

Null controllability of degenerate parabolic equations of hypoelliptic type

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CIRM, Novembre 2015

Degenerate parabolic operator of hypoelliptic type

$A(x) \geq 0$ in $\bar{\Omega}$, not > 0 everywhere

$$\begin{cases} \partial_t f - \operatorname{div}[A(x)\nabla f] + b(t, x) \cdot \nabla f = 1_\omega(x)u(t, x) & \text{in } (0, T) \times \Omega \\ + \text{ B.C.} & \text{on } (0, T) \times \Gamma \end{cases}$$

$L = \sum_{j=1}^r X_j^2 + X_0 + c$ where X_0, \dots, X_r are C^∞ first-order differential operators and there exists n operators, among $[X_{j_1}, [X_{j_2}, [\dots, X_{j_k}]\dots]]$ which are linearly independent at any given point $x \in \Omega$.

Ex of application : Boundary layers in fluids (Prandtl/Crocco eqs).

Question : Null controllability? (usual sense)

“ $\forall f_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \Omega)$ such that $f(T) = 0$ ”

Comparison parabolic : $\forall \omega \quad \forall T$
hyperbolic : GCC, $T_{min} > 0$

Goal : Understand mechanisms on simple equations/geometries

$$\begin{array}{ll} \text{control system} & \text{adjoint system} \\ \left\{ \begin{array}{l} (\partial_t + L)f(t) = u(t)1_\omega \\ f(0) = f_0 \end{array} \right. & \left\{ \begin{array}{l} (-\partial_t + L^*)g(t) = 0 \\ g(T) = g_T \end{array} \right. \end{array}$$

- **approximate controllability** in time $T > 0$ is equivalent to **unique continuation** :

$$g \equiv 0 \text{ on } (0, T) \times \omega \quad \Rightarrow \quad g \equiv 0 \text{ in } Q_T$$

- **null controllability** in time $T > 0$ is equivalent to **observability** :

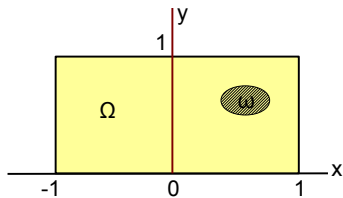
$$\exists C_T > 0, \quad \|g_0\|_{L^2(\Omega)}^2 \leq C_T^2 \int_0^T \int_\omega |g(t, x)|^2 dx dt, \quad \forall g_T \in L^2(\Omega).$$

Difficulty : No proof of appropriate Carleman estimates for observability

- ① Grushin-type equations
- ② Heisenberg heat equation
- ③ Kolmogorov-type equations

PART 1 : Grushin-type equations

$$\begin{cases} \left(\partial_t - \partial_x^2 - |x|^{2\gamma} \partial_y^2 \right) f(t, x, y) = 1_{\omega}(x, y) u(t, x, y) & \text{in } \Omega \quad \gamma > 0 \\ f(t, \dots) = 0 & \text{on } \partial\Omega \end{cases}$$



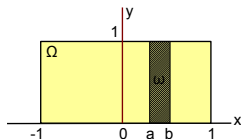
For $\gamma \in \mathbb{N}$, $L = X_1^2 + X_2^2$ with $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}$

$$[X_1, X_2] = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \quad [X_1, [X_1, X_2]] = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \dots$$

Rk : Unique continuation holds $\forall \gamma > 0, \forall \omega, \forall T > 0$.

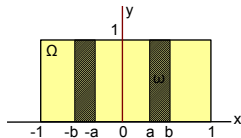
Results

$$(\mathbf{G}) : \begin{cases} (\partial_t - \partial_x^2 - |x|^{2\gamma} \partial_y^2) f = 1_\omega u(t, x, y) \\ f(t, \cdot, \cdot) = 0 \text{ on } \partial\Omega \end{cases}$$



Thm : [KB-Cannarsa-Guglielmi 2012]

- If $\gamma < 1$ then (\mathbf{G}) is controllable $\forall T > 0, \forall \omega$.
- If $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$ then (\mathbf{G}) is controllable only in large time : $T_{min} > 0$
- If $\gamma > 1$ then (\mathbf{G}) is not controllable.



