Controllability of linear parabolic systems: New phenomena

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GOAL:

The general aim of this talk is to show some phenomenona which arise when we deal with the null controllability properties of **coupled parabolic** systems:

1. **First phenomenon:** **Boundary controllability** is not equivalent to **distributed controllability** for coupled parabolic systems.
2. **Second phenomenon:** The **null controllability** properties are not equivalent to the **approximated controllability** of these problems.
3. **Third phenomenon:** **Minimal time of controllability**. The null controllability only holds if is T is large enough.
4. **Fourth phenomenon:** The null controllability of parabolic system depends on the **position of the control open set**.

Contents

1. The parabolic scalar case

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Let us fix $T > 0$, $\Omega \subset \mathbb{R}^N$, a regular bounded domain, $\omega \subset \Omega$, an open subset, and $\gamma \subset \partial\Omega$, a relative open subset. We consider the scalar parabolic problem:

$$
(1) \qquad \boxed{\begin{cases} y_t - \Delta y = u 1_\omega & \text{in } Q := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}}
$$

$$
(2) \qquad \boxed{\begin{cases} y_t - \Delta y = 0 & \text{in } Q, \\ y = v 1_\gamma & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}}
$$

In (1) and (2), 1_ω and 1_γ are, resp., the characteristic functions of the sets ω and γ, $y(x,t)$ is the state, $y_0 \in L^2(\Omega)$ (or $y_0 \in H^{-1}(\Omega)$) is the initial datum and $v \in L^2(\Sigma)$ and $u \in L^2(Q)$ are scalar control functions.

1. The parabolic scalar case

Remark

We have two different concepts of controllability in the parabolic framework:

1. **Approximate controllability**.
2. **Exact controllability to zero**.

And two different ways of acting on the system:

1. **Distributed controls**.
2. **Boundary controls**.

1. The parabolic scalar case

Theorem (Approximate controllability)

Assume Ω, ω, γ and T as before. Then,

1. *System* (1) *is approximately controllable at time T (distributed case).*
2. *System* (2) *is approximately controllable at time T (boundary case).*

Theorem (Null controllability)

Assume Ω, ω, γ and T as before. Then,

1. *System* (1) *is exactly controllable to zero at time T (distributed case).*
2. *System* (2) *is exactly controllable to zero at time T (boundary case).*

[Lebeau-Robbiano] (1996), [Fursikov-Imanuvilov] (1996), ...

Remark

The previous results are valid for any Ω, ω, γ and $T > 0$.

Scalar systems: Summary

1. The same positive results for the **distributed** and **boundary control** problems.
2. The same positive results for the **approximate** and **null controllability** problems.
3. The positive results are valid for any time $T > 0$ (**no minimal time for controlling**).
4. The controllability results do not depend on the position of ω and γ (**no geometrical conditions**).

Non-scalar systems

Are this properties valid in the case of **non-scalar parabolic systems**?

OBJECTIVE

Analyze the controllability properties of **non-scalar parabolic systems** in the case of distributed and boundary controls. To this end, we will consider simple systems (2×2 **parabolic linear systems**).

IMPORTANT

We have systems of two coupled heat equations and we want to control these systems (two states) only acting on the second equation.

2. First phenomenon: Boundary and distributed controllability

Let us consider the 2×2 linear reaction-diffusion system

$$(3) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_1 y = Bu1_\omega & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Here $\omega = (a,b) \subset (0,\pi)$, $T > 0$, $y_0 \in L^2((0,\pi);\mathbb{R}^2)$, $u \in L^2(Q)$ and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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One has

Theorem

System (3) is exactly controllable to trajectories at time T if and only if

$$\det\,[B\,,\, A_1 B] \neq 0 \Longleftrightarrow a_{12} \neq 0.$$

Proof: $\boxed{\Longrightarrow}$: If $a_{12} = 0$, then y_1 is independent of u.

$\boxed{\Longleftarrow}$: The controllability result for system (3) is equivalent to the observability inequality: $\exists C > 0$ such that

$$\|\varphi_1(\cdot,0)\|_{L^2}^2 + \|\varphi_2(\cdot,0)\|_{L^2}^2 \le C \iint_{\omega\times(0,T)} |\varphi_2(x,t)|^2 \, dx\, dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega;\mathbb{R}^2)$ of the adjoint problem:

$$(4) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} + A_1^*\varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot,T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

It is a consequence of well-known global Carleman estimates for parabolic equations.

