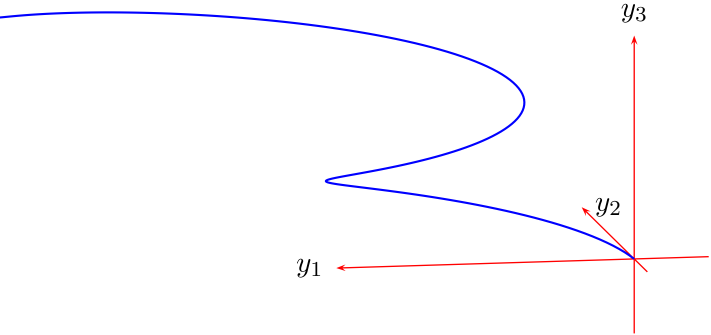
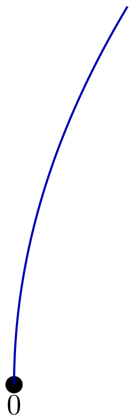


If $n \leqslant 3$, a small perturbation of the motion of the curve does not remove the crossing. If $n > 3$, using the controllability of the linearized control systems, one can perturb the motion of the curve in order to avoid the crossing.

Step 3: A neighborhood of a curve in $\mathbb{R}^n \setminus \{0\}$ sent to 0



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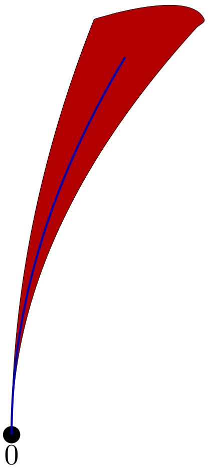
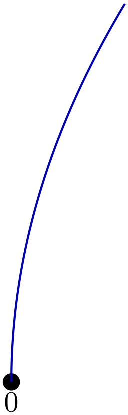


The red neighborhood can be sent to 0 in finite time by means of a time-varying feedback law.

Step 4: A magic curve



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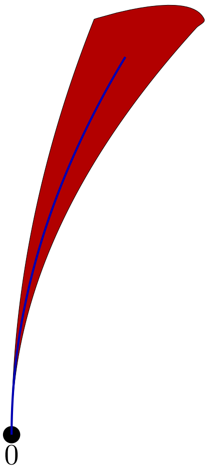

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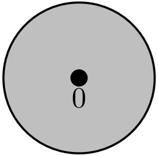

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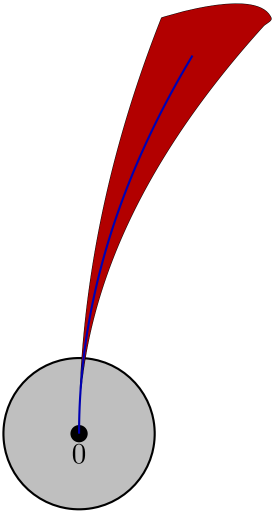

The grey ball can be sent in finite time into the red neighborhood by means of a time-varying feedback law.

Stabilization of the under-actuated satellite

(1) $$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i,\ \dot{\eta} = A(\eta)\omega,$$

We consider again the case where $m = 2$ and assume that

$$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3.$$

Then $0 \in \mathbb{R}^6$ is locally continuously reachable in small-time for the control system the control system (1) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws.

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Then $0 \in \mathbb{R}^6$ is locally continuously reachable in small-time for the control system the control system (1) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws. Construction of explicit time-varying stabilizing feedback laws:

- Special cases: G. Walsh, R. Montgomery and S. Sastry (1994); P. Morin, C. Samson, J.-B. Pomet and Z.-P. Jiang (1995).
- General case: JMC and E.-Y. Keraï (1996); P. Morin and C. Samson (1997).

Damping and stabilization

For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies.

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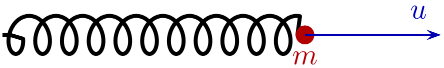
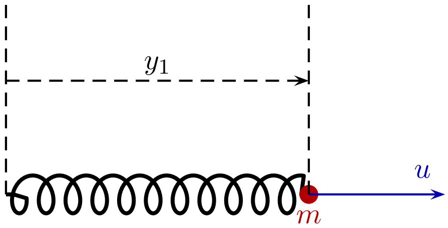
For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies. Let us go back to spring-mass control system.





The control system is

(Spring-mass)
$$\dot{y}_1 = y_2,\ \dot{y}_2 = -\frac{k}{m}y_1 + \frac{u}{m},$$

where, as already mentioned, m is the mass of the point attached to the spring, y_1 is the displacement of the mass (on a line), y_2 is the speed of the mass, k is the spring constant, and u is the external force applied to the mass. The state is $(y_1, y_2)^{\mathrm{tr}} \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$.

The total energy E of the system is

$$E = \frac{1}{2}(ky_1^2 + my_2^2). \tag{1}$$

One has

$$\dot{E} = uy_2. \tag{2}$$

Hence if $y_2 = 0$, one cannot have $\dot{E} < 0$. However it tempting to consider the following feedback laws

$$u := -My_2, \tag{3}$$

where $M > 0$. Using the LaSalle invariance principle, one gets that these feedback laws globally asymptotically stabilize the spring-mass control system.

Application: Orbit transfer with low-thrust systems (JMC and L. Praly (1996))

Electric propulsion is characterized by a low-thrust acceleration level but a high specific impulse. They can be used for large amplitude orbit transfers if one is not in a hurry.
The state of the control system is the position of the satellite (here identified to a point: we are not considering the attitude of the satellite) and the speed of the satellite. Instead of using Cartesian coordinates, one prefers to use the "orbital" coordinates. The advantage of this set of coordinates is that, in this set, the first five coordinates remain unchanged if the thrust vanishes: these coordinates characterize the Keplerian elliptic orbit. When thrust is applied, they characterize the Keplerian elliptic osculating orbit of the satellite. The last component is an angle which gives the position of the satellite on the Keplerian elliptic osculating orbit of the satellite.

A usual set of orbital coordinates is

$$\begin{aligned} p &:= a(1-e^2), \\ e_x &:= e\cos\tilde{\omega}, \quad \text{with} \quad \tilde{\omega} = \omega + \Omega, \\ e_y &:= e\sin\tilde{\omega}, \\ h_x &:= \tan\frac{i}{2}\cos\Omega, \\ h_y &:= \tan\frac{i}{2}\sin\Omega, \\ L &:= \tilde{\omega} + v, \end{aligned}$$

where a, e, ω, Ω, i characterize the Keplerian osculating orbit:

1. a is the semi-major axis,
2. e is the eccentricity,
3. i is the inclination with respect to the equator,
4. Ω is the right ascension of the ascending node,
5. ω is the angle between the ascending node and the perigee,

and where v is the true anomaly.

$$\dot{p} = 2\sqrt{\frac{p^3}{\mu}}\frac{1}{Z}S,$$

$$\dot{e}_x = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[Z(\sin L)Q + AS - e_y(h_x \sin L - h_y \cos L)W\right],$$

$$\dot{e}_y = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[-Z(\cos L)Q + BS - e_x(h_x \sin L - h_y \cos L)W\right],$$

$$\dot{h}_x = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\cos L)W,\ \dot{h}_y = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\sin L)W,$$

$$\dot{L} = \sqrt{\frac{\mu}{p^3}}Z^2 + \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left(h_x \sin L - h_y \cos L\right)W,$$

where $\mu > 0$ is a gravitational coefficient depending on the central gravitational field, Q, S, W, are the radial, orthoradial, and normal components of the thrust and where

$$Z := 1 + e_x \cos L + e_y \sin L,\ A := e_x + (1+Z)\cos L,$$
$$B := e_y + (1+Z)\sin L,\ X := 1 + h_x^2 + h_y^2.$$

We study the case, useful in applications, where

$$Q = 0,$$

and, for some $\varepsilon > 0$,

$$|S| \leqslant \varepsilon \text{ and } |W| \leqslant \varepsilon.$$

Note that ε is small, since the thrust acceleration level is low.
The goal: give feedback laws, which (globally) asymptotically stabilize a given Keplerian elliptic orbit characterized by the coordinates $\bar{p}, \bar{e}_x, \bar{e}_y, \bar{h}_x, \bar{h}_y$.
In order to simplify the notations (this is not essential for the method), we restrict our attention to the case where the desired final orbit is geostationary, that is,

$$\bar{e}_x = \bar{e}_y = \bar{h}_x = \bar{h}_y = 0.$$

We start with a change of "time". One describes the evolution of (p, e_x, e_y, h_x, h_y) as a function of L instead of t. Then our system reads

$$\begin{cases} \dfrac{dp}{dL} = 2KpS, \\ \dfrac{de_x}{dL} = K[AS - e_y(h_x \sin L - h_y \cos L)W], \\ \dfrac{de_y}{dL} = K[BS - e_x(h_x \sin L - h_y \cos L)W], \\ \dfrac{dh_x}{dL} = \dfrac{K}{2}X(\cos L)W, \ \dfrac{dh_y}{dL} = \dfrac{K}{2}X(\sin L)W, \\ \dfrac{dt}{dL} = K\sqrt{\dfrac{\mu}{p}}Z, \end{cases}$$

with

$$K = \left[\frac{\mu}{p^2}Z^3 + (h_x \sin L - h_y \cos L)W\right]^{-1}. \tag{1}$$