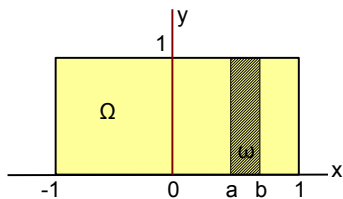
Thm : [KB-Miller-Morancey 2014] $\gamma \geq 1$

- If $\gamma = 1$ then $T_{min} = a^2/2$.
- Otherwise, the null controllable initial conditions are functions analytic in y , with L^2 -values in x such that

$$\sum_{n \in \mathbb{N}^*} e^{2d\pi n} \|f_n^0\|_{L^2}^2 < \infty \text{ where } d := \begin{cases} \frac{a^{\gamma+1}}{\gamma+1} & \text{if } \gamma > 1 \\ \frac{a^2}{2} - T & \text{if } \gamma = 1, T < \frac{a^2}{2}. \end{cases}$$

- If $\omega = (0, b) \times (0, 1)$ then (\mathbf{G}) is null controllable $\forall T > 0, \forall \gamma > 0$.

Proof when ω is a strip : Fourier decomposition



$$g(t, x, y) = \sum_{n=1}^{\infty} g_n(t, x) \sin(n\pi y)$$

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 - |x|^{2\gamma} \partial_y^2 \right) g = 0 \\ g(t, \cdot, \cdot) = 0 \quad \text{on } \partial\Omega \end{cases} \quad (G_n^*) \begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 \\ g_n(t, \pm 1) = 0 \end{cases}$$

Goal : $\int_{\Omega} g(T, x, y)^2 dx dy \leq C \int_0^T \int_{\omega} g(t, x, y)^2 dx dy dt$

$$\Leftrightarrow \sum_{n=1}^{\infty} \int_{-1}^1 g_n(T, x)^2 dx \leq C \sum_{n=1}^{\infty} \int_0^T \int_a^b g_n(t, x)^2 dx dt$$

\Leftrightarrow Uniform observability of (G_n^*) wrt n :

$$\int_{-1}^1 g_n(T, x)^2 dx \leq C \int_0^T \int_a^b g_n(t, x)^2 dx dt$$

Uniform observability when $\gamma \in (0, 1)$ or $[\gamma = 1, T \text{ large}]$

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 \\ g_n(t, \pm 1) = 0 \end{cases}$$

- ① **Explicit decay rate** for the Fourier components (rescaling)

$$\int_{-1}^1 g_n(T, x)^2 dx \leq e^{-cn^{\frac{2}{1+\gamma}}(T-t)} \int_{-1}^1 g_n(t, x)^2 dx, \quad \forall t \in [0, T]$$

- ② Global **Carleman estimates** with weight $e^{\frac{nT^2 \psi(x)}{t(T-t)}}$

$$\int_{T/3}^{2T/3} \int_{-1}^1 g_n(t, x)^2 dx dt \leq e^{Cn} \int_0^T \int_a^b g_n(t, x)^2 dx dt$$

$$\Rightarrow \frac{T}{3} \int_{-1}^1 g_n(T, x)^2 dx \leq e^{Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3}} \int_0^T \int_a^b g_n(t, x)^2 dx dt$$

$Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3} \rightarrow -\infty$ when $[0 < \gamma < 1]$ or $[\gamma = 1, T > 3/c]$

Rk : The 2D-Grushin-equation is not conjugated by a weight fn !