1. Using some appropriate global Carleman inequalities for the adjoint problem (4), we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

1. Using some appropriate global Carleman inequalities for the adjoint problem (4), we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}[t(T-t)]^{-3}\left(|\varphi_1|^2 + |\varphi_2|^2\right),$$

$\forall s \geq s_1 = \sigma_1(\Omega,\omega_0)(T+T^2)$.

2. We now use the second equation in (4), $\boxed{a_{12}\varphi_1 = \varphi_{2,t} + d_2\varphi_{2,xx} - a_{22}\varphi_2}$, to prove ($\varepsilon > 0$):

$$s^3 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}[t(T-t)]^{-3}|\varphi_1|^2 \leq \varepsilon\mathcal{I}(\varphi_1) + \frac{C_2}{\varepsilon}s^7 \iint_{\omega\times(0,T)} e^{-2s\alpha}[t(T-t)]^{-7}|\varphi_2|^2.$$

$\forall s \geq s_1 = \sigma_1(\Omega,\omega_0)(T+T^2)$.

From the two previous inequalities (global Carleman estimate)

$$\boxed{\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_2 s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_2|^2,}$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$. Combining this inequality and energy estimates for system (4) we deduce the desired observability inequality.

$$(3) \qquad \begin{cases} y_t - Dy_{xx} + A_1 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Remark

- *System* (3) *is always controllable if we exert a control in each equation (two controls).* ***Important***: *Two equations and* D *is a diagonal matrix.*
- *The controllability result for system* (3) *is independent of the diffusion matrix* D. *This positive controllability result is also valid in the* N*-dimensional case.*
- *The same result can be obtained for the approximate controllability at time* T. *Therefore,* ***approximate*** *and* ***null controllability*** *are equivalent concepts.*

References

- DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000).
- AMMAR KHODJA, BENABDALLAH, DUPAIX, KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003).
- G.-B., PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006).
- G.-B., DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Port. Math. 67 (2010).

$$(5) \qquad \boxed{\begin{cases} y_t - D y_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

where $\boxed{A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ $v \in L^2(0,T)$: scalar control.

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Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d_1 = d_2 > 0$. Assume μ_1, μ_2 are the eigenvalues of A_1. Then system (5) is null controllable at time T if and only $\boxed{\det\,[B\,,\,A_1B] = a_{12} \neq 0}$ *and*

$$\boxed{\mu_1 - \mu_2 \neq j^2 - k^2} \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j.$$

- FERNÁNDEZ-CARA, G.-B., DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010).

$$
(5) \qquad \begin{cases} y_t - Dy_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

First phenomenon

The boundary and distributed controllability properties of the system

$$
y_t - Dy_{xx} + A_1 y
$$

are different and not equivalent.

- AMMAR KHODJA, BENABDALLAH, G.-B., DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (2011).

$$
(5) \qquad \boxed{\begin{cases} y_t - D y_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}
$$

Remark

The same result can be obtained for the approximate controllability at time T. Therefore, ***approximate*** *and* ***null controllability*** *are equivalent concepts.*

3. Second phenomenon: Approximate and null controllability

3. Second phenomenon: Approximate/null controllability

$$
(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}
$$

where $D = \text{diag}\,(d_1, d_2)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

We will assume that $d_1 \neq d_2$ and, for instance, $d_1 = 1$, $d_2 = d \neq 1$.

GOAL

Given $T > 0$, does there exist $v \in L^2(0,T)$ s.t. $y(T) = 0$?

Remark

Recall that the parabolic system $y_t - Dy_{xx} + A_0 y = u 1_\omega$ is approximate and null controllable at time T for any $T > 0$.

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Approximate controllability:

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d \neq 1$. *Then system* (6) *is approximately controllable at time* $T > 0$ *if and only if* $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

$$\boxed{D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}}, \qquad d \neq 1.$$

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

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Is this problem null controllable at a given time $T > 0$ when $\boxed{\sqrt{d} \notin \mathbb{Q}}$???

No:

3. Second phenomenon: Approximate/null controllability

$$(6)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Theorem (Luca, de Teresa, (2012))

There exists $d > 0$ *with* $\boxed{\sqrt{d} \notin \mathbb{Q}}$ *such that system* (6) *is not null controllable at any time* $T > 0$.

