Control and stabilization of degenerate evolution equations in one space dimension

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CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Outline

1. Control of degenerate parabolic equations
   - Locally distributed control
   - Boundary control

2. Control of degenerate hyperbolic equations
   - Boundary observability
   - Boundary controllability
   - Boundary stabilization

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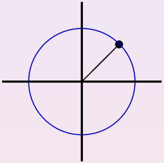
Example: Budyko-Sellers model

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = q(t,x)\,\beta(u) - \gamma(u) & x \in (-1,1) \\ (1-x^2)u_{x|_{x=\pm 1}} = 0 \end{cases}$$



- $u(t,x)$= sea-level zonally averaged temperature
- $q(t,x)$= solar input
- $\beta(u)$= co-albedo
- $\gamma(u)$= outgoing infrared radiation

Degenerate parabolic problems in 1D

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = f & \textit{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) & u(t,1) = 0 \ + \text{ b.c. at } \ x = 0 \end{cases}$$

$u_0 \in L^2(0,1)\,,\ f \in L^2(Q_T)$

weakly degenerate case $1/a \in L^1(0,1)$

$$u(t,0) = 0$$

strongly degenerate case $1/a \notin L^1(0,1)$

$$\lim_{x\downarrow 0} a(x)u_x(t,x) = 0$$

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The space $H^1_a(0,1)$

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$$H^1_a(0,1) = \Big\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2 \, dx < \infty \Big\}$$

- weakly degenerate case $\boxed{1/a \in L^1(0,1)}$

$$H^1_a(0,1) \subset C([0,1])$$

with compact embedding

$$H^1_{a,0}(0,1) = \Big\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2 \, dx < \infty \ \& \ u(0) = 0 = u(1) \Big\}$$

- strongly degenerate case $\boxed{1/a \notin L^1(0,1)}$

$$H^1_a(0,1) \not\subset L^\infty(0,1)$$

because for $a(x) = x$ one has $u(x) = \log\big|\log(2x)\big| \in H^1_a(0,1)$

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The degenerate operator as a generator

$$H^2_a(0,1) = \Big\{ u \in H^1_a(0,1) \;\big|\; au_x \in H^1(0,1) \Big\}$$

- $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$

$$\begin{cases} D(\mathcal{A}) = H^2_a(0,1) \cap H^1_{a,0}(0,1) \\ \mathcal{A}u = \big(au_x\big)_x \end{cases}$$

  is densely defined, self-adjoint, and dissipative

- in the strongly degenerate case $u \in D(\mathcal{A}) \implies au_x \overset{(x\to 0)}{\longrightarrow} 0$

Hardy's and Poincare's inequalities

Let $a(x) = x^\alpha, \alpha \in [0, 2)$. Then any $v \in H^1_{a,0}(0, 1)$ satisfies

- Hardy's inequality

$$\frac{(1-\alpha)^2}{4} \int_0^1 x^{\alpha-2} v^2 \, dx \leqslant \int_0^1 x^\alpha v_x^2 \, dx$$

- Poincare's inequality

$$\int_0^1 v^2 \, dx \leqslant \min\left\{4, \frac{1}{2-\alpha}\right\} \int_0^1 x^\alpha v_x^2 \, dx$$

Similar results for general a provided that

$$\mu_a := \limsup_{x \downarrow 0} \frac{x|a'(x)|}{a(x)} < 2$$

Well-posedness

- $\mathcal{A}$ generates analytic semigroup in $L^2(0,1)$
- $u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}f(s)ds$ unique solution

$$u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H_a^1(0,1))$$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = f & \text{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) \\ u(t,1) = 0 \text{ and } \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ au_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

- maximal regularity

$$u_0 \in H_a^1(0,1) \implies u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

(needed to justify integration by parts)

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Null controllability: locally distributed control

$$u^f \leftrightarrow \begin{cases} u_t - \big(a(x)u_x\big)_x = \chi_\omega f & \text{in } Q_T = (0,T)\times(0,1) \\ u(0,x) = u_0(x) \\ u(t,1) = 0 \ \text{ and } & \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ au_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases} \qquad (C)$$