Proof when ω is arbitrary

Lebeau-Robbiano's strategy : $0 = T_0 < T_1 < \dots < T_j \rightarrow T$:

- on $[T_j, T_{j+1/2}]$, one applies a control that steers the 2^j -first components to zero,
- on $[T_{j+1/2}, T_{j+1}]$, no control \rightarrow dissipation.

[Benabdallah-Dermenjian-Le Rousseau 07]

Lebeau-Robbiano's inequality : $\exists C > 0$ such that $\forall N, \forall b \in l^2$

$$\int_0^{2\pi} \left| \sum_{k=1}^N b_k \sin(ky) \right|^2 dy = \sum_{k=1}^N |b_k|^2 \leq e^{cN} \int_c^d \left| \sum_{k=1}^N b_k \sin(ky) \right|^2 dy$$

Key point :

dissipation $N^{\frac{2}{1+\gamma}}$ $\gg \gg$ cost N and Lebeau-Robbiano's cst N
 \rightarrow only when $\gamma < 1$

No uniform observability when $[\gamma > 1]$ or $[\gamma = 1, T \leq a^2/2]$

Consider the solution of the Grushin eq $g_n(t, x) := w_n(x)e^{-\lambda_n t}$ built with the first eigenfunction

$$\begin{cases} -w_n''(x) + (n\pi)^2|x|^{2\gamma}w_n = \lambda_n w_n(x), & x \in (-1, 1), n \in \mathbb{N}^*, \\ w_n(\pm 1) = 0, & w_n \geq 0. \end{cases}$$

Then

$$\frac{\int_0^T \int_a^b g_n(t, x)^2 dx dt}{\int_{-1}^1 g_n(T, x)^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b w_n(x)^2 dx \leq C e^{c|n|^{\frac{2}{1+\gamma}} T} e^{-2\pi \frac{a\gamma+1}{\gamma+1} |n|}$$

converges to zero.

Key point : Agmon estimates with $h = \frac{1}{n\pi}$ and $V(x) = |x|^{2\gamma}$

$$-h^2 \Delta w + V(x)w = \lambda w \quad \Rightarrow \quad w(x) \leq C e^{-\frac{d(x)-\epsilon}{h}}$$

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 \\ g_n(t, \pm 1) = 0 \end{cases}$$

Via 1D transmutation and lateral propagation of energy method on the resulting 1D wave equation, we obtain the **optimal cost estimate**

$$\|g_n(T)\|_{L^2(-1,1)} \leq C e^{D^+ |n|} \left(\int_0^T \int_{(-b,-a) \cup (a,b)} |g_n(t,x)|^2 dx \right)^{1/2}$$

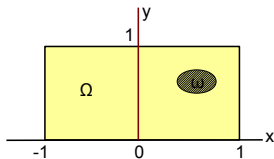
$$\text{where } D := \begin{cases} \pi \frac{a^{\gamma+1}}{\gamma+1} & \text{if } \gamma > 1, \\ \pi \left(\frac{a^2}{2} - T \right) & \text{if } \gamma = 1. \end{cases}$$

This proves :

- uniform observability wrt n iff $T > T_{min} := a^2/2$ when $\gamma = 1$,
- characterization of null controllable initial conditions :

$$\|u\|_{L^2((0,T) \times \Omega)}^2 = \sum_{n \in \mathbb{Z}} \|u_n\|_{L^2((0,T) \times (-1,1))}^2 \leq \sum_{n \in \mathbb{Z}} C^2 e^{2D^+ |n|} \|f_n^0\|_{L^2(-1,1)}^2$$

$$\left(\partial_t - \partial_x^2 - |x|^{2\gamma} \partial_y^2\right) f(t, x, y) = \mathbf{1}_\omega(x, y) u(t, x, y)$$



- Generalizations to $x \in \Omega_1$, $y \in \Omega_2$ with coeff $\|x\|^{2\gamma} b(x)$
+ Inverse-source problem [KB-Cannarsa-Yamamoto 2013]
- **Open** : $\gamma = 1$ and ω arbitrary
- **Open** : general geometry (Ω, ω)
- $\gamma \geq 1$: For given control support ω and time T , the regularity of initial conditions that are null controllable (with L^2 -controls) depends on ω and T \rightarrow GCC : **precise dependence** ?