- LUCA, DE TERESA, *Control of coupled parabolic systems and Diophantine approximations*, SeMA J. 61 (2013).

Second phenomenon

For system (6): Approximate controllability $\boxed{\not\Leftrightarrow}$ null controllability.

4. Third phenomenon: Minimal time of controllability

4. Third phenomenon: Minimal time

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where $D = \operatorname{diag}(1,d)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Assumption

In the sequel, $\boxed{D = \operatorname{diag}(1,d)}$ with $\boxed{d \neq 1}$ and $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Goal

Analyze the null controllability properties at time $T > 0$ of system (6).

4. Third phenomenon: Minimal time

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 \in H_0^1(0,\pi)^2 & \text{in } (0,\pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0,t)\, dt$$

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$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0,t)\, dt$$

Thus $\boxed{y(T) = 0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t) B^* D \varphi_x(0,t)\, dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

Fattorini-Russell Method

Fattorini-Russell Method

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1} \left\{k^2, dk^2\right\} := \bigcup_{k\geq 1} \left\{\lambda_{k,1}, \lambda_{k,2}\right\}$.
- $\{\Phi_{k,i}\}$ a (Riesz) basis of $H_0^1(0,\pi)^2$, where $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1, 2$ are eigenfunctions of the operator $\boxed{-D\partial_{xx}^2 + A_0^*}$.
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $\boxed{k^2 D + A_0^*}$ associated to the eigenvalues k^2, dk^2.

(6) $$\boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Objective: Existence of $v \in L^2(0,T)$ s.t.

$$\boxed{\int_0^T v(t) B^* D \varphi_x(0,t)\, dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

4. Third phenomenon: Minimal time

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$$\boxed{\int_0^T v(t) B^* D \varphi_x(0,t)\,dt = -\langle y_0, \varphi(0)\rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

- Choosing $\varphi_0 = \Phi_{k,i}$, we have $\varphi(\cdot,t) = e^{-\lambda_{k,i}(T-t)} \Phi_{k,i}$ and

$$\varphi(x,0) = e^{-\lambda_{k,i} T} \Phi_{k,i}(x), \quad \varphi_x(0,t) = k e^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes (**moment problem**)

$$\boxed{k B^* D V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\,dt = -e^{-\lambda_{k,i} T} \langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)}$$

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Approximate controllability: a necessary condition (I)

- $$\boxed{\boxed{kB^*DV_{k,i}} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)}$$

$$(6)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

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- A necessary condition: $B^*DV_{k,i} \neq 0$ for all $k \geq 1, i = 1, 2$
- Recall $\boxed{d \neq 1}$,

$$B^* = (0,1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here $\boxed{B^*DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1, 2}$ (**algebraic Kalman condition**)

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^* DV_{k,1} \int_0^T v(T-t) e^{-\lambda t}\, dt = -e^{-\lambda T} \langle y_0, \Phi_{k,1} \rangle \\ jB^* DV_{j,2} \int_0^T v(T-t) e^{-\lambda t\, dt} = -e^{-\lambda T} \langle y_0, \Phi_{j,2} \rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \Longleftrightarrow \boxed{\sqrt{d} \notin \mathbb{Q}}$$

4. Third phenomenon: Minimal time

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$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \Longleftrightarrow \boxed{\sqrt{d} \notin \mathbb{Q}}$$

In the sequel, we will assume $\sqrt{d} \notin \mathbb{Q}$, i.e., the eigenvalues of $-D\partial_{xx}^2 + A_0^*$ with Dirichlet boundary conditions are pairwise distinct.

$$(6)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

$$\boxed{kB^*DV_{k,i} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k,i)}$$

Summarizing

Let $m_{k,i} = -\langle y_0, \Phi_{k,i} \rangle$, $b_{k,i} = kB^*DV_{k,i}$ (for any $\varepsilon > 0$, $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_{k,i}}}$ and $\boxed{|b_{k,i}| \geq C_\varepsilon e^{-\varepsilon \lambda_{k,i}}}$),

$$\exists\,?\, v \in L^2(0,T) \;:\; \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = \frac{m_{k,i}}{b_{k,i}} e^{-\lambda_{k,i}T}, \quad \forall k \geq 1,\; i = 1,2$$