Typically, one consider the following control Lyapunov function

$$V(p, e_x, e_y, h_x, h_y) = \frac{1}{2}\left(\frac{(p-\bar{p})^2}{p} + \frac{e^2}{1-e^2} + h^2\right),$$

with $e^2 = e_x^2 + e_y^2 < 1$ and $h^2 = h_x^2 + h_y^2$. The time derivative of V along a trajectory of our control system is is given by

$$\dot{V} = K(\alpha S + \beta W),$$

with

$$\begin{aligned}
\alpha &:= 2p\frac{\partial V}{\partial p} + A\frac{\partial V}{\partial e_x} + B\frac{\partial V}{\partial e_y}, \\
\beta &:= (h_y \cos L - h_x \sin L)\left(e_y\frac{\partial V}{\partial e_x} + e_x\frac{\partial V}{\partial e_y}\right) \\
&\qquad + \frac{1}{2}X\left((\cos L)\frac{\partial V}{\partial h_x} + (\sin L)\frac{\partial V}{\partial h_y}\right).
\end{aligned}$$

Following the damping method, one defines

$$
\begin{array}{rcl}
S & := & -\sigma_1(\alpha), \\
W & := & -\sigma_2(\beta)\sigma_3(p, e_x, e_y, h_x, h_y),
\end{array}
$$

where $\sigma_1 : \mathbb{R} \to \mathbb{R}, \sigma_2 : \mathbb{R} \to \mathbb{R}$ and $\sigma_3 : (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \to (0, 1]$ are such that

$$
\sigma_1(s)s > 0,\ \sigma_2(s)s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\},
$$
$$
\| \sigma_1 \|_{L^\infty(\mathbb{R})} < \varepsilon,\ \| \sigma_2 \|_{L^\infty(\mathbb{R})} < \varepsilon,
$$
$$
\sigma_3(p, e_x, e_y, h_x, h_y) \leqslant \frac{1}{1+\varepsilon} \frac{\mu}{p^2} \frac{(1-|e|)^3}{|h|}.
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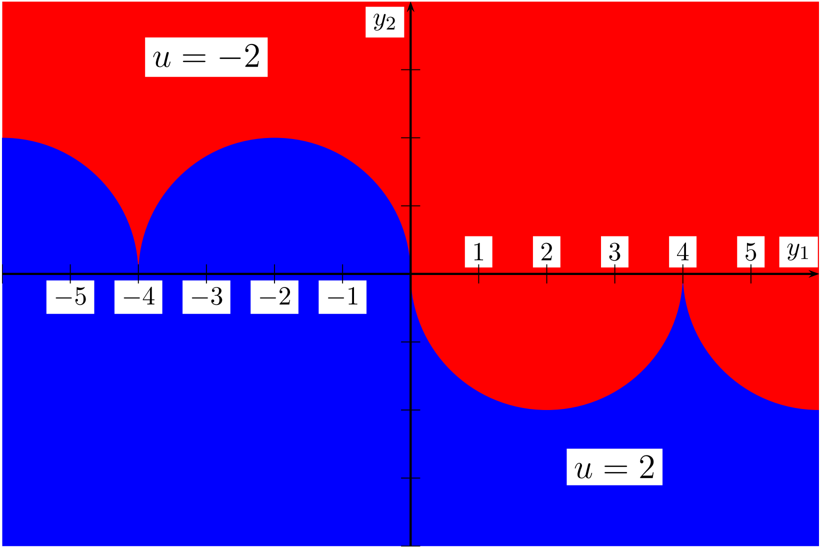
$$\sigma_3(p, e_x, e_y, h_x, h_y) \leqslant \frac{1}{1+\varepsilon} \frac{\mu}{p^2} \frac{(1-|e|)^3}{|h|}.$$

It works!

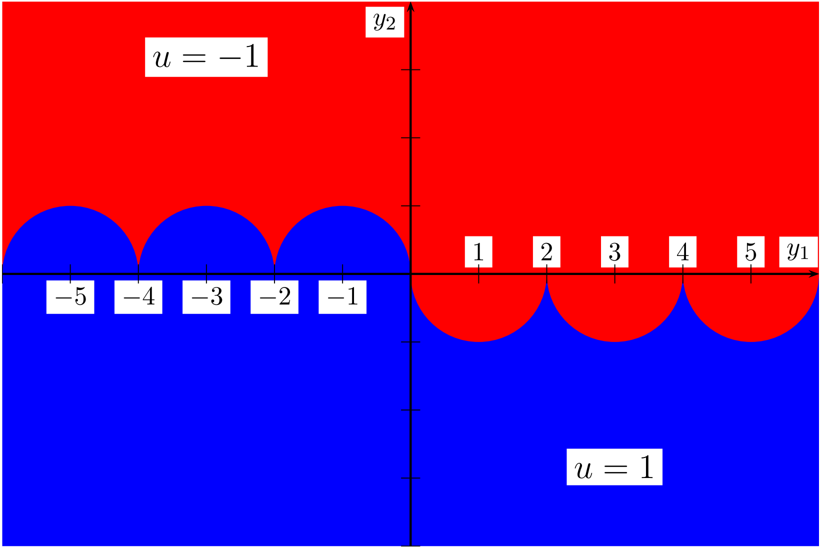
Comparison with optimal control

It is interesting to compare the feedback constructed here to the open-loop optimal control for the minimal time problem (reach $(\bar{p}, 0, 0, 0, 0)$ in a minimal time with the constraint $|u(t)| \leqslant M$). Numerical experiments show that the use of the previous feedback laws (with suitable saturations σ_i, $i \in \{1, 2, 3\}$) gives trajectories which are nearly optimal if the state is not too close to $(\bar{p}, 0, 0, 0, 0)$. Note that our feedback laws are quite easy to compute compared to the optimal trajectory and provide already good robustness properties compared to the open-loop optimal trajectory (the optimal trajectory in a closed-loop form being, at least for the moment, out of reach numerically). However, when one is close to the desired target, our feedback laws are very far from being optimal.

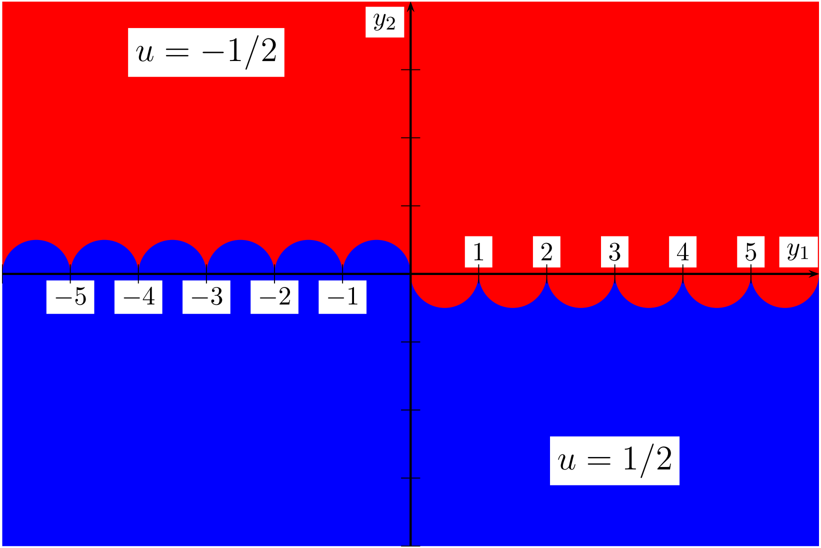
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ |u| \leqslant 2$



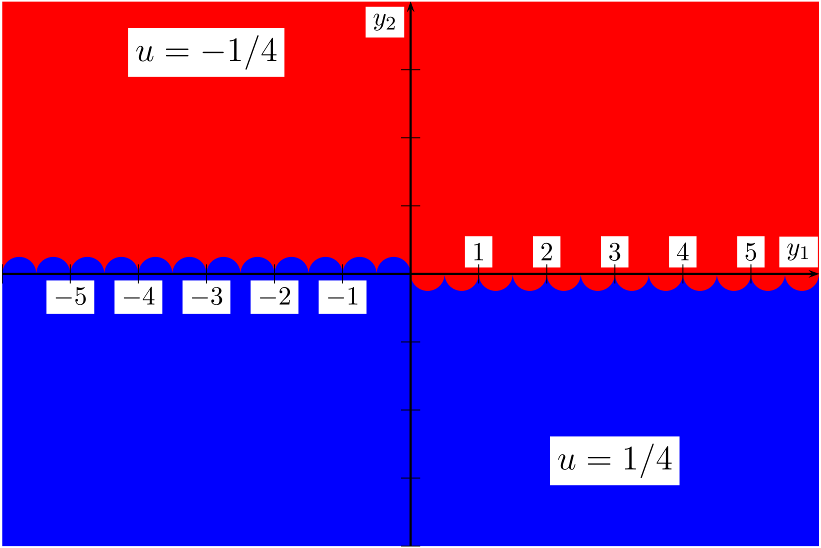
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ |u| \leqslant 1$



$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ |u| \leqslant 1/2$$



$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ |u| \leqslant 1/4$



An important limitation of the damping method

Let us consider the spring-mass control system with normalized physical constants $(k = m = g = 1)$

$$\dot{y}_1 = y_2, \, \dot{y}_2 = -y_1 + u.$$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$V(y) = y_1^2 + y_2^2, \forall y = (y_1, y_2)^{\text{tr}} \in \mathbb{R}^2.$$

One has $\dot{V} = 2y_2 u$. and it is tempting to take $u := -My_2$, where M is some fixed positive real number. An a priori guess would be that, if we let M be quite large, then we get a quite good convergence, as fast as we want.