PART 2 : Heisenberg heat equation

$$\left[\partial_t - \left(\partial_{x_1} - \frac{x_2}{2} \partial_{x_3} \right)^2 - \left(\partial_{x_2} + \frac{x_1}{2} \partial_{x_3} \right)^2 \right] g(t, x_1, x_2, x_3) = 1_\omega u(t, x_1, x_2, x_3)$$

$$L = X_1^2 + X_2^2 \text{ with } X_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{x_2}{2} \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1}{2} \end{pmatrix} \quad [X_1, X_2] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Fourier series in x_3 lead to a 2D heat-equation with complex-valued coefficients, on which the previous strategy fails.

Change of variables : $x = x_1$ $y = x_2$ $z = x_3 + \frac{x_1+x_2}{2}$

$$\left(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2 \right) g(t, x, y, z) = 1_{\tilde{\omega}} \tilde{u}(t, x, y, z)$$

This formulation is well adapted to Fourier transform in variables (y, z) .
We will study this eq on

- the rectangle $(x, y, z) \in (-1, 1) \times \mathbb{T} \times \mathbb{T}$
- the unbounded domain $(x, y, z) \in (-1, 1) \times \mathbb{T} \times \mathbb{R}$

$$(H) : \begin{cases} (\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2) f(t, x, y, z) = 1_\omega u(t, x, y, z) \\ f(t, \pm 1, y, z) = 0, \end{cases}$$

Thm : [KB-Cannarsa 2015]

- On the rectangle $\Omega = (-1, 1) \times \mathbb{T} \times \mathbb{T}$ with $\omega = (a, b) \times \omega_y \times \mathbb{T}$,
- On $\Omega = (-1, 1) \times \mathbb{T} \times \mathbb{R}$ with $\omega = (a, b) \times \omega_y \times \mathbb{R}$,

(H) is null controllable in large time $T_{min} \geq \frac{1}{8} \max\{(1+a)^2, (1-b)^2\}$.

Remarks :

- Unique continuation holds $\forall T > 0$ (Holmgren principle).
- T_{min} is always > 0 and related to the distance between $\partial\Omega$ and ω .

Key points of the proof

$$\begin{cases} (\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2)g(t, x, y, z) = 0, & (x, y, z) \in (-1, 1) \times \mathbb{T} \times \mathbb{T} \\ g(t, \pm 1, y, z) = 0, \end{cases}$$

Fourier series : $y \mapsto n \in \mathbb{Z}$ $z \mapsto p \in \mathbb{Z}$

$$\begin{cases} (\partial_t - \partial_x^2 + (px + n)^2)g_{n,p}(t, x) = 0, & x \in (-1, 1) \\ g_{n,p}(t, \pm 1) = 0, \end{cases}$$

① **Decay rate** : $\lambda_{n,p} \geq \begin{cases} c|p| \\ cn^2 \end{cases}$ if $|n| \geq 2|p|$

② **Optimal weight** for Carleman estimates : $e^{(|n|+|p|)\frac{T^2\psi(x)}{\epsilon(T-\epsilon)}}$

With p fixed :

dissipation $N^2 \gg \gg$ cost N **and** Lebeau-Robbiano's cst N

- Lebeau-Robbiano's strategy wrt n , with p fixed : localization wrt y
- Quantification of the cost $e^{|p|(c_1 - c_2 T)}$: uniform wrt p for large T

$$\left(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\right)g(t, x, y, z) = 0$$