- null controllable in time $T > 0$

$$\forall u_0 \in L^2(0,1) \quad \exists f \in L^2(Q_T) \ : \ \begin{cases} u^f(T,\cdot) \equiv 0 \\ \int_{Q_T} |f|^2 \leqslant C_T \int_0^1 |u_0|^2 \end{cases}$$

- observability on $(0,T)\times\omega$

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The simplest example of degeneracy

$\omega = (a, b) \subset\subset (0, 1)$

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Theorem (C – Martinez – Vancostenoble, 2008)

null controllability $\begin{cases} \textit{false} & \alpha \geq 2 \quad (\to \textit{regional null controllability}) \\ \textit{true} & 0 \leq \alpha < 2 \begin{cases} \textit{any b.c.} & 0 \leq \alpha < 1 & \textit{weak} \\ \textit{Neumann b.c.} & 1 \leq \alpha < 2 & \textit{strong} \end{cases} \end{cases}$

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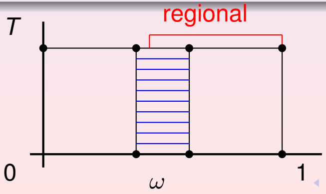

Lack of null controllability for $\alpha \geq 2$

The classical Liouville change of variable

$$y(x) = \int_x^1 \frac{ds}{s^{\alpha/2}} \qquad U(y(x), t) = x^{\alpha/4} u(x, t)$$

transforms the equation into

$$U_t - U_{yy} + c_\alpha(y) U = \chi_{\widetilde{\omega}} F \qquad 0 < y < \infty$$

with $\widetilde{\omega} =]\widetilde{b}, \widetilde{a}[$ bounded and $c(y) = \frac{\alpha(3\alpha-4)}{4[2+(\alpha-2)y]^2}$ bounded for $\alpha \geq 2$

which is NOT null controllable

Ref: Micu, Zuazuza (2001) and Escauriaza, Seregin, Sverak (2003, 2004)

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Carleman estimate $0 < \alpha < 2$

$$\begin{cases} w_t + \left(x^\alpha w_x\right)_x = f \quad \text{in } Q_T \\ w(t,1) = 0 \ \text{ and } \begin{cases} w(t,0) = 0 & \text{weakly degenerate} \\ x^\alpha w_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \end{cases}$$

let $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \qquad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C – Martinez – Vancostenoble, 2008)

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^\alpha w_x^2 + \tau^3\theta^3 x^{2-\alpha} w^2\right) e^{2\tau\varphi}\, dxdt$$

$$\leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}\, dxdt + C \int_0^T \left\{\tau\theta w_x^2 e^{2\tau\varphi}\right\}_{|x=1} dt$$

Observability on $(0, T) \times \omega$

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- $t \mapsto \int_0^1 x^\alpha v_x^2 \, dx$ increasing
- integrate & use Carleman's estimate

$$\begin{aligned} \int_0^1 x^\alpha v_x^2(x,0)\,dx &\leqslant \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x^2(x,t)\,dxdt \\ &\leqslant C_T \int\int_{Q_T} \theta(t) x^\alpha v_x^2(x,t) e^{2s\phi(x,t)}\,dxdt \\ &\leqslant C_T \int_0^T\int_\omega v^2(x,t)\,dxdt \end{aligned}$$

- use Hardy's inequality

$$\int_0^1 x^{\alpha-2} v^2(x,0)\,dx \le \int_0^1 x^\alpha v_x^2(0,x)\,dx \le C\int_0^T\int_\omega v^2(x,t)dxdt$$

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Further results in 1D: locally distributed control