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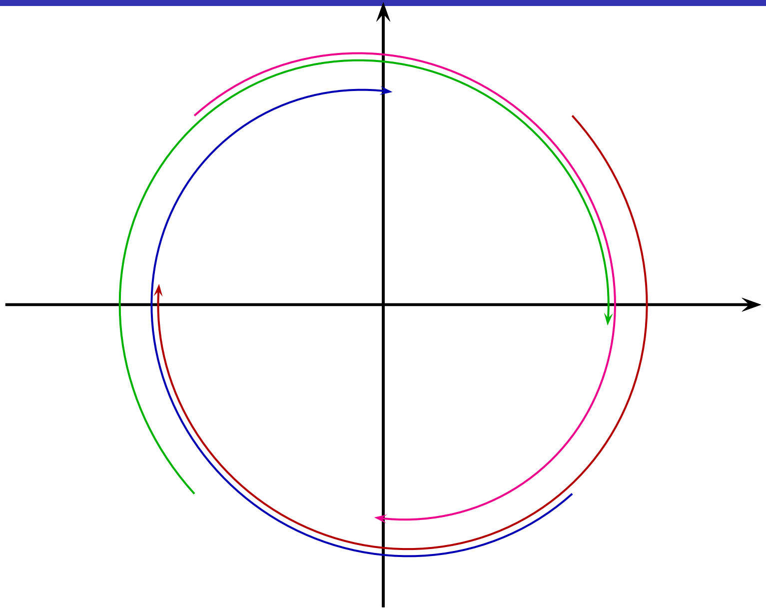
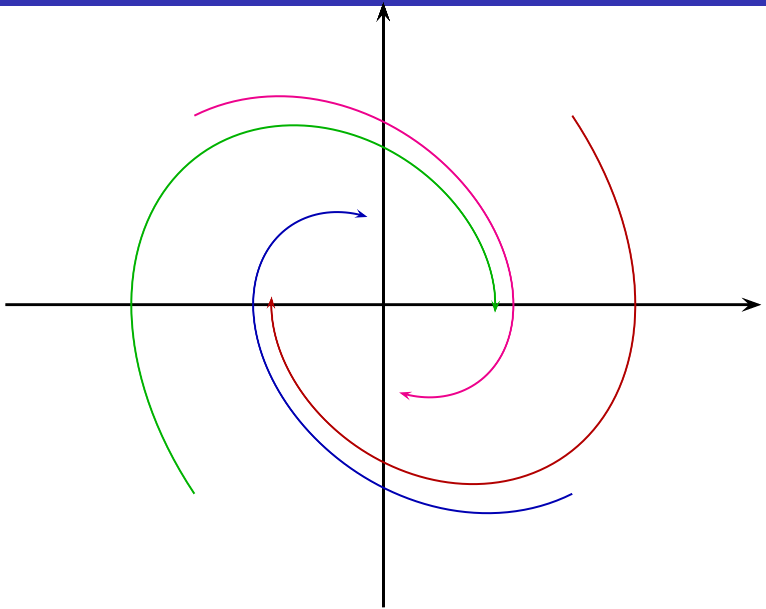
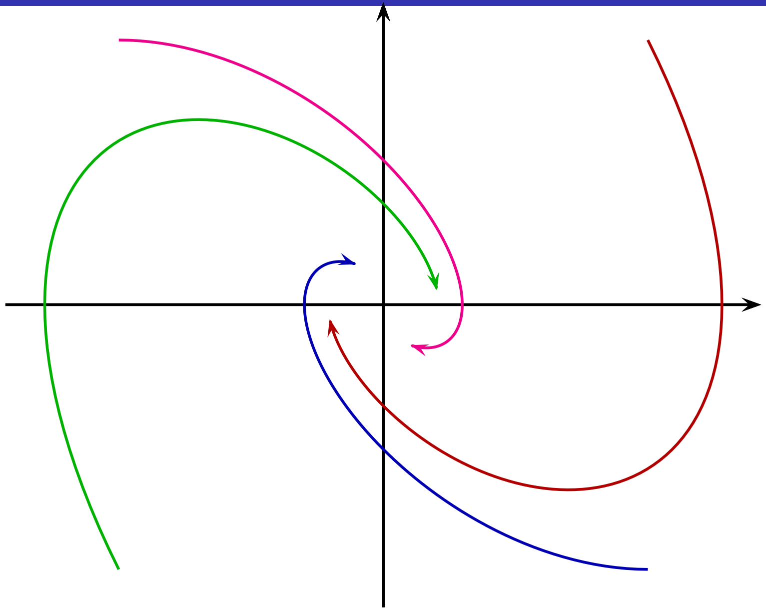
$\dot{y}_1 = y_2, \, \dot{y}_2 = -y_1 - (1/10)y_2$



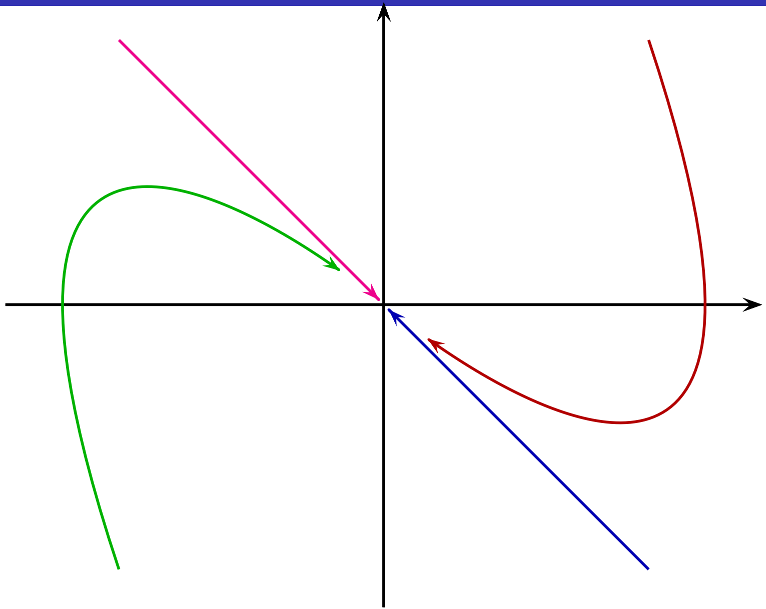
$\dot{y}_1 = y_2$, $\dot{y}_2 = -y_1 - (1/2)y_2$



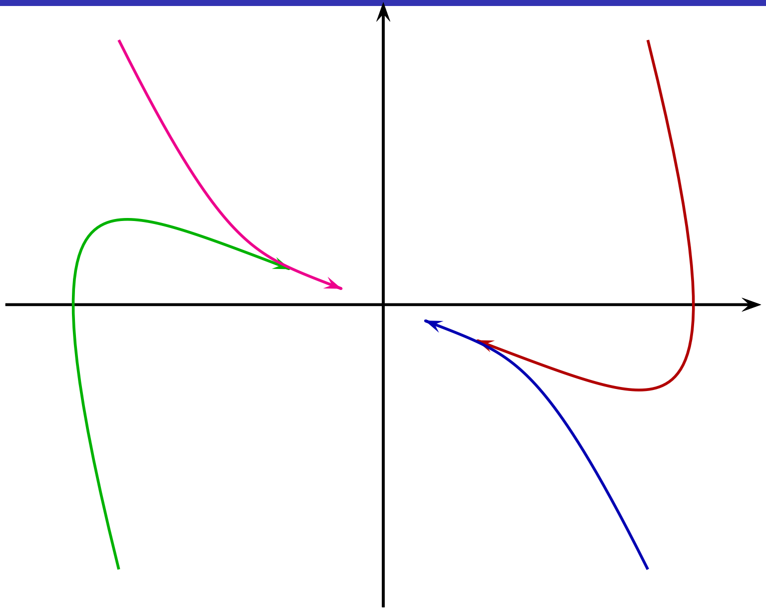
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - y_2$



