Observation of solutions of the Neutron Transport Equation in the diffusion limit.

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Luminy : Contrôle des EDP et Applications.

Observation of the Initial Data for the Transport Equation.

In a convenient parabolic scaling the solution of the Transport Equation converges to the solution of the Diffusion Equation

- Since the problem is linear, for well prepared initial data one has a good control of the distance between the solution of the transport equation and the solution of the diffusion equation.
- One can use the now classical estimates on the solution of the diffusion equation to obtain observation estimates for the initial data of the neutron transport equation.
- This may have practical application in particular for non destructive remote sensing
- This may give some intuition to support the corresponding results for the diffusion equation????

Transport Equation.

$$\partial_t f(x,v,t) + v\cdot\nabla_x f = \int_{\mathbf{R}^N} [k(x,v',v)f(v') - k(x,v,v')f(v)]dv'$$

$$\text{Detailed balance} \int_{\mathbf{R}^N} k(x,v,v')dv' = \int_{\mathbf{R}^N} k(x,v',v)dv'$$

$(x,v,t)\mapsto f(x,v,t)$ is a function defined on the phase space $\Omega\times\mathbf{R}^N$ and describes the density of particles which at the point x and time t do have the velocity v.
It was as introduced by Lorentz (1905) for a gas of electrons.
Many further applications, electrons in semi conductors , plasma , radiative transfer and interaction of neutrons with uranium kernels.

An elementary version which contains most of the mathematical subtleties $x \in \Omega \quad \omega \in \mathbf{S}^{N-1}$

$$\partial_t u_\epsilon + \frac{\omega \cdot \nabla_x u_\epsilon}{\epsilon} + \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon) = 0\,, \quad \overline{u} = \frac{1}{|\mathbf{S}^{N-1}|}\int u(x,\omega)\overline{d\omega}$$

- All the particles have the same kinetic energy $v = \omega \in \mathbf{S}^{N-1}$
- $\Omega \subset \mathbf{R}^N_x$ is either the torus $\mathbf{T}^N = \mathbf{R}^N/\mathbf{Z}^N$ or a domain with smooth boundary and exterior normal $\vec{n}_x$ In this second case a boundary condition has to be prescribed:

Specular :

$$x \in \partial\Omega \Rightarrow u(x,\omega) = u(x, \mathcal{R}(\omega))\,, \mathcal{R}(\omega) = \omega - 2\vec{n}_x(\omega \cdot \vec{n}_x)$$

Absorbing :

$$u(x,\omega,t)\Big|(\Gamma_-) = 0\,, \Gamma_- = \{(x,\omega) \in \partial\Omega \times \mathbf{S}^{N-1}\,, \omega \cdot \vec{n}_x < 0\}$$

Well posed problem, spectral properties well understood Jurgens, Ghidouche Point Ukai...Han-Kwan Léautaud.

Scaling

ϵ is a scaling parameter which will play an important role later and $\sigma_\epsilon(x) \geq 0$ represent the opacity of the media.
The behavior of the solution results from the competition between the advection term $\omega \cdot \nabla_x$ and the relaxation term $\sigma(x)(u - \overline{u})$ and the role of ϵ is to measure the relative strenght of these two effects. More over such ϵ is motivated by physic.

Examples:

- scattering cross-section of neutrons collisions of non fission type in uranium oxide $\simeq 10\text{cm}^{-1}$
- scattering cross-section of neutrons collisions in water 0.1cm^{-1} a
- 100 times more in uranium than in water.

Transport and Diffusion;What I want to consider.

$$
\partial_t u_\epsilon + \frac{\omega \cdot \nabla_x u_\epsilon}{\epsilon} + \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon) = 0\,,
$$

$$
\overline{d\omega} = \frac{\overline{d\omega}}{|\mathbf{S}^{N-1}|} \quad \overline{u} = \int u(x,\omega)\overline{d\omega}\,,
$$

$$
\lim_{\epsilon\to 0}\sigma_\epsilon(x) = \sigma(x) \ \lim_{\epsilon\to 0}\frac{1}{|\mathbf{S}^{N-1}|}u_\epsilon(x,\omega,t) = \rho(x,t)\,,
$$

$$
\partial_t\rho - \nabla_x\Big(\frac{1}{N\sigma(x)}\nabla_x\rho\Big) = 0\ ,\ \ \rho(x,0) = \lim_{\epsilon\to 0}\int u_\epsilon(x,\omega,0)\overline{d\omega}
$$

Plus boundary condition if $\partial\Omega \neq \emptyset$.

- Absorbing $\Rightarrow$ Dirichlet.
- Specular $\Rightarrow$ Neumann.

Convergence to the Diffusion. Estimates for $t > 0$ valid for all type of standard boundary condition.

$$\partial_t u_\epsilon + \frac{\omega \cdot \nabla_x u_\epsilon}{\epsilon} + \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon) = 0\,, \quad \overline{u} = \frac{1}{|\mathbf{S}^{N-1}|}\int u(x,\omega)\overline{d\omega}$$

- Maximum principle:

$$\inf_{(x,\omega)\in\Omega\times\mathbf{S}^{N-1}} u_\epsilon(x,\omega,0) \le u_\epsilon(x,\omega,t) \le \sup_{(x,\omega)\in\Omega\times\mathbf{S}^{N-1}} u_\epsilon(x,\omega,0)\,;$$

- Energy estimate:

$$\frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |u_\epsilon(x,\omega,t)|^2 dx\overline{d\omega} + \int_{\Omega\times\mathbf{S}^{N-1}} \frac{1}{\epsilon^2}\sigma_\epsilon(x)(u_\epsilon - \overline{u}_\epsilon)^2 dx\overline{d\omega} \le \frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}} |u_\epsilon(x,\omega,0)|^2 dx\overline{d\omega}\,.$$

Hence the solution is described (in any $L^p\,, 1 < p \le \infty$) by a contraction semi group $e^{t\mathcal{T}_\epsilon}$.