For general a

$$\limsup_{x\downarrow 0} \frac{x|a'(x)|}{a(x)} < 2$$

- divergence form
  - Martinez – Vancostenoble (2006) $\partial_t u - \partial_x\big(a(x)\partial_x u\big) = \chi_\omega f$
  - Alabau – C – Fragnelli (2006) $\partial_t u - \partial_x\big(a(x)\partial_x u\big) + g(u) = \chi_\omega f$
  - Flores – de Teresa (2010) $\partial_t u - \partial_x\big(x^\alpha \partial_x u\big) + x^\beta b(t,x)\partial_x u = \chi_\omega f$
- nondivergence form C – Fragnelli – Rocchetti (2007, 2008)

$$\partial_t u - a(x)\partial_x^2 u - b(x)\partial_x u = \chi_\omega f$$

- degenerate/singular Vancostenoble – Zuazua (2008), Vancostenoble (2009)

$$\partial_t u - \partial_x\big(x^\alpha \partial_x u\big) - \frac{\lambda}{x^\beta} u = \chi_\omega f$$

Further results

- C – de Teresa (2009)   cascade 2×2

$$\omega \cap \mathcal{O} \neq \varnothing \qquad \begin{cases} \partial_t u - \partial_x(x^\alpha \partial_x u) + c(t,x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x(x^\alpha \partial_x v) + d(t,x)v = \chi_{\mathcal{O}} u \end{cases}$$

- Ben Hassi – Ammar Khodja – Hajjaj – Maniar (2011, 2013)

$$\omega \cap \mathcal{O} \neq \varnothing \qquad \begin{cases} \partial_t u - \partial_x\big(a_1(x)\partial_x u\big) + c(t,x)u = \xi + \chi_\omega h \\ \partial_t v - \partial_x\big(a_2(x)\partial_x v\big) + d(t,x)v = \chi_{\mathcal{O}} u \end{cases}$$

- inverse problems C – Tort – Yamamoto (2010)
- interior degeneracy Fragnelli – Mugnai (2013)
- Neumann boundary conditions and inverse problems
  Boutaayamou – Fragnelli – Maniar (2014)

Outline

1. Control of degenerate parabolic equations
   - Locally distributed control
   - Boundary control
2. Control of degenerate hyperbolic equations
   - Boundary observability
   - Boundary controllability
   - Boundary stabilization

Boundary control at $x = 1$

$0 \leqslant \alpha < 2\,, \; T > 0$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in}(0,1)\times(0,T) \\ u(1,t) = g(t) & \text{and} \quad \begin{cases} u(t,0) = 0 & \text{weakly degenerate} \\ au_x(t,\cdot)_{|x=0} = 0 & \text{strongly degenerate} \end{cases} \\ u(x,0) = u_0(x) \end{cases}$$

follows from locally distributed result, but can also be derived by the flatness approach

- Martin, Roiser, Rouchon ($0 \leqslant \alpha < 1$)
- Moyano ($1 \leqslant \alpha < 2$)

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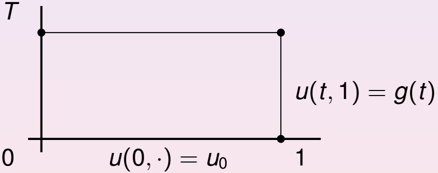

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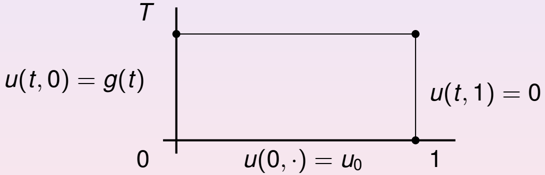

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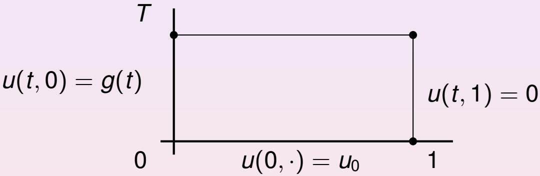

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The associated eigenvalue problem

$$\begin{cases} -(x^\alpha \phi'(x))' = \lambda \phi(x) & x \in (0,1) \\ \phi(0) = 0, \quad \phi(1) = 0 \end{cases} \tag{1}$$