Let $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset (0,\infty)$ be a sequence with **pairwise distinct elements**:

$$\boxed{\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty}$$

Goal: Given $\{m_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \subset \mathbb{R}$ satisfying $\boxed{|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}}$ and $\boxed{|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}}$, find $v \in L^2(0,T)$ s.t.

$$\int_0^T v(T-t) e^{-\Lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k \geq 1.$$

Recall that the assumption

$$\boxed{\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty}$$

implies:

Theorem

Under the previous assumptions, $\left\{e^{-\Lambda_k t}\right\}_{k\geq 1} \subset L^2(0,T)$ admits a ***biorthogonal family*** *$\{q_k\}_{k\geq 1}$ in $L^2(0,T)$, i.e.:*

$$\int_0^T e^{-\Lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k, l \geq 1$$

A formal solution to

$$\boxed{\int_0^T v(T-t)e^{-\Lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\Lambda_k T}, \quad \forall k \geq 1,}$$

is v given by: $\boxed{v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k}e^{-\Lambda_k T} q_k(t)}$,

Question: $v \in L^2(0,T)$?, i.e., is the series $\boxed{\sum_{k\geq 1} \frac{m_k}{b_k}e^{-\Lambda_k T} q_k(t)}$ convergent in $L^2(0,T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

Theorem

Assume that $\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty$ *and (**gap condition**)*

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k - j|, \quad \forall k, j}.$$

Then, for any $\varepsilon > 0$ *one has*

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Lambda_k}, \quad \forall k \geq 1,$$

and, for $T > 0$, *the control* $v(T - t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0,T)$.

Theorem

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and, for $T > 0$*, the control* $v(T-t) = \sum_{k\geq 1}\frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0,T)$.

Recall that in our case $\Lambda = \{\Lambda_k\}_{k\geq 1} = \{j^2, dj^2\}_{j\geq 1}$, and the property

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k-j|, \quad \forall k,j},$$

does not hold.

How does this fact affect our problem??

Theorem

Assume $\boxed{\displaystyle\sum_{k\geq 1}\frac{1}{|\Lambda_k|}<\infty}$. *Then, for any* $\varepsilon>0$ *one has*

$$C_{1,\varepsilon}\frac{e^{-\varepsilon\Lambda_k}}{|W'(\Lambda_k)|}\leq \|q_k\|_{L^2(0,T)}\leq C_{2,\varepsilon}\frac{e^{\varepsilon\Lambda_k}}{|W'(\Lambda_k)|},\quad \forall k\geq 1,$$

where $W(z)$ *is the Blaschke product:*

$$W(z)=\prod_{k=1}^{\infty}\frac{1-z/\Lambda_k}{1+z/\Lambda_k},\qquad \boxed{W'(\Lambda_k)=-\frac{1}{2\Lambda_k}\prod_{j\neq k}^{\infty}\frac{1-\Lambda_k/\Lambda_j}{1+\Lambda_k/\Lambda_j}}$$

Definition

The **condensation index** of $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\log |W'(\Lambda_k)|}{\Re(\Lambda_k)} \in [0, +\infty] \,.$$

Corollary

For any $\varepsilon > 0$ *one has*

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\Lambda_k}, \quad \forall k \geq 1.$$

Recall that we had m_k s.t. $|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}$, $|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

$$\boxed{\int_0^T v(T-t) e^{-\Lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k},$$

We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

Recall that we had m_k s.t. $|m_k| \le C_\varepsilon e^{\varepsilon\Lambda_k}$, $|b_k| \ge C_\varepsilon e^{-\varepsilon\Lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

$$\boxed{\int_0^T v(T-t)e^{-\Lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\Lambda_k T}, \quad \forall k},$$

We took $v(T-t) = \sum_{k\ge 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

From the previous result: Given $\varepsilon > 0$:

$$\left|\frac{m_k}{b_k}\right| e^{-\Lambda_k T} \, \|q_k\|_{L^2(0,T)} \le C_\varepsilon e^{-\Lambda_k (T - c(\Lambda) - \varepsilon)}$$

Then

$$T > c(\Lambda) \Longrightarrow v(T-t) = \sum_{k\ge 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0,T).$$

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

In our case,

$$\Lambda_d := \{\Lambda_k\}_{k\geq 1} = \left\{j^2, dj^2\right\}_{j\geq 1}.$$

Then

If $T > c(\Lambda_d)$, system (6) is null controllable at time T, where $c(\Lambda_d)$ is the **condensation index** of the sequence Λ_d.