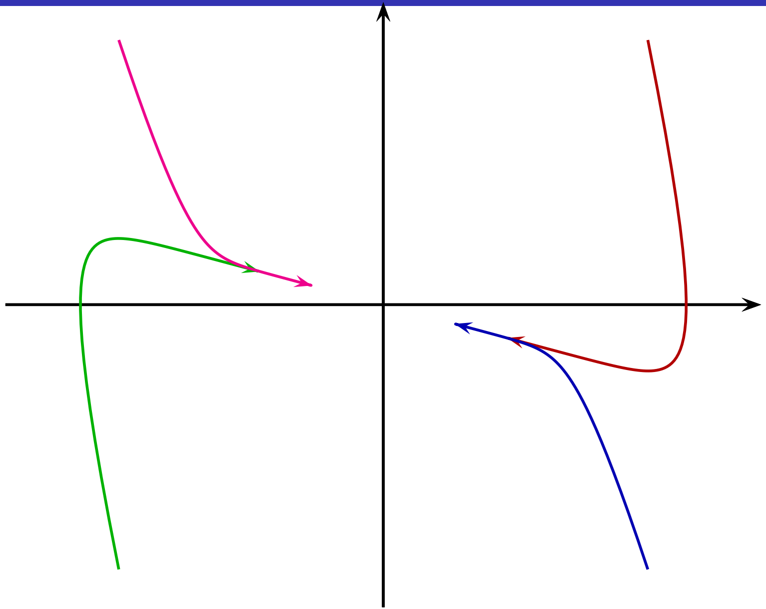
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 2y_2$



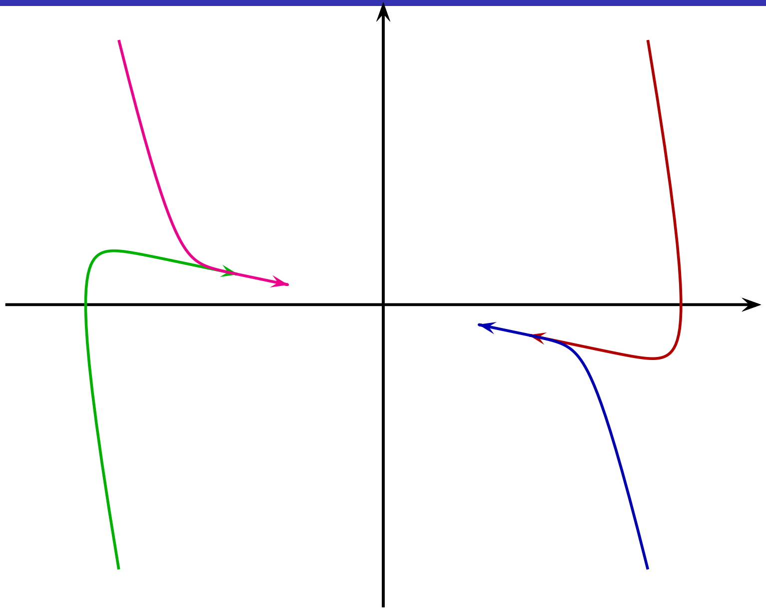
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 3y_2$



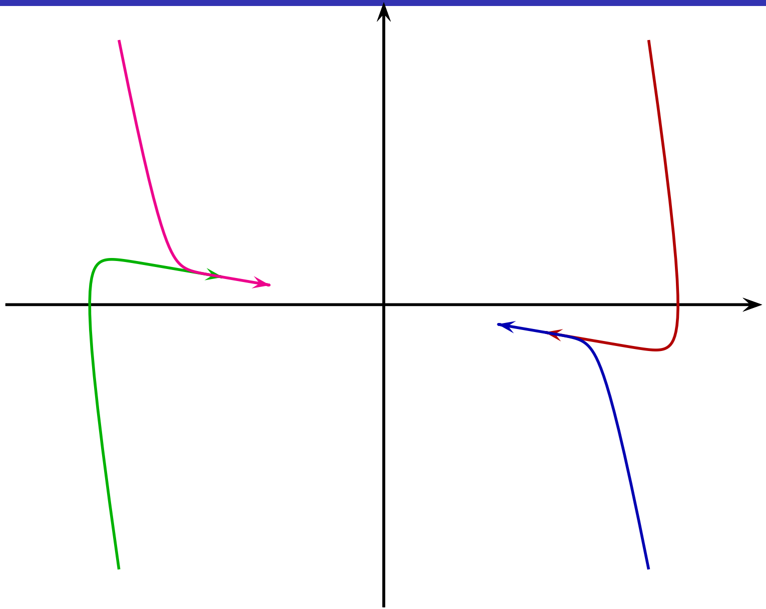
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 4y_2$



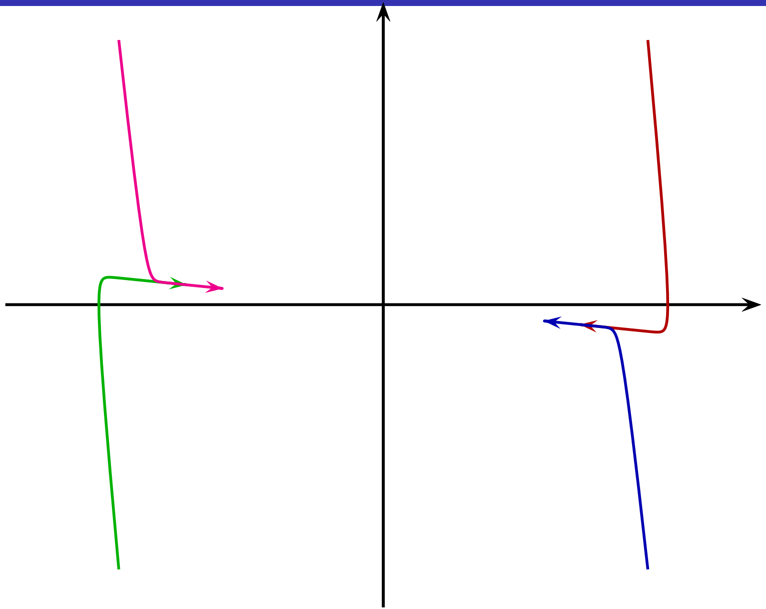
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 5y_2$



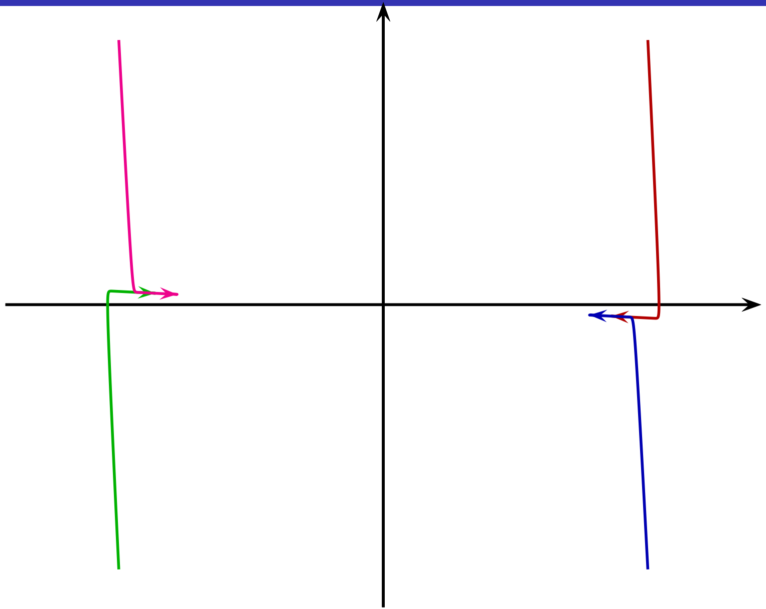
$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 6y_2$



$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 10y_2$



$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 - 20y_2$



Linear transformations and rapid exponential stabilization

Let us now present a method which can be used to get rapid exponential stabilization even for control systems modeled by means of partial differential equations.

We first start with a linear control in finite dimension with a control of dimension 1. We consider the following control system

$$\dot{y} = Ay + Bu, \tag{1}$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}$. We assume that

(2) the control system (1) is controllable.

Let $\lambda \in \mathbb{R}$. Let $\mathsf{GL}(n, \mathbb{R})$ be the set of invertible elements of $\mathbb{R}^{n\times n}$. We are looking for $T \in \mathsf{GL}(n, \mathbb{R})$ and $K \in \mathbb{R}^{1\times n}$ such that, if $y = Tz$ and $u = Kz + v$, then (1) is equivalent to

$$\dot{z} = (A - \lambda \mathsf{Id})z + Bv, \tag{3}$$

where Id is the identity matrix in $\mathbb{R}^{n\times n}$. Clearly, if such T and K exists for every $\lambda \in \mathbb{R}$ the control system $\dot{y} = Ay + Bu$ satisfies the rapid exponential stabilization property.