Let

$$\nu_\alpha = \frac{1-\alpha}{2-\alpha} \qquad \text{and} \qquad \kappa_\alpha = \frac{2-\alpha}{2}$$

Denote by J_ν the Bessel function of first kind of order ν:

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} \qquad y \in (0,+\infty)$$

and by $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \ldots$ the sequence of positive zeros of J_ν

Then the eigenvalues of problem (1) are given by

$$\lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2 \qquad \forall n \geq 1 \tag{2}$$

and the corresponding normalized eigenfunctions take the form

$$\Phi_{\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) \qquad x \in (0,1)$$

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A sufficient condition for attainability

For any $v \in L^2(0,1)$ let $\widehat{v}_{\alpha,n} = \int_0^1 v(x)\Phi_{\alpha,n}(x)\,dx$

For any $K > 0$ define $\mathcal{P}_{\alpha,K} = \left\{ v \in L^2(0,1) \ : \ \sum_{n\geq 1} n^{3/2} |\widehat{v}_{\alpha,n}| e^{K\kappa_\alpha \pi n} < \infty \right\}$

Theorem (C–Martinez-Vancostenoble)

There exists $K^* > 0$ such that, given any $\alpha \in [0,1)$, $T > 0$, $u_0 \in L^2(0,1)$, and $v \in \mathcal{P}_{\alpha,K^*}$ one can find a control $g \in H^1(0,T)$ such that the solution of

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- Theorem implies approximate and null controllability
- $\mathcal{P}_{\alpha,K^*}$ is independent of T, see also Seidman (1979)
- Similar results for the heat equation
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Regularity of attainable states

Fattorini and Russel (1971) noted that, for the heat equation, any attainable state is the restriction to $[0, 1]$ of an analytic function

Proposition (C–Martinez-Vancostenoble)

Let $(\mu_n)_{n\geq 1}$ be such that, for some $K > 0$, the sequence $(\mu_n e^{Kn})_{n\geq 1}$ is bounded. Then

$$v(x) := \sum_{n=1}^{\infty} \mu_n \Phi_{\alpha,n}(x) \qquad (x \in [0, 1])$$

has the following property: there exists an even function F_α, holomorphic in the strip $\left\{z \in \mathbb{C} \;:\; |\Im z| < \frac{K}{\pi}\right\}$ such that $v(x) = x^{1-\alpha} F_\alpha(x^{\kappa_\alpha})$

Corollary

If $v \in \mathcal{P}_{\alpha,K^*}$, then (v is attainable and) there exists an even function F_α, holomorphic in the strip $\left\{z \in \mathbb{C} \;:\; |\Im z| < \frac{K^*}{\pi}\right\}$ such that

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The question of uniformly attainable targets

Recall that, for every $\alpha \in [0,1)$,

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The cost of null controllability

We want to measure the cost to steer any u_0 to 0 in time T with respect to α
Given $u_0 \in L^2(0,1)$ let

$$\mathcal{G}(\alpha, u_0) := \{g \in H^1(0,T) \ : \ u(T,\cdot) = 0\}$$

where

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } (0,1) \times (0,T) \\ u(0,t) = g(t), & u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Theorem (C–Martinez-Vancostenoble)

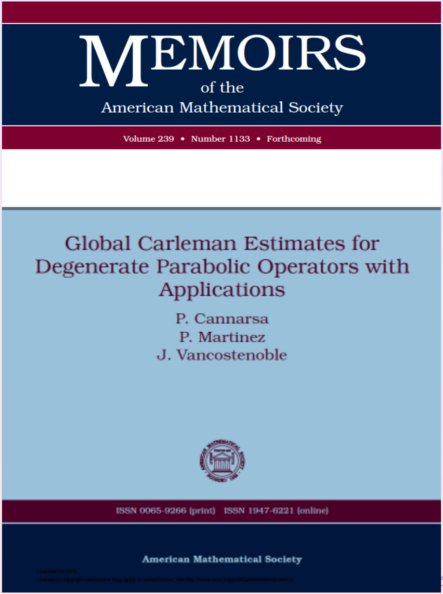
(a) *There exists $M_1(u_0)$ and M_2, independent of α, such that*

$$\frac{M_1(u_0)}{1-\alpha} \le \inf_{g \in \mathcal{G}(\alpha, u_0)} \|g\|_{H^1(0,T)} \le \frac{M_2}{1-\alpha} \|u_0\|_{L^2}$$