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

$$D = \operatorname{diag}(1, d), \quad \Lambda_d = \{k^2, dk^2\}_{k\geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ such that if $\boxed{T > T_0}$ then system (6) is null controllable at time T

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

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We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0,+\infty]$ such that if $\boxed{T > T_0}$ then system (6) is null controllable at time T

$\boxed{T > c(\Lambda_d)}$ is a sufficient condition for the null controllability of system (6) at time T. But,

what happens if $\boxed{T < c(\Lambda_d)}$?

One can prove:

Theorem

Let us take

$$\boxed{T_0 = c(\Lambda_d) \in [0, +\infty]}.$$

Then, if $\boxed{T < T_0}$, *system* (6) *is not null controllable at time* T.

Idea of the proof

By contradiction:

- The null controllability at time T is equivalent to: $\exists C_T > 0$ s.t.

$$\sum_{n,i} e^{-2\lambda_{n,i}T} |a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* DV_{n,i} e^{-\lambda_{n,i}t} a_{n,i} \right|^2 dt, \ \forall \{a_{n,i}\}_{n,i} \in \ell^2.$$

- Argument: Use the overconvergence of Dirichlet series.

$$
(6) \qquad \boxed{\begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}
$$

The controllability result

1. $\forall T > 0$: **Approximate controllability** at time T if and only if $\boxed{\sqrt{d} \notin \mathbb{Q}}$.
2. Assume $\boxed{\sqrt{d} \notin \mathbb{Q}}$, $\exists T_0 = c(\Lambda_d) \in [0, +\infty]$ such that
   1. the system is null controllable at time T if $\boxed{T > T_0}$
   2. Even if $\sqrt{d} \notin \mathbb{Q}$, if $\boxed{T < T_0}$ the system is **not null controllable** at time T!

$$(6)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

In fact, the good minimal time is

$$T_0 = \limsup_{k\to\infty} \frac{-\left(\log|b_k| + \log|W'(\Lambda_k)|\right)}{\Re(\Lambda_k)} \in [0,\infty]$$

4. Third phenomenon: Minimal time

$$
(6) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

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$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

Theorem

For any $\tau \in [0, +\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau$.

Remark

- There exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = +\infty$ (Luca, de Teresa).
- $c(\Lambda_d) = 0$ for almost $d \in (0, \infty)$ such that $\sqrt{d} \notin \mathbb{Q}$.
- For any $\tau \in [0, +\infty]$, the set $\{d \in (0, \infty) : c(\Lambda_d) = \tau\}$ is dense in $(0, +\infty)$.

4. Third phenomenon: Minimal time

$$(6) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

where $D = \text{diag}\,(1, d)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Third phenomenon

For system (6): If $\sqrt{d} \notin \mathbb{Q}$, then,

1. **Approximate controllability**: System (6) is approximately controllable at any time $T > 0$.
2. **Null controllability**: System (6) is null controllable is $T > T_0 = c(\Lambda_d)$ and is not if $T < T_0 = c(\Lambda_d)$.

4. Third phenomenon: Minimal time

Remark

This minimal time also arises in other parabolic problems (degenerated problems):
BEAUCHARD, CANNARSA, GUGLIELMI, *Null controllability of Grushin-type operators in dimension two. J. Eur. Math. Soc. (JEMS) (2014).*
BEAUCHARD, MILLER, MORANCEY, *2d Grushin-type equations: Minimal time and null controllable data, J. Differential Equations 259 (2015), no. 11*

Reference

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014).

http://personal.us.es/manoloburgos

5. Fourth phenomenon: Dependence on the position of the control set

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$. We consider the coupled parabolic systems:

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q := (0,\pi) \times (0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

In (7), 1_ω is the characteristic function of the set ω, $y(x,t)$ is the state, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ is the initial datum and

- $q \in L^\infty(0,\pi)$ is a given function, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$ is a constant matrix and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a constant vector of $\mathbb{R}^2$;
- $u \in L^2(Q)$ is a scalar control function.