Existence and uniqueness of T and K

The equivalence between $\dot{y} = Ay + Bu$ and $\dot{z} = (A - \lambda \mathrm{Id})z + Bv$ with $y = Tz$ and $u = Kz + v$ holds if and only if

(1) $$AT + BK = TA - \lambda T,$$

(2) $$TB = B.$$

One has the following theorem.

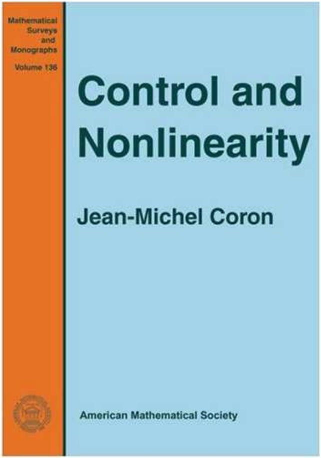
Theorem

If $\dot{y} = Ay + Bu$ is controllable, there exists one and only one $(T, K) \in GL(n, \mathbb{R}) \times \mathbb{R}^{1\times n}$ such that (1) and (2) hold.

A quick history on backstepping

1. Backstepping was initially a recursive method to stabilize finite dimensional control system of the form $\dot{x} = f(x, y)$, $\dot{y} = u$.
2. First application to PDE: JMC and B. d'Andréa-Novel (1998).

For more details



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.

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3. This method has been used on the discretization of partial differential equations by D. Bošković, A. Balogh and M. Krstic in 2003.
4. A key modification of the method by using a Volterra transformation of the second kind is introduced by D. Bošković, M. Krstic and W. Liu in 2001.
5. For a survey on this method with Volterra transformations of the second kind, see the book by M. Krstic and A. Smyshlyaev in 2008.

The historical example for a heat equation

We consider the heat control system

$$(1) \qquad y_t - y_{xx} = 0,\ y(t,0) = 0,\ y(t,1) = u(t),\ t \in [0,+\infty),\ x \in [0,1],$$

where, at time $t \in [0,+\infty)$, the state is $y(t) \in L^2(0,1)$, $x \in (0,1) \mapsto y(t)(x) := y(t,x)$ and the control is $u(t) \in \mathbb{R}$. We are interested in the rapid exponential stabilization of this linear (controllable) control system. We try the the approach by means of linear transform. Let $\lambda \in \mathbb{R}$. Consider the following controlled system (called the target system)

$$(2) \quad z_t - z_{xx} = -\lambda z,\ z(t,0) = 0,\ z(t,1) = v(t),\ t \in [0,+\infty),\ x \in [0,1],$$

where, at time $t \in [0,+\infty)$, the state is $z(t) \in L^2(0,1)$, $x \in (0,1) \mapsto z(t)(x) := z(t,x)$ and the control is $v(t) \in \mathbb{R}$. We look for two linear maps $T^{-1} : L^2(0,1) \to L^2(0,1)$ $y \mapsto z$ and $K : L^2(0,1) \to \mathbb{R}$, $z \mapsto Kz$ such that the target system (2) is transformed into the initial system (1) if $u = Kz + v$.

D. Bošković, M. Krstic and W. Liu in 2001 proposed to look for T^{-1} in the class of Volterra transform of the second kind:

(1) $$z(x_1) := y(x_1) - \int_0^{x_1} k(x_1, x_2) y(x_2) dx_2.$$

One of the advantages of the Volterra transforms of the second kind is that there are invertible (if k is smooth enough, for example in $L^2((0,1) \times (0,1))$). Note that, once T is defined, we must take

(2) $$Kz = -\int_0^1 k(1, x_2) y(x_2) dx_2.$$

Moreover, the feedback law $u(y) := \int_0^1 k(1,s) y(s) ds$ leads for z to the closed loop system

(3) $$z_t - z_{xx} = -\lambda z, \; z(t,0) = z(t,1) = 0$$

which leads to exponential stability for z with an exponential decay rate of λ (in $L^2(0,1)$).

Since $y \in L^2(0,1) \to z \in L^2(0,1)$ is an isomorphism the same holds for the closed loop system

(1) $$y_t - y_{xx} = 0,\ y(t,0) = 0,\ y(t,1) = \int_0^1 k(1,s)y(s)ds,$$

which shows the rapid exponential stabilizability of the initial heat control system (with a method to compute a feedback law leading to an exponential stability with an exponential decay rate as large as we want).

Kernel equation

Straightforward computations show that the y system is equivalent to the z system if and only if k satisfies the following equation, called the kernel equation,

$$
\begin{cases}
k_{11} - k_{22} = \lambda k, & 0 < x_2 < x_1 < 1, \\
k(x_1, 0) = 0, & 0 < x_1 < 1, \\
k(x, x) = -\dfrac{\lambda}{2} x, & 0 < x < 1,
\end{cases}
\tag{1}
$$

$k_{ii} := \partial^2_{x_i x_i} k$, $i \in \{1, 2\}$.

A method to prove the existence of k

D. Bošković, M. Krstic and W. Liu in 2001 proposed the following iterative scheme. Let us make the following change of variables $t = x_1 - x_2$, $s = x_1 + x_2$ and define $G(s,t) := k(x_1, x_2)$ on $\mathcal{T}_0 := \{(s,t);\ t \in [0,1], s \in [t, 2-t]\}$. Then k satisfies the kernel equation if and only if

$$(1) \qquad \left\{ \begin{array}{rcll} G_{st} & = & -\dfrac{\lambda}{4} G, & \text{in } \mathcal{T}_0, \\ G(s,s) & = & 0, & \text{in } [1,2], \\ G(s,0) & = & \dfrac{\lambda}{4} s, & \text{in } [0,2]. \end{array} \right.$$

One integrates the first equation of (1) with respect to t from 0 to t. One gets, using also the third equality of (1),

$$(2) \qquad G_s(s,t) = G_s(s,0) - \frac{\lambda}{4} \int_0^t G(s,t_1) dt_1 = \frac{\lambda}{4} - \frac{\lambda}{4} \int_0^t G(s,t_1) dt_1.$$

We integrate this equation with respect to s from t to s. Using also the second equation of (1), we get

$$
(1) \qquad \begin{aligned} G(s,t) &= G(t,t) + \frac{\lambda}{4}(s-t) - \frac{\lambda}{4}\int_t^s \int_0^t G(s_1,t_1)dt_1 ds_1 \\ &= \frac{\lambda}{4}(s-t) - \frac{\lambda}{4}\int_t^s \int_0^t G(s_1,t_1)dt_1 ds_1 \end{aligned}
$$

One defines inductively $G^n : \mathcal{T}_0 \to \mathbb{R}$, $n \in \mathbb{N} \setminus \{0\}$, by requiring

$$
(2) \qquad G^1(s,t) = 0,
$$

$$
(3) \qquad G^{n+1}(s,t) = \frac{\lambda}{4}(s-t) - \frac{\lambda}{4}\int_t^s \int_0^t G^n(s_1,t_1)dt_1 ds_1
$$

One gets, by induction on n, that

$$
(4) \qquad G^n(s,t) = -\sum_{k=1}^{n} \frac{(s-t)s^{k-1}t^{k-1}(-\lambda)^k}{(k-1)!k!4^k},
$$

a sum which converges as $n \to +\infty$.

Formulas for G and k

Let

(1) $$I(x) := \sum_{k=1}^{+\infty} \frac{(-x)^{2k-1}}{(k-1)!k!2^{2k-1}}.$$

Then

(2) $$G(s,t) = \frac{\lambda}{2}(s-t)\frac{I(\sqrt{\lambda st})}{\sqrt{\lambda st}},$$

(3) $$k(x_1,x_2) = \frac{\lambda}{2}x_2\frac{I(\sqrt{\lambda(x_1^2-x_2^2)})}{\sqrt{\lambda(x_1^2-x_2^2)}}.$$

How to recover the null controllability with the backstepping method (JMC and H.-M. Nguyen (2015))

From now on we assume that $\lambda > 1$. Looking at the explicit expression of the kernel k, one sees that

(1) $$|k|_{H^1(\Delta)} \leqslant Ce^{C\sqrt{\lambda}},$$

where

(2) $$\Delta := \{(x_1, x_2);\ 0 < x_2 < x_1 < 1\}.$$

The inverse transform of

(3) $$z(x_1) := y(x_1) - \int_0^{x_1} k(x_1, x_2) y(x_2) dx_2,$$

has the form

(4) $$y(x_1) := z(x_1) - \int_0^{x_1} l(x_1, x_2) z(x_2) dx_2.$$

The exact expression of l shows that

(5) $$|l|_{H^1(\Delta)} \leqslant C\lambda.$$

So if we apply the backstepping for λ and during the interval of time $[0, \tau]$, we have

$$(6) \quad |y(\tau)|_{L^2} \leqslant C\lambda |z(\tau)|_{L^2} \leqslant C\lambda e^{-\lambda\tau} |z(0)|_{L^2} \leqslant C\lambda e^{-\lambda\tau} e^{C\sqrt{\lambda}} |y(0)|_{L^2}.$$

Similar estimates holds for the control $y(t, 1)$. Let $T > 0$, and for $n \in \mathbb{N} \setminus \{0, 1\}$, let $t_n = T(1 - 1/n^2)$ and $\lambda_n = n^8$. Let $t_1 := 0$ and $\lambda_1 := 1$. During the interval $[t_n, t_{n+1})$ we apply the feedback law coming from the backstepping with $\lambda := \lambda_n$

Proposition (H.-M. Nguyen and JMC (2015))

$$(7) \quad \lim_{t \to T_-} |y(t, \cdot)|_{L^2} = 0,$$

$$(8) \quad \lim_{t \to T_-} u(t) = 0.$$

Hence this is a new method to prove the null controllability of the heat equation in small time.