(b) *There exist $M_1, M_2 > 0$, independent of α, such that*

$$\frac{M_1}{1-\alpha} \le \sup_{\|u_0\|=1} \inf_{g \in \mathcal{G}(\alpha, u_0)} \|g\|_{H^1(0,T)} \le \frac{M_2}{1-\alpha}$$

Higher dimension



Degenerate wave equations in 1D

From now on: joint work with **F. Alabau-Boussouira** and **G. Leugering**

$$\begin{cases} u_{tt} - \big(a(x)u_x\big)_x = 0 & t > 0 \quad x \in (0,1) \\ u(0,x) = u_0(x) \quad u_t(0,x) = u_1(x) & x \in (0,1) \end{cases}$$

We keep using the degeneracy parameter μ_a as

$$\mu_a = \limsup_{x \downarrow 0} \frac{x|a'(x)|}{a(x)}$$

The above wave equation is

- weakly degenerate if $0 \leqslant \mu_a < 1$
- strongly degenerate if $1 \leqslant \mu_a < 2$

We can impose the boundary conditions $\quad u(t,1) = 0 \quad$ and

$$\begin{cases} u(t,0) = 0 & \text{if } 0 \leqslant \mu_a < 1 \\ \lim_{x\downarrow 0} a(x)\, u_x(t,x) = 0 & \text{if } 1 \leqslant \mu_a < 2 \end{cases}$$

Conservation of the energy

Given the solution u of

$$\begin{cases} u_{tt} - \big(a(x)u_x\big)_x = 0 & t > 0 \quad x \in (0,1) \\ u(0,x) = u_0(x) \quad u_t(0,x) = u_1(x) & x \in (0,1) \end{cases}$$

with initial conditions $(u_0, u_1) \in H^1_a(0,1) \times L^2(0,1)$ and boundary conditions

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we define the energy of u by

$$E_u(t) := \frac{1}{2} \int_0^1 \left\{ u_t^2(t,x) + a(x) u_x^2(t,x) \right\} dx$$

Proposition

$$E_u(t) = E_u(0) \quad \forall\, t \geqslant 0$$

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Lemma (direct inequality)

For every $T \geqslant 0$

$$\int_0^T u_x^2(t,1)dt \leqslant C_a(T)E_u(0)$$

for some constant $C_a(T) > 0$

The proof uses the multiplier xu_x

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   - Boundary controllability
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Observability

The degenerate wave system is said to be *observable in time* $T > 0$ if there exists a constant $C > 0$ such that for any $(u_0, u_1) \in H^1_a(0,1) \times L^2(0,1)$ the solution satisfies

$$\int_0^T u_x^2(t,1)\,dt \geqslant C\,E_u(0)$$

Theorem

Assume

$$\mu_a = \sup_{0<x\leqslant 1} \frac{x|a'(x)|}{a(x)} < 2$$

Then, for every $T \geqslant 0$, the solution satisfies

$$a(1)\int_0^T u_x^2(t,1)\,dt \geqslant \left\{(2-\mu_a)T - \frac{4}{\min\{1, a(1)\}} - 2\,\mu_a\sqrt{C_a}\right\}E_u(0)$$

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Observability cost

Definition

- *Any constant C satisfying the observability inequality is called an observability constant for the degenerate wave equation in time T*
- *The supremum of all observability constants for the degenerate wave system is denoted by C_T*
- *The inverse $c_T = 1/C_T$ is sometimes called the cost of observability (or the cost of control) in time T*