- Lipschitz stability estimate for the inverse source-problem [KB-Cannarsa 2015]
- **Open problems :**
 - explicit value of the minimal time
 - ω arbitrary (localization in z)
 - observability inequality with $\Omega = \mathbb{R} \times \mathbb{T} \times \mathbb{T}$
 $\lambda_{n,p} = c|p|$ (invariance under translation in x)
→ no localization in y should be possible.
- **Conjecture :** If the first Lie bracket is sufficient to satisfy Hörmander's condition, then null controllability holds in large time. (without boundary)

PART 3 : Kolmogorov-type equations

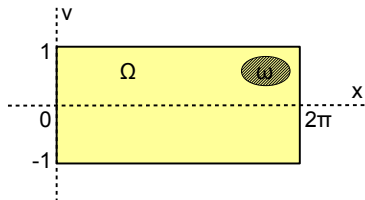
$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right) f(t, x, v) = 1_\omega(x, v) u(t, x, v)$$

For $\gamma \in \mathbb{N}^*$, $L = X_1^2 + X_0$ with $X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $X_0 = \begin{pmatrix} v^\gamma \\ 0 \end{pmatrix}$

~ linearizations of Prandtl and Crocco equations for fluids.

We will consider this equation on

- the **whole space** : $(x, v) \in \mathbb{R} \times \mathbb{R}$ with $\gamma = 1$
- the **rectangle** : $(x, v) \in \mathbb{T} \times (-1, 1)$
 - with Dirichlet boundary conditions in v , $\forall \gamma \in \mathbb{N}^*$
 - with periodic-boundary conditions in x with $\gamma = 1$



$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right) f(t, x, v) = 1_\omega(x, v) u(t, x, v)$$

Thm : [KB 2014 + KB-Helffer-Henry-Robbiano] $(x, v) \in \mathbb{T} \times (-1, 1)$

- Periodic/ $v + (\gamma = 1)$: null controllable $\forall T > 0, \forall \omega$,
- Dirichlet/ $v + (\omega = \mathbb{T} \times (a, b))$
 - $\gamma = 1$: null controllable $\forall T > 0$
 - $\gamma = 2$: null controllable only in large time : $T_{min} \geq \frac{a^2}{2}$
 - $\gamma \geq 3$: not null controllable.

Rk : Unique continuation holds $\forall \gamma$, but not for all zero-order C^∞ -perturbation [Alinhac-Zuily 1980]

Thm : [Moyano-Le Rousseau 2015] $\gamma = 1$ $(x, v) \in \mathbb{R}^{2d}$ $\omega = \omega_x \times \omega_v$
 If both ω_x and ω_v satisfy

$$\exists \delta, r > 0 / \forall y \in \mathbb{R}^d, \exists y' \in \omega_x / B(y', r) \subset \omega_x \text{ and } \|y - y'\| < \delta.$$

then null controllability holds $\forall T > 0$.

Key points of the proof on the rectangle

$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right)g(t, x, v) = 0 \quad (x, v) \in \mathbb{T} \times (-1, 1)$$

Fourier series : $x \mapsto n \in \mathbb{Z}$

$$\left(\partial_t + inv^\gamma - \partial_v^2\right)g_{n,p}(t, x, v) = 0 \quad (x, v) \in \mathbb{T} \times (-1, 1)$$

- ① **Decay rate** : periodic/ v and $\gamma = 1$: $|n|^2$ (explicit)
Dirichlet/ v : $|n|^{\frac{2}{2+\gamma}}$ (subtle spectral analysis)

- ② **Optimal weight** for Carleman estimates : $e^{\sqrt{|n|} \frac{T^2 \psi(x)}{t(T-t)}}$

Periodic/ v and $\gamma = 1$:

dissipation $N^2 \gg \gg$ cost \sqrt{N} **and** Lebeau-Robbiano's cst N
 \rightarrow observability $\forall \omega$

Dirichlet/ v :

dissipation $N^{2/3}$ or \sqrt{NT} $\gg \gg$ cost \sqrt{N} **but not** L-R's cst N
 \rightarrow this strategy does not allow localization in variable x .