The estimates

$$|k|_{H^1(\Delta)} \leqslant Ce^{C\sqrt{\lambda}}, \tag{1}$$

$$|l|_{H^1(\Delta)} \leqslant C\lambda. \tag{2}$$

are crucial for this method. Note that one can find related estimates in G. Lebeau and L. Robbiano (1995) (in every space dimension). Let us consider the case of the control system

$$\left\{\begin{array}{ll} y_t(t,x) = (a(x)y_x(t,x))_x + c(x)y(t,x) & \text{in } (\tau_1,\tau_2)\times[0,1], \\ y(t,0)=0,\ y(t,1)=u(t) & \text{for } t\in(\tau_1,\tau_2), \end{array}\right. \tag{3}$$

and the target system

$$\left\{\begin{array}{ll} z_t(t,x) = (a(x)z_x(t,x))_x + c(x)z(t,x) - \lambda z & \text{in } (\tau_1,\tau_2)\times[0,1], \\ z(t,0)=0,\ y(t,1)=u(t) & \text{for } t\in(\tau_1,\tau_2). \end{array}\right. \tag{4}$$

We assume that $a\in H^2(0,1)$, $a>0$ in $[0,1]$, and that $c\in H^1(0,1)$.

Proposition (H.-M. Nguyen and JMC (2015))

There exists a kernel k which allows to transform the initial y system into the z system and one has, for $\lambda \in [1, +\infty)$,

$$|k|_{H^1(\Delta)} \leqslant Ce^{C\sqrt{\lambda}}, \tag{1}$$

$$|l|_{H^1(\Delta)} \leqslant C\lambda. \tag{2}$$

Remark

Our proof is different from the iterative scheme mentioned above. We interpret the kernel equation on k (and l) as a wave equation defined in $[0,1]^2$. Estimates (1) and (2) follow from an energy type estimate for the wave equation which is somehow nonstandard in the sense that the energy not only contains the gradient of the solutions but also the solutions; the standard energy estimate only gives the exponent λ in (1).

Stabilization in finite time

However the above strategy do not lead to stabilization in finite time. This is due to the fact that $u(t,y)$ is small along the trajectories starting from the time 0 but is quite large for a given y and $t \to T_-$.

We look for time-varying feedback laws $(t, y) \in \mathbb{R} \times L^2(0,1) \mapsto u(t,y) \in \mathbb{R}$ satisfying the following three properties.

$(\mathcal{P}_1)$. The feedback law u is T-periodic with respect to time:

(1) $$u(t,y) = u(t+T, y) \text{ for every } (t,y) \in \mathbb{R} \times L^2(0,1).$$

$(\mathcal{P}_2)$. There exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that

(2) $$t_0 = 0,$$

(3) $$\lim_{n\to+\infty} t_n = T,$$

(4) $$u \text{ is of class } C^1 \text{ in } [t_n, t_{n+1}) \times L^2(0,1) \text{ for every } n \in \mathbb{N}.$$

$(\mathcal{P}_3)$. The map u vanishes on $\mathbb{R} \times \{0\}$ and there exists a continuous function $M : [0,T) \to [0,+\infty)$ such that

(5) $$|u(t,y_2) - u(t,y_1)| \leqslant M(t)|y_2 - y_1|_{L^2} \quad \forall\, (t, y_1, y_2) \in [0,T) \times L^2(0,1) \times L^2(0,1).$$

Proposition

Assume that F satisfies Properties $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, and $(\mathcal{P}_3)$. Let $0 \leqslant s < T$. There exists $T_0 = T_0(s) > 0$ such that, for every $y^0 \in L^2(0,1)$, there exists a unique solution $y \in C^0\big([s, s+T_0); L^2(0,1)\big)$ of

(1)
$$\begin{cases} y_t(t,x) = y_{xx}(t,x) & \text{for } (t,x) \in (s,\tau) \times [0,1], \\ y(t,0) = 0,\ y(t,1) = u(t, y(t,\cdot)) & \text{for } t \in (s,\tau), \\ y(s,\cdot) = y^0 & \text{for } x \in [0,1], \end{cases}$$

Moreover,

(2)
$$|y(t,\cdot)|_{L^2} \leqslant C|y^0|_{L^2} \text{ for } t \in (s, s+T_0),$$

for some positive constant $C = C(s)$ independent of y^0 and the functions $T_0 : [0,T) \to (0,+\infty)$ and $C : [0,T) \to [0,+\infty)$ can be chosen such that, for every $\delta \in (0,T]$,

(3) $\inf\{T_0(s); s \in [0, T-\delta]\} > 0$ *and* $\sup\{C(s); s \in [0, T-\delta]\} < +\infty.$

Proposition

Assume that F satisfies Properties $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, and $(\mathcal{P}_3)$ and that there exist $C>0$ and $\bar{T}\in(0,T)$ such that

$$(\mathcal{P}_4) \qquad |u(t,y)| \leqslant C|y|_{L^2}^{1/2},\ \forall\,(t,y)\in[\bar{T},T)\times L^2(0,1).$$

Then, for every $s\in\mathbb{R}$ and for every $y^0\in L^2(0,1)$, there exists a unique solution $y\in C^0\big([s,+\infty);L^2(0,1)\big)$ of

$$(1) \quad \begin{cases} y_t(t,x)=y_{xx}(t,x) & \text{for } (t,x)\in(s,+\infty)\times[0,1], \\ y(t,0)=0,\ y(t,1)=u(t,y(t,\cdot)) & \text{for } t\in(s,+\infty), \\ y(s,\cdot)=y^0 & \text{for } x\in[0,1]. \end{cases}$$

Notation $\phi(t,s,y^0):=y(t,\cdot)$.

Theorem (JMC and H.-M. Nguyen (2015))

Let $T>0$ *and* $\Gamma>0$. *There exists a time-varying feedback laws* $(t,y)\in\mathbb{R}\times L^2(0,1)\mapsto u(t,y)\in\mathbb{R}$ *satisfying Properties* ($\mathcal{P}_1$), ($\mathcal{P}_2$), *and* ($\mathcal{P}_3$) *such that*

$$\text{(1)}\quad \Phi(t+2T,t,y^0)=0 \text{ for every } (t,y^0)\in\mathbb{R}\times L^2(0,1) \text{ such that } |y^0|_{L^2}\leqslant\Gamma,$$

and such that the following uniform stability condition

$$\text{(2)}\quad \begin{cases} \forall\,\varepsilon>0,\ \exists\eta>0 \text{ such that, } \forall\,t'\in\mathbb{R},\ \forall\,t\in[t',+\infty),\\ \text{and } \forall\,y^0\in L^2(0,1),\ \big(|y^0|_{L^2}\leqslant\eta\big)\Rightarrow\big(|\Phi(t,t',y^0)|\leqslant\varepsilon\big)\end{cases}$$

holds.

Construction of u

For $n \in \mathbb{N}$, let λ_n and t_n be defined by

(1) $$\lambda_n = (n+1)^8 \text{ for every } n \in \mathbb{N},$$

(2) $$t_0 = 0,$$

(3) $$t_n = T\left(1 - \frac{1}{2n^2}\right) \text{ for every } n \in \mathbb{N} \setminus \{0\}.$$

Let α be a real number such that

(4) $$4 < \alpha < 5.$$

Let $(\mu_n)_{n \in \mathbb{N}}$ be defined by

(5) $$\mu_n := e^{-n^\alpha}, \forall\, n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, we choose a function $\varphi_n \in C^1(\mathbb{R})$ such that $0 \leqslant \varphi_n \leqslant 1$, $\varphi_n(s) = 1$ for $s \leqslant \mu_n$ and $\varphi_n(s) = 0$ if $s \geq 2\mu_n$. Fix N a large positive integer.