Remark

From this definition we have also that the degenerate wave system is observable if

$$C_T = \inf_{(u_0,u_1)\neq(0,0)} \frac{\int_0^T u_x^2(t,1)\,dt}{E_u(0)} > 0$$

Observability time

By the lower bound

$$a(1)\int_0^T u_x^2(t,1)\,dt \geqslant \Big\{(2-\mu_a)T - \frac{4}{\min\{1,a(1)\}} - 2\,\mu_a\sqrt{C_a}\Big\}E_u(0)$$

we obtain

Corollary

Assume $\mu_a < 2$. Then the degenerate wave system is observable in time T if

$$T > T_a := \frac{1}{(2-\mu_a)}\Big(\frac{4}{\min\{1,a(1)\}} + 2\,\mu_a\sqrt{C_a}\Big)$$

In this case

$$C_T \geqslant \frac{1}{a(1)}\Big\{(2-\mu_a)T - \frac{4}{\min\{1,a(1)\}} - 2\,\mu_a\sqrt{C_a}\Big\}$$

Optimal observability time for $a(x) = x^\theta$

- Gueye 2014: $0 \leqslant \theta < 1$ and observation at $x = 0$
- M. Zhang-H. Gao: $0 \leqslant \theta < 2$ and observation at $x = 1$

Failure of observability

Example

Assume that $a(x) = x^\theta$, $x \in [0,1]$ *with* $\theta \geqslant 2$. *Then* $\mu_a = \theta \geqslant 2$.

Consider the degenerate wave system

$$\begin{cases} u_{tt} - \left(x^\theta u_x\right)_x = 0 & \text{in }]0,T[\times]0,1[\\ u(t,1) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} x^\theta\, u_x(t,x) = 0 & 0 < t < T \\ \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} & x \in]0,1[\end{cases}$$

where u_0 *and* u_1 *are smooth functions with compact support in* $]0,1[$

Assume that $T > 0$ *is given. Then the above system is not observable, that is there exist nonvanishing initial data in the energy space for which* $u_x(.,1) \equiv 0$

Sketch of the proof

Again by the Liouville transform

$$u(t,x) = x^{-\theta/4} v(t,\varphi(x)) \quad t>0\ x\in]0,1[$$

with the new space variable

$$\varphi(x) = \begin{cases} \ln(\frac{1}{x}) & \text{if } \theta = 2 \\ \frac{2(x^{1-\theta/2}-1)}{\theta-2} & \text{if } \theta > 2 \end{cases}$$

the wave equation takes the form

$$\begin{cases} v_{tt} - v_{yy} + p(y)v = 0 & \text{in }]0,T[\times]0,\infty[\\ v(t,0) = 0 & t\in]0,T[\\ (v,v_t)(0,y) = (v_0,v_1)(y) & y\in]0,\infty[\end{cases}$$

where p is a bounded positive potential, and v_0, v_1 are also smooth functions with compact support in $]0,\infty[$, and the observation point is $y=0$ for the unknown v
Since v_0, v_1 have compact support, the finite speed propagation of the support for the wave equation with bounded potential implies that $v_y(\cdot,0) = 0$ on $[0,T]$ when the supports of v_0, v_1 are sufficiently far away from $y=0$

Thus, there is non unique observability, and the original problem in u is not observable

Blow-up of observability time

Let $0 \leqslant \theta < 2$ be given and consider the degenerate wave system

$$\begin{cases} u_{tt} - \left(x^{\theta} u_x\right)_x = 0 & \text{in }]0,T[\times]0,1[\\ u(t,1) = 0 \;\text{ and }\; \lim_{x\downarrow 0} x^{\theta}\, u_x(t,x) = 0 & 0 < t < T \\ u(0,x) = u_0(x)\,,\, u_t(0,x) = u_1(x) & x \in]0,1[\end{cases}$$

Then thanks to our previous results, the above system is observable through the boundary $x = 1$ for all

$$T > T_{\theta} := \frac{1}{2-\theta}\left(4 + 2\theta \min\left\{2, \frac{1}{\sqrt{2-\theta}}\right\}\right)$$