Quantification wrt n of the decay rate of Fourier modes

Goal : $\partial_t g_n + inv^\gamma g_n - \partial_v^2 g_n = 0, \quad v \in (-1, 1), \quad g_n(t, \pm 1) = 0$

$$\Rightarrow \exists K, c > 0 \quad \|g_n(t)\|_{L^2(-1,1)} \leq K \|g_n^0\|_{L^2(-1,1)} e^{-cn^{\frac{2}{2+\gamma}} t}, \quad \forall n, g_n(0)$$

After rescaling $g_n(t, v) = h(\tau = n^{\frac{2}{2+\gamma}} t, y = n^{\frac{1}{2+\gamma}} v)$, $R = n^{\frac{1}{2+\gamma}}$, we need

$$\partial_\tau h + iy^\gamma h - \partial_y^2 h = 0, y \in (-R, R) \quad h(\tau, \pm R) = 0$$

$$\Rightarrow \exists K, c > 0 \quad \|h(\tau)\|_{L^2(-R,R)} \leq K \|h^0\|_{L^2(-R,R)} e^{-c\tau}, \quad \forall R, h_0$$

Strategy : From resolvent bounds to semigroup bounds
[Helffer-Sjöstrand 2009] a quantitative Gearhardt-Prüss statement

Key points : The Davies operator $\left(-\frac{d^2}{dy^2} + iy^2\right)$ on the whole line \mathbb{R} and the Dirichlet realization of the complex Airy operator $\left(-\frac{d^2}{dy^2} + iy\right)$ on the half line $(0, \infty)$ have discrete spectrum and ad hoc resolvent estimates.
For $\gamma = 1$: transformations $y \mapsto \pm y \pm R$ only affect $\Im(\text{eigenvalues})$.

Key ideas of the proof on the whole space

[Moyano-Le Rousseau 2015]

$$\left(\partial_t + v\partial_x - \partial_v^2\right)g(t, x, v) = 0, \quad (x, v) \in \mathbb{R}^2$$

Fourier transform : $x \mapsto \xi \in \mathbb{R}$

$$\left(\partial_t + i\xi v - \partial_v^2\right)g_\xi(t, v) = 0, \quad (x, v) \in \mathbb{R}^2$$

① **Decay rate** : $e^{-\xi^2 t^3}$ (explicit solution)

② **Optimal weight** for Carleman estimates : $e^{\sqrt{|\xi|} \frac{T^2 \psi(x)}{t(T-t)}}$

dissipation $N^2 \gg \gg$ cost \sqrt{N} **and** Lebeau-Robbiano's cst N

Spectral inequality : There exists $C > 0$ such that, for every $N > 0$,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx \leq e^{CN} \int_{\omega_x} |f(x)|^2 dx, \quad \forall f \in L^2(\mathbb{R}^d) / \text{Supp}(\hat{f}) \subset B_{\mathbb{R}^d}(0, N)$$

$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right)g(t, x, v) = 0 \quad (x, v) \in \mathbb{T} \times (-1, 1)$$

- **Conjecture** : Null controllability does not hold with $\omega = (a, b) \times (-1, 1)$ and Dirichlet/ $v \rightarrow$ **GCC**
Proof on a toy model $\partial_x \leftarrow \sqrt{-\Delta_x}$: [\[KB-Helffer-Henry-Robbiano\]](#)
- Characterization of **null controllable data** in term of Gevrey-2 regularity with $\omega = \mathbb{T} \times (a, b)$ (optimal costs)
- For given control support ω and time T , the regularity of initial conditions that are null controllable (with L^2 -controls) depends on ω and T . **Precise dependence ?**
- **General geometry ?** Direct 2D approach ?
- **Without Fourier transform ?** $x\partial_v$
- + degenerate diffusion (Linearized Crocco eq)
+ nonlinearity ...