We define u in the following way for $t_n \leqslant t < t_{n+1}$,

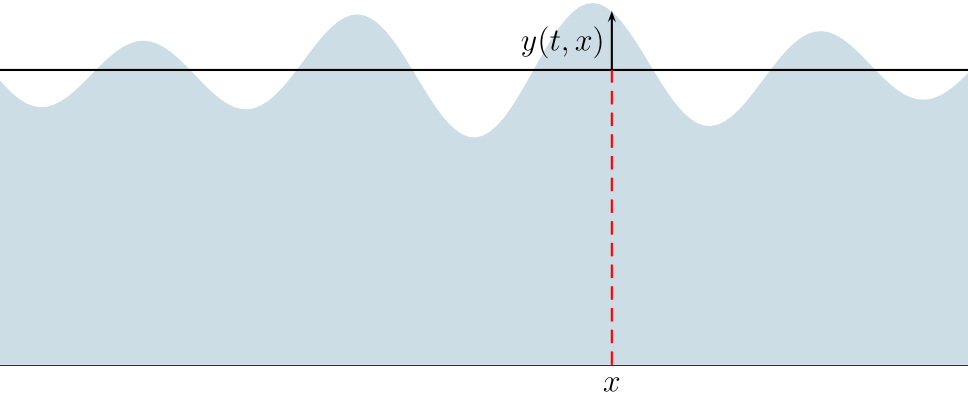
(1) $$u(t,y) := \int_0^1 k_n(1,x)y(x)\,dx, \ \forall\, t \in [t_n, t_{n+1}) \text{ with } n \leqslant N-1,$$

(2)
$$u(t,y) := \varphi_n\big(\|y\|_{L^2}\big) \int_0^1 k_n(1,x)y(x)\,dx, \ \forall\, t \in [t_n, t_{n+1}) \text{ with } n \geq N.$$

Remark

Using G. Lebeau-L. Robbiano (1995) instead of the backstepping approach, one can get the existence of the feedback law in dimension $n > 1$. Open problem: Is it possible to stabilize in small time the heat equation by means of stationary feedback laws? May be one can try to use the kernel $k_{\lambda(y)}$ instead of the kernel $k_{\lambda(t)}$ with $\lambda(y)$ converging to $+\infty$ as $y \to 0$.

$y(t,x)$



A KdV control system

(1) $$y_t + y_x + y_{xxx} + yy_x = 0,\ t \in [0, T],\ x \in [0, L],$$

(2) $$y(t, 0) = y(t, L) = 0,\ y_x(t, L) = u(t),\ t \in [0, T].$$

where, at time $t \in [0, T]$, the control is $u \in \mathbb{R}$ and the state is $y(t, \cdot) \in L^2(0, L)$.

Definition of the local controllability

Our control system is

(1) $$y_t + y_x + y_{xxx} + yy_x = 0,\ t \in [0,T],\ x \in [0,L],$$

(2) $$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T].$$

Definition of the local controllability of (1)-(2)

Let $T > 0$. The control system (1)-(2) is locally controllable in time T if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $y^0 \in L^2(0,L)$ and for every $y^1 \in L^2(0,L)$ satisfying $|y^0|_{L^2(0,L)} < \eta$ and $|y^1|_{L^2(0,L)} < \eta$, there exists $u \in L^2(0,T)$ satisfying $|u|_{L^2(0,T)} < \varepsilon$ such that the solution $y \in C^0([0,T]; L^2(0,L))$ of (1)-(2) satisfying the initial condition $y(0,x) = y^0(x)$ is such that $y(T,x) = y^1(x)$.

Question: Let $T > 0$, is is true that (1)-(2) is locally controllable.

Controllability of the linearized control system

The linearized control system (around 0) is

(1) $$y_t + y_x + y_{xxx} = 0,\ t \in [0,T],\ x \in [0,L],$$

(2) $$y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T].$$

where, at time $t \in [0,T]$, the control is $u \in \mathbb{R}$ and the state is $y(t,\cdot) \in L^2(0,L)$.

Definition of the controllability of (1)-(2)

Let $T > 0$. The linear control system (1)-(2) is controllable in time T if, for every $y^0 \in L^2(0,L)$ and for every $y^1 \in L^2(0,L)$, there exists $u \in L^2(0,T)$ such that the solution $y \in C^0([0,T];L^2(0,L))$ of (1)-(2) satisfying the initial condition $y(0,x) = y^0(x)$ is such that $y(T,x) = y^1(x)$.

Controllability of the linearized control system

Theorem (L. Rosier (1997))

For every $T > 0$, the linearized control system is controllable in time T if and only

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},\, k \in \mathbb{N}^*,\, l \in \mathbb{N}^* \right\}.$$

Application to the nonlinear system

Theorem (L. Rosier (1997))

For every $T > 0$, the KdV control system is locally controllable in time T if $L \notin \mathcal{N}$.

Question: Does one have controllability if $L \in \mathcal{N}$?

Controllability when $L \in \mathcal{N}$

Theorem (JMC and E. Crépeau (2004))

If $L = 2\pi$ (which is in $\mathcal{N}$: take $k = l = 1$), for every $T > 0$ the KdV control system is locally controllable in time T.

Theorem (E. Cerpa (2007), E. Cerpa and E. Crépeau (2008))

For every $L \in \mathcal{N}$, there exists $T > 0$ such that the KdV control system is locally controllable in time T.

Stabilization and damping

Let us consider the linear KdV control system

(1) $y_t + y_x + y_{xxx} = 0,\ y(t,0) = y(t,L) = 0,\ y_x(t,L) - y_x(t,0) = u(t),$

where, at time $t \geqslant 0$, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $u(t) \in \mathbb{R}$. Simple integrations by parts show that, along the solution of (1), one has

(2) $$\frac{d}{dt}\int_0^L y^2 dx = u(y_x(t,L) + y_x(t,0)).$$

Hence it is tempting to consider the feedback law

(3) $$u(y) = -M(y_x(t,L) + y_x(t,0)),$$

with $M > 0$. It has been proved by G.P. Menzala, C.F. Vasconcellos and E.Zuazua in 2002 that this feedback law leads to exponential stability of the closed loop system if the length is not critical *even for the nonlinear KdV equation*. Unfortunately letting $M \to +\infty$ do not lead to rapid exponential stabilization, i.e. to an exponential decay rate as large as one wants.

A second KdV control system

(1) $$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\ t \in (0,T),\ x \in (0,L), \\ y(t,0) = u(t),\ y(t,L) = 0,\ y_x(t,L) = 0,\ t \in (0,T). \end{cases}$$

The control system (1) is locally null controllable: L. Rosier (2004).

Theorem (E. Cerpa and JMC (2013))

For every $\lambda > 0$, there exist $C > 0$, $r > 0$ and a feedback law $y \mapsto u(y)$ such that, for this feedback law,

(2) $$\left(|y(0)|_{L^2(0,L)} \leqslant r\right) \Rightarrow \left(|y(t)|^2_{L^2(0,L)} \leqslant Ce^{-\lambda t}|y(0)|^2_{L^2(0,L)},\ \forall t > 0.\right)$$

Proof: With M. Krstic's backstepping approach

We look for a transformation $y \in L^2(0,L) \mapsto z \in L^2(0,L)$ defined by

(1) $$z(x_1) := y(x_1) - \int_{x_1}^{L} k(x_1,x_2)y(x_2)dx_2,$$

such that the trajectory y of

(2) $$y_t + y_x + y_{xxx} = 0,\ y(t,0) = u(t),\ y(t,L) = 0,\ y_x(t,L) = 0,$$

with the feedback law $u(t) := \int_0^L k(0,x_2)y(t,x_2)dx_2$ is mapped into the trajectory $z = z(t,x)$, solution of the linear system

(3) $$z_t + z_x + z_{xxx} + \lambda z = 0,\ z(t,0) = 0,\ z(t,L) = 0,\ z_x(t,L) = 0.$$

Note that, for (3), one has (just multiply (3) by z and do some integrations by parts):

(4) $$|z(t)|^2_{L^2(0,L)} \leqslant e^{-\lambda t}|z(0)|^2_{L^2(0,L)},\ \forall t \geqslant 0.$$

Kernel equation

This property for the transformation $y \mapsto z$ holds if (and only if)

$$(1) \quad \left\{ \begin{array}{rcll} k_{111} + k_1 + k_{222} + k_2 & = & -\lambda k, & \text{for } 0 < x_1 < x_2 < L, \\ k(x_1, L) & = & 0, & \text{in } [0, L], \\ k(x_1, x_1) & = & 0, & \text{in } [0, L], \\ k_1(x_1, x_1) & = & \dfrac{\lambda}{3}(L - x_1), & \text{in } [0, L]. \end{array} \right.$$

with $k_i := \partial_{x_i} k$, $k_{iii} := \partial^3_{x_i x_i x_i} k$. Moreover, if k is smooth enough (Lipschitz is sufficient), one can check that the same feedback law provides for the initial nonlinear KdV control system (local) asymptotic stability with an exponential decay rate at least equal to λ.