For any $C > 0$ denote by $T_{\theta}^{*}(C)$ the infimum of all times $T > 0$ such that C is an observability constant for the above wave system in time T
($T_{\theta}^{*}(C) = \infty$ if no such time exists)
Then

$$\frac{C}{2-\theta} \leqslant T_{\theta}^{*}(C)$$

Therefore the minimal control time blows up as $\theta \to 2^{-}$ with the same order as T_{θ}

Outline

1. Control of degenerate parabolic equations
   - Locally distributed control
   - Boundary control
2. Control of degenerate hyperbolic equations
   - Boundary observability
   - Boundary controllability
   - Boundary stabilization

Boundary controllabity

We consider the following controlled degenerate system

$$y_{tt} - \big(a(x)y_x\big)_x = 0 \quad \text{in } \;]0,\infty[\times]0,1[$$

$$\begin{cases} y(t,1) = f(t) \;\text{ and } \begin{cases} y(t,0) = 0 & \text{if } \; \mu_a \in [0,1[\\ \lim_{x\downarrow 0} a(x)\, y_x(t,x) = 0 & \text{if } \; \mu_a \in [1,2[\end{cases} & 0 < t < \infty \\ \begin{cases} y(0,x) = y_0(x) \\ y_t(0,x) = y_1(x) \end{cases} & x \in]0,1[. \end{cases}$$

Theorem

Assume $0 \leqslant \mu_a < 2$. *Then for any* $T > T_a$ *and*

$$(y_0, y_1) \in L^2(0,1) \times H_a^{-1}(0,1) \quad \textit{and} \quad (y_0^T, y_1^T) \in L^2(0,1) \times H_a^{-1}(0,1)$$

there exists a control $f \in L^2(0,T)$ *such that the solution of the above system (in the sense of transposition) satisfies* $(y, y_t)(T,\cdot) \equiv (0,0)$.

The proof is based on our observability result and the HUM

Outline

1. Control of degenerate parabolic equations
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Boundary stabilization

Let $\rho : \mathbb{R} \mapsto \mathbb{R}$ be a nondecreasing continuous function such that $\rho(0) = 0$ and assume there exist constants $c_1 > 0$, $c_2 > 0$ and an odd, continuously differentiable, strictly increasing function g on $[-1, 1]$ such that

$$\begin{aligned} c_1 g(|s|) \leqslant |\rho(s)| \leqslant c_2 g^{-1}(|s|) \quad \forall\, |s| \leqslant 1, \\ c_1 |s| \leqslant |\rho(s)| \leqslant c_2 |s| \quad \forall\, |s| \geqslant 1. \end{aligned}$$

As before, let a satisfy the above assumptions with $\mu_a \in [0, 2[$
We consider the degenerate wave equation

$$u_{tt} - \big(a(x) u_x\big)_x = 0 \quad \text{in }]0, T[\times]0, 1[$$

with the nonlinear boundary damping

$$\begin{cases} \rho(u_t(t,1)) + u_x(t,1) + \beta u(t,1) = 0\,, & \begin{cases} u(t,0) = 0 & \text{if } \mu_a \in [0,1[\\ \lim_{x\downarrow 0} a(x)\, u_x(t,x) = 0 & \text{if } \mu_a \in [1,2[\end{cases} \\ u(0,x) = u_0(x)\,, \quad u_t(0,x) = u_1(x) \end{cases}$$

where $\beta \geqslant 0$ is given

Energy dissipation

We define the space $\mathcal{H}_\beta = W_a^1(0,1) \times L^2(0,1)$ where

$$W_a^1(0,1) = \begin{cases} H_a^1(0,1) & \text{if } \mu_a \in [1,2[\\ \{u \in H_a^1(0,1)\,,\, u(0)=0\} & \text{if } \mu_a \in [0,1[\end{cases}$$

Then one can show that the above nonlinear system is well-posed in the framework of nonlinear semigroups in $\mathcal{H}_\beta$