5. Fourth phenomenon: geometrical dependence

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Remark

If $q \in L^\infty(0,\pi)$ satisfies: There exist an open subset $\omega_0 \subseteq \omega$ and a constant $\delta > 0$ s.t.

$$\boxed{q \geq \delta > 0 \text{ a.e. } \omega_0} \quad \text{or} \quad \boxed{q \leq -\delta < 0 \text{ a.e. } \omega_0}$$

$\left(\Longrightarrow \boxed{\operatorname{Supp} q \cap \omega \neq \emptyset} \right)$, then it is possible to repeat the arguments of section 2 and prove:

Theorem

Under the previous assumption, system (7) is approximately and exactly controllable to zero at any time $T > 0$.

5. Fourth phenomenon: geometrical dependence

Let us consider the 2×2 linear reaction-diffusion system

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

where $q \in L^\infty(Q)$, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$,

$$\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},}$$

$\boxed{\omega = (a,b)} \subset (0,\pi)$ and $u \in L^2(Q)$ is a scalar control function.

5. Fourth phenomenon: geometrical dependence

Let us consider the 2×2 linear reaction-diffusion system

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x) A_0 y = Bu 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

where $q \in L^\infty(Q)$, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$,

$$\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},}$$

$\boxed{\omega = (a,b)} \subset (0,\pi)$ and $u \in L^2(Q)$ is a scalar control function.

No sign conditions on q.

$$\boxed{\omega \cap \operatorname{Supp} q = \emptyset}$$

5. Fourth phenomenon: geometrical dependence

$$(7) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Theorem (Ammar Khodja, Benabdallah, G-B, de Teresa (2011))

Assume $I_k(q) \neq 0$ *for any* $k \geq 1$, *where*

$$(8) \qquad I_k(q) := \int_0^\pi q(x)|\sin(kx)|^2\,dx,$$

and

$$\int_0^\pi q(x)\,dx \neq 0.$$

Then, for any $T > 0$, *system* (7) *is* ***null controllable*** *at time* T.

5. Fourth phenomenon: geometrical dependence

$$
(7) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Null controllability properties of system (7) when

$$\int_0^\pi q(x)\,dx = 0?$$

In order to simplify the problem, we will assume the **geometrical assumption**:

Assumption (A1)

The function q satisfies $\operatorname{Supp} q \subset [0,a]$ or $\operatorname{Supp} q \subset [b,\pi]$ ($\omega = (a,b)$).

$$
(7) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Proposition (Boyer and Olive (2014))

Under the geometrical assumption (A1), system (7) *is* ***approximately controllable*** *at time* $T > 0$ *if and only if*

$$
I_k(q) \neq 0, \quad \forall k \geq 1.
$$

5. Fourth phenomenon: geometrical dependence

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Proposition (Boyer and Olive (2014))

Under the geometrical assumption (A1), system (7) *is* ***approximately controllable*** *at time* $T > 0$ *if and only if*

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

Remarks

1. The approximate controllability of system (7) does not depend on T.
2. Again, condition
$$I_k(q) \neq 0, \quad \forall k \geq 1.$$
is necessary for the null controllability of system (7) at time $T > 0$

$$
(7) \qquad \left\{ \begin{array}{ll} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{array} \right.
$$

We have a Riesz basis $\mathcal{B} := \left\{ \Phi^*_{k,1}, \Phi^*_{k,2} \right\}_{k \geq 1}$ of eigenfunctions and generalized eigenfunctions of the operator $L^* := -\frac{d^2}{dx^2} + q(x) A_0^*$ associated to the eigenvalue k^2 (**simple**).

Idea:

We will work with controls $u(x,t) = f(x) v(t)$ with $v \in L^2(0,T)$ and $f \in L^2(0,\pi)$ (appropriate) satisfies $\operatorname{Supp} f \subset \omega$.

Objective

Apply Fattorini-Russell method: **moment problem**

$$(7)\qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

The moment problem

Find $v \in L^2(0,T)$ s.t.

$$\begin{cases} \displaystyle\int_0^T v(T-t)\boxed{e^{-k^2 t}}\,dt = \frac{m_{k,1}}{f_k} e^{-k^2 T}, & \forall k \geq 1, \\ \displaystyle\int_0^T v(T-t)\boxed{te^{-k^2 t}}\,dt = \frac{m_{k,2}}{I_k(q) f_k} e^{-k^2 T}, & \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}}$ and $\boxed{|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon \lambda_k}}$ $(i = 1, 2)$.