Proof of the existence of k

Let us make the following change of variables $t = x_2 - x_1$, $s = x_1 + x_2$ and define $G(s,t) := k(x_1, x_2)$ on $\mathcal{T}_0 := \{(s,t);\, t \in [0,L], s \in [t, 2L-t]\}$. Then k satisfies the kernel equation if and only if

$$(1) \qquad \begin{cases} 6G_{tts} + 2G_{sss} + 2G_s = -\lambda G, & \text{in } \mathcal{T}_0, \\ G(s, 2L-s) = 0, & \text{in } [L, 2L], \\ G(s,0) = 0, & \text{in } [0, 2L], \\ G_t(s,0) = \dfrac{\lambda}{6}(s-2L), & \text{in } [0,2L]. \end{cases}$$

We transform this equation by integrating twice with respect to t and then once with respect to s. We get that (1) is equivalent to

$$(2) \quad G(s,t) = -\frac{\lambda t}{6}(2L-t-s) + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss} + 2G_s + \lambda G\Big)(\eta,\xi)\,d\xi\, d\tau\, d\eta.$$

To prove that such a function $G = G(s,t)$ exists, we use a method of successive approximations. We define

$$g^1(s,t) = -\lambda(2L - t - s)/6 \tag{1}$$

and define the recursive formula as follows,

$$g^{n+1}(s,t) = (1/6)\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2g^n_{sss} + 2g^n_s + \lambda g^n\Big)(\eta,\xi)\,d\xi\, d\tau\, d\eta. \tag{2}$$

Performing some computations, we get for instance

$$g^2(s,t) = 1/(108)\Big\{t^3\big(\lambda - \lambda^2 L + \frac{\lambda^2 t}{4}\big)\big(2L - t - s\big) + \frac{t^3\lambda^2}{4}\big[(2L-t)^2 - s^2\big]\Big\}, \tag{3}$$

More generally, one has the following formula

(1) $$g^k(s,t) = \sum_{i=1}^{k} \left(a_k^i t^{2k-1} + b_k^i t^{2k}\right)\left[(2L-t)^i - s^i\right],$$

where the coefficients satisfy $b_k^k = 0$ and, more importantly, there exist positive constants M, B such that, for any $k \geq 1$ and any $(s,t) \in \mathcal{T}_0$

(2) $$\left|g^k(s,t)\right| \leqslant M \frac{B^k}{(2k)!}(t^{2k-1} + t^{2k}).$$

This implies that the series $\sum_{n=1}^{\infty} g^n(s,t)$ is uniformly convergent in $\mathcal{T}_0$. Therefore the series defines a continuous function $G : \mathcal{T}_0 \to \mathbb{R}$

(3) $$G(s,t) = \sum_{n=1}^{\infty} g^n(s,t).$$

Then, one checks that G is a solution of our integral equation and that is C^1 on $\mathcal{T}_0$.

Return to the initial KdV control system

$$
(1) \qquad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \; t \in (0,T), \; x \in (0,L), \\ y(t,0) = 0, \; y(t,L) = 0, \; y_x(t,L) = u(t) \, t \in (0,T). \end{cases}
$$

We assume that

$$
(2) \qquad L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, \; k \in \mathbb{N}^*, \; l \in \mathbb{N}^* \right\}.
$$

Then the linearized control system around 0 is controllable and the nonlinear control system is locally controllable in small-time. We are interested in the rapid exponential stabilization of the nonlinear system. Unfortunately the backstepping approach is not working.

Rapid exponential stabilization of the initial KdV-control system

Theorem (JMC and Q. Lü (2013))

Let us assume that $L \notin \mathcal{N}$ *For every* $\lambda > 0$, *there exist* $C > 0$, $r > 0$ *and a feedback law* $y \mapsto u(y)$ *such that, for this feedback law,*

$$(1) \quad \left(|y(0)|_{L^2(0,L)} \leqslant r\right) \Rightarrow \left(|y(t)|^2_{L^2(0,L)} \leqslant Ce^{-\lambda t}|y(0)|^2_{L^2(0,L)}, \ \forall t > 0.\right)$$

Remarks 1. For the linearized KdV-Rosier control system, such a result was obtained before by E. Cerpa and E. Crépeau in 2009 for the H^1 norm. But it seems that, due to some regularity issues, this does not allow to get the rapid exponential stabilization for the nonlinear KdV-Rosier system.
2. The feedback law $u = 0$ already provides (local) exponential stability if $L \notin \mathcal{N}$: G. Perla Menzala, C. Vasconcellos and E. Zuazua (2002).
3. If $L = 2\pi$ (the first critical length) and $u = 0$, one does not have asymptotic stability for the linearized system; however, one has asymptotic stability for the nonlinear system: JMC, J. Chu and P. Shang (2012). The decay rate is not exponential (it is $1/\sqrt{t}$).

Proof of the rapid exponential stabilizability

The backstepping approach does not work. We need to use more general transformations: $y \in L^2(0,L) \mapsto z \in L^2(0,L)$ is now defined by

$$(1) \qquad z(x_1) := y(x_1) - \int_0^L k(x_1,x_2)y(x_2)dx_2.$$

(Every linear transformation $y \in L^2(0,L) \mapsto z \in L^2(0,L)$ can been written in this form). Again, we want that the trajectory y of

$$(2) \qquad y_t + y_x + y_{xxx} = 0,\ y(t,0) = 0,\ y(t,L) = 0,\ y_x(t,L) = u(t),$$

with the feedback law $u(t) := \int_0^L k_{x_1}(0,x_2)y(t,x_2)dx_2$ is mapped into the trajectory $z = z(t,x)$, solution of the linear system

$$(3) \qquad z_t + z_x + z_{xxx} + \lambda z = 0,\ z(t,0) = 0,\ z(t,L) = 0,\ z_x(t,L) = 0.$$

Kernel equation

This property for the transformation $y \mapsto z$ holds if (and only if)

$$
(1) \qquad \begin{cases} k_{111} + k_1 + k_{222} + k_2 + \lambda k = \lambda \delta(x_1 - x_2), & \text{on } (0,L)^2, \\ k(x_1,0) = k(x_1,L) = k_2(x_1,0) = k_2(x_1,L) & \text{on } (0,L), \\ k(0,x_2) = k(L,x_2) = 0 & \text{on } (0,L), \end{cases}
$$

where $\delta(x_1 - x_2)$ is the Dirac mass on the diagonal of the square $[0,L] \times [0,L]$.
Next step: Prove the existence of a solution to the kernel equation (1).

How to prove the existence of k

Let us define an unbounded linear operator $A : D(A) \subset L^2(0, L)$ $\to L^2(0, L)$ as follows.

$$D(A) := \{\varphi;\ \varphi \in H^3(0, L),\ \varphi(0) = \varphi(L) = 0,\ \varphi_x(0) = \varphi_x(L)\}, \quad (1)$$

$$A\varphi := -\varphi_{xxx} - \varphi_x. \quad (2)$$

The operator A is a skew-adjoint operator and has compact resolvent. Furthermore, since $L \notin \mathcal{N}$, $L \notin 2\pi\mathbb{N}$, which, as one easily checks, implies that 0 is not an eigenvalue of A. Denote by $\{i\mu_j\}_{j\in\mathbb{Z}}$, $\mu_j \in \mathbb{R}$, the eigenvalues of A, which are organized in the following way:

$$\ldots \leqslant \mu_{-2} \leqslant \mu_{-1} < 0 < \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots. \quad (3)$$

Since the control is of dimension 1 and the linearized control system is controllable, all these eigenvalues are simple. Let us write $\{\varphi_j\}_{j\in\mathbb{Z}}$ for the corresponding eigenfunctions with $|\varphi_j|_{L^2(0,L)} = 1$ $(j \in \mathbb{Z})$. It is well known that $\{\varphi_j\}_{j\in\mathbb{Z}}$ constitutes an orthonormal basis of $L^2(0, L)$.

For $j \in \mathbb{Z}$, let $\psi_j : [0, L] \to \mathbb{C}$ be the solution of

$$\begin{cases} \psi_j''' + \psi_j' + \lambda\psi_j - i\mu_j\psi_j = 0 \text{ in } (0, L), \\ \psi_j(0) = \psi_j(L) = 0, \\ \psi_j'(L) - \psi_j'(0) = 1. \end{cases} \tag{1}$$

The idea is to search k in the following form

$$k(x_1, x_2) = \sum_{j \in \mathbb{Z}} c_j \psi_j(x_1)\varphi_j(x_2). \tag{2}$$

...
Then prove that $y \in L^2(0, L) \mapsto z \in L^2(0, L)$ defined by

$$z(x_1) := y(x_1) - \int_0^L k(x_1, x_2)y(x_2)dx_2. \tag{3}$$

is invertible.

An open problem and a work in progress

- An open problem: Try to get a good enough estimate on k (and on the l for the inverse transform) in order to recover the local controllability in small time and prove the stabilizability in finite time by means of time-varying feedback laws.
- Work in progress: One dimensional Schrödinger control systems (JMC L. Gagnon, M. Morancey)