Moreover the natural energy of the solutions, defined by

$$E_u(t) =: \frac{1}{2}\Big[\int_0^1 \Big(u_t^2 + a u_x^2\Big)\,dx + \beta a(1) u^2(t,1)\Big]$$

is dissipative:

$$\frac{dE_u}{dt}(t) = -a(1)u_t(t,1)\rho(u_t(t,1)) \leqslant 0 \qquad \forall\, t \geqslant 0$$

Optimal-weight convexity method

We follow the framework of the optimal-weight convexity method (Alabau-Boussouira 2005, 2010)

- Introduce the function $H : [0, r_0^2] \to [0, \infty)$ by
$$H(x) = \sqrt{x} g(\sqrt{x}) \quad x \in [0, r_0^2],$$
where $r_0 \leqslant 1$ is assumed to be sufficiently small
- Assume that H is strictly convex on $[0, r_0^2]$ and extend H to $\widehat{H}$ on $[0, \infty)$ by setting $\widehat{H}(x) = +\infty$ when $x \notin [0, r_0^2]$
- Then define a function L on $[0, \infty)$ by
$$L(y) = \begin{cases} \dfrac{\widehat{H}^*(y)}{y} & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$
- where $\widehat{H}^*$ stands for the convex conjugate of $\widehat{H}$ defined by $\widehat{H}^*(y) = \sup_{x \in \mathbb{R}} \{xy - \widehat{H}(x)\}$
- Define a function Λ_H on $[0, r_0^2]$ $\quad \Lambda_H(x) = \dfrac{H(x)}{xH'(x)}$

Stabilization result

Theorem

We assume the above hypotheses on a and on ρ, g and H, and that $\beta > 0$ is given. Let $(u_0, u_1) \in \mathcal{H}_\beta$ be given such that $E_u(0) > 0$, and u be the corresponding solution of the above nonlinear system. Let $\gamma > \max(\frac{E_u(0)}{2L(H'(r_0^2))}, C^*)$, then the energy E_u of u satisfies the following estimate:

$$E_u(t) \leqslant 2\gamma L\Big(\frac{1}{\psi_0^{-1}(\frac{t}{M})}\Big), \quad \forall\, t \geq \frac{M}{H'(r_0^2)}$$

where

$$\psi_0(x) = \frac{1}{H'(r_0^2)} + \int_{1/x}^{H'(r_0^2)} \frac{1}{y^2(1 - \Lambda_H((H')^{-1}(\theta)))}\, dy$$

Furthermore, if $\limsup_{x\to 0^+} \Lambda_H(x) < 1$*, then E satisfies the following simplified decay rate*

$$E_u(t) \leq 2\gamma\Big(H'\Big)^{-1}\Big(\frac{\kappa M}{t}\Big)$$

for t sufficiently large, and where $\kappa > 0$ is a constant independent of $E(0)$, C^* is an explicit constant depending on the data

Examples of decay rates

Same decay rates for the degenerate and nondegenerate damped wave equation

Let us now give some examples of the resulting decay rates

- For the polynomial case for which $g(x) = |x|^{p-1}x$ in a neighborhood of $x = 0$ with $p > 1$,

$$E_u(t) \leqslant C_{E_u(0)}\gamma t^{-\frac{2}{p-1}} \text{ for sufficiently large } t$$

- For $g(x) = |x|^{p-1}x \ln^q(\frac{1}{|x|})$ in a neighborhood of $x = 0$ with $p > 1$, $q > 0$,

$$E_u(t) \leqslant C_{E_u(0)}\gamma t^{-\frac{2}{p-1}}(\ln(t))^{-2q/(p-1)} \text{ for sufficiently large } t$$

- For $g(x) = sign(x)e^{-1/x^2}$ in a neighborhood of $x = 0$,

$$E_u(t) \leqslant C_{E_u(0)}\gamma e^{-2(\ln(t))^{1/p}} \text{ for sufficiently large } t$$

Happy Birthday!

merci de votre attention

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