(7)
$$\boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

The moment problem

Find $v \in L^2(0,T)$ s.t.

$$\begin{cases} \displaystyle\int_0^T v(T-t)e^{-k^2 t}\,dt = \frac{m_{k,1}}{f_k} e^{-k^2 T}, \quad \forall k \geq 1, \\ \displaystyle\int_0^T v(T-t)te^{-k^2 t}\,dt = \frac{m_{k,2}}{\boxed{I_k(q)}\, f_k} \boxed{e^{-k^2 T}}, \quad \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}}$ and $\boxed{|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon \lambda_k}}$ $(i = 1, 2)$.

$$(7)\qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Conclusion

We can obtain the positive controllability result if
$T > \widetilde{T}_0(q) = \limsup \frac{-\log |I_k(q)|}{k^2}$,

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$. Then, if $T > \widetilde{T}_0(q)$, system (7) is null-controllable at time T.

Does the minimal time depend on the choice $u(x,t) = f(x)v(t)$?

What happens if $\boxed{T < \widetilde{T}_0(q)}$?

$$(7)\qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

As before, the null controllability property for system (7) is equivalent to the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx \, dt,$$

for the solutions to **the adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} + q(x)A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

$$\boxed{\|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt,}$$

If $T < \widetilde{T}_0(q)$, we can prove that the inequality does not hold **reasoning by contradiction**: Then system

(7)
$$\begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

is not null controllable at time T.

$$
(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}
$$

$$
\boxed{\omega \cap \operatorname{Supp} q = \emptyset}
$$

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$ and let:

$$
\widetilde{T}_0(q) := \limsup \frac{-\log |I_k(q)|}{k^2} \in [0, +\infty]
$$

Then,

1. *If $T > \widetilde{T}_0(q)$, then system (7) is null-controllable at time T.*
2. *If $\operatorname{Supp} q \subset [0,a]$ or $\operatorname{Supp} q \subset [b,\pi]$, for any $T < \widetilde{T}_0(q)$, the system is not null-controllable at time T.*

$$(7)\qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Remarks

1. The previous results cannot be obtained using Carleman inequalities.
2. Due to the geometrical assumption

   $\boxed{\text{The function } q \text{ satisfies } \operatorname{Supp} q \subset [0,a] \text{ or } \operatorname{Supp} q \subset [b,\pi] \ (\omega = (a,b))}$

   the boundary and distributed null controllability results coincide.

$$(7)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

General case

$$\omega = (a,b) \subset (0,\pi) \text{ and } \operatorname{Supp} q \cap \omega = \emptyset.$$

The condition $I_k(q) \neq 0$ is no longer necessary:

$$I_{1,k}(q) := \int_0^a q(x)|\sin(kx)|^2\,dx; \quad I_{2,k}(q) := \int_b^1 q(x)|\sin(kx)|^2\,dx$$

$$I_k(q) = I_{1,k}(q) + I_{2,k}(q) = \int_0^\pi q(x)|\sin(kx)|^2\,dx;$$

$$
(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}
$$

Proposition (Boyer and Olive (2014))

If $\omega = (a,b)$, system (7) is ***approximately controllable*** *at time $T > 0$ if and only if*

$$
|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.
$$

The proof uses the independence of the functions $\sin(kx)$ and $\cos(kx)$ in ω.

$$(7) \qquad \begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Remarks

1. The approximate controllability of system (7) does not depend on T.
2. Again, condition

$$|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.$$

is necessary for the null controllability of system (7) at time $T > 0$.

Null controllability of system (7)???

$$(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

In this case we can have $I_k(q) = 0$, and then,

$$L := -\frac{d^2}{dx^2} + q(x)A_0 : L^2(0,\pi;\mathbb{R}^2) \longrightarrow L^2(0,\pi;\mathbb{R}^2)$$

has eigenvalues (k^2) of multiplicity 2.

Idea

Apply Fattorini-Russell's method with control under the form:

$$\boxed{u(x,t) = f_1(x)v_1(t) + f_2(t)v_2(t)}$$

with $\mathrm{Supp}\, f_1, \mathrm{Supp}\, f_2 \subset (a,b)$

$$
(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}
$$

Theorem

Let $\omega = (a,b) \subset (0,\pi)$ *and* $q \in L^\infty(Q)$ *satisfying* $\omega \cap \operatorname{Supp} q = \emptyset$,

$$
|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0 \; (\iff \; |I_{1,k}(q)|^2 + |I_k(q)|^2 \neq 0).
$$

and

$$
T_0(q) = \limsup \frac{\min\left[-\log|I_{1,k}(q)|, -\log|I_k(q)|\right]}{k^2}
$$

Then,

1. *If* $T > T_0(q)$*, then system (7) is null-controllable at time* T*.*
2. *For any* $T < T_0(q)$*, the system is not null-controllable at time* T*.*

$$
(7) \qquad \boxed{\begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}
$$

Remark

If

$$|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0$$

and

$$\int_0^a q(x)\,dx \neq 0 \quad \text{or} \quad \int_b^\pi q(x)\,dx \neq 0 \quad \text{or} \quad \int_0^\pi q(x)\,dx \neq 0,$$

Then $T_0(q) = 0$ (Null controllability of system (7) for every $T > 0$).

Idea of the proof:

1. The reasoning for $T < T_0(q)$ is by contradiction.
2. For proving the positive controllability result for $T > T_0(q)$ we have to "mesure" the linear independence of $\boxed{B^*\Phi^*_{k,1} := \psi_k}$ and $\boxed{B^*\Phi^*_{k,2} := \sin(kx)}$ in ω ($\Phi^*_{k,1}$ and $\Phi^*_{k,2}$ are the eigenfunctions or the eigenfunction and the generalized eigenfunction of $L^* := -\frac{d^2}{dx^2} + q(x)A_0^*$ associated to k^2). Thanks to the assumption $\omega \cap \operatorname{Supp} q = \emptyset$ and the expression of ψ_k in ω this amounts to prove

$$\det \begin{pmatrix} f_{1,k} & f_{2,k} \\ \widetilde{f}_{1,k} & \widetilde{f}_{2,k} \end{pmatrix} \geq \frac{C}{k^m} \frac{I_{1,k}(q)}{I_k(q)}, \text{ when } I_{1,k}(q) \neq 0 \text{ and } I_k(q) \neq 0$$

where $C > 0$, $m \geq 1$, $f_{i,k}$ is the Fourier coefficient of f_i and

$$\widetilde{f}_{i,k} = \int_0^\pi f_i(x)\psi_k(x)\,dx, \quad k \geq 1, \quad i = 1, 2.$$

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.

Example

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1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.
2. $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x)\, dx = 0$, $\forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin\left(k\left(a_1 + a_2 + \ell\right)\right) \sin\left(k(a_2 - a_1)\right) \sin\left(k\ell\right)$$

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.
2. $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x)\, dx = 0$, $\forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin\left(k\left(a_1 + a_2 + \ell\right)\right) \sin\left(k(a_2 - a_1)\right) \sin\left(k\ell\right)$$

- **Aprox. Contr.** $T > 0 \iff \boxed{(a_1 + a_2 + \ell)/\pi}, \boxed{(a_2 - a_1)/\pi}, \boxed{\ell/\pi} \notin \mathbb{Q}$.
- Given $\tau \in [0, \infty]$, $\exists a_1$, a_2 y ℓ satisfying the previous property s.t. $\boxed{T_0(q) = \tau}$. **Minimal time** of null controllability which could be $\boxed{T_0(q) = \infty}$.

$$
(7) \qquad \begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Fourth phenomenon

For system (7): $\omega = (a,b) \subset (0,\pi)$ and $\omega \cap \operatorname{Supp} q = \emptyset$, then,

1. The **approximate controllability** is not equivalent to the **null controllability**.
2. **Null controllability**: The controllability result depends on the relative position of ω with respect to $\operatorname{Supp} q$.

Summarizing

Scalar case versus systems (parabolic problems)

	SCALAR CASE	SYSTEMS
boundary $\Leftrightarrow$ distributed control	Yes	No
approximate $\Leftrightarrow$ null controllability	Yes	No
minimal time for controling	No	Yes
geometrical conditions	No	Yes

Thank you for your attention!!