For weak limit u of subsequence $u_\epsilon \to u$ with $\sigma_\epsilon \to \sigma$ converging strongly

$$\min(\sigma(x)) \int_0^T \int_\Omega |u - \overline{u}|^2 dx\overline{d\omega}dt \leq \lim_{\epsilon \to 0} \int_0^T \int_\Omega \sigma(x)|u_\epsilon - \overline{u_\epsilon}|^2 dx\overline{d\omega}dt$$

$$\leq \epsilon^2 \frac{1}{2} \int_{\Omega \times \mathbf{S}^{N-1}} |u(x,\omega,0)|^2 dx\overline{d\omega} \Rightarrow u = \overline{u} = \rho(x,t).$$

$$\partial_t \overline{u_\epsilon} + \nabla_x (\frac{\overline{\omega u_\epsilon}}{\epsilon}) = 0 \ \text{ Fick Law}$$

$$\frac{1}{\sigma_\epsilon(x)} \partial_t \overline{\omega u_\epsilon} + \frac{1}{\sigma_\epsilon(x)} \nabla_x \overline{\omega \otimes \omega u_\epsilon} + \frac{\overline{\omega u_\epsilon}}{\epsilon} = 0$$

$$\partial_t \overline{u_\epsilon} - \nabla_x (\frac{1}{\sigma_\epsilon(x)} \nabla_x \overline{\omega \otimes \omega u_\epsilon}) = \epsilon \nabla_x (\frac{1}{\sigma_\epsilon(x)} \partial_t \overline{\omega u_\epsilon}) \to 0$$

$$\partial_t \rho - \nabla_x (\frac{1}{N\sigma(x)} (\nabla_x \rho)) = 0\,.$$

Hence the limit of the average density is given by the semi group $e^{-t\mathcal{A}}$

$$\partial_t \rho - \nabla_x (\frac{1}{N\sigma(x)} (\nabla_x \rho)) = 0$$

$$\partial_t \rho + \mathcal{A}\rho = 0\,,$$

With Dirichlet or Neumann Boundary conditions.

Strong Convergence

Theorem

Under the same hypothesis $\overline{u}_\epsilon(x,\omega,t) \to \rho(x,t)$ *in* $L^\infty(\mathbf{R}_t^+; L^2(\Omega \times \mathbf{S}^{N-1}))$ *strong.*

Proof Start from:

$$\sqrt{\sigma_\epsilon(x)}\frac{u_\epsilon - \overline{u_\epsilon}}{\epsilon} \text{ bounded in } L^\infty(\mathbf{R}_t^+; L^2(\Omega \times \mathbf{S}^{N-1}))$$

and converges weakly to

$$\frac{1}{\sqrt{\sigma(x)}}\omega\nabla_x\rho\,.$$

Then by comparison.

$$\frac{1}{2}\int_\Omega |\rho(x,t)|^2 dx + \frac{1}{N\sigma(x)}|\nabla_x \rho(x,s)|^2 dxds = \frac{1}{2}\int_\Omega |\rho(x,0)|^2 dx$$

$$\frac{1}{2}\int_\Omega |\rho(x,t)|^2 dx \leq \limsup \frac{1}{2}\int_\Omega |u_\epsilon(x,\omega,t)|^2 dx\overline{d\omega}$$

$$\int_0^t \int_\Omega \frac{1}{N\sigma(x)}|\nabla_x \rho(x,s)|^2 dxds = \int_0^t \int_{\Omega\times \mathbf{S}^{N-1}} |\frac{1}{\sqrt{\sigma(x)}}\omega\nabla_x \rho(x,s)|^2 dx\overline{d\omega}$$

$$\leq \limsup \int_0^t \int_{\Omega\times \mathbf{S}^{N-1}} (\sqrt{\sigma_\epsilon(x)}\frac{u_\epsilon - \overline{u_\epsilon}}{\epsilon})^2 dx\overline{d\omega}ds$$

$$\frac{1}{2}\int_\Omega |u_\epsilon(x,\omega,t)|^2 dx\overline{d\omega} + \int_0^t \int_{\Omega\times \mathbf{S}^{N-1}} \frac{\sigma_\epsilon(x)}{\epsilon^2}(u_\epsilon - \overline{u_\epsilon})^2 dx\overline{d\omega}ds$$

$$\leq \frac{1}{2}\int_\Omega |\rho(x,0)|^2 dx\,,$$

$$= \frac{1}{2}\int_\Omega |\rho(x,0)|^2 dx\,.$$

A "soft relation " between observation for Transport equation equation and Diffusion equation

Theorem.

Under the above hypothesis, for any $\tilde{\Omega} \subset \Omega$ *and any* $s > 0$ *there exists a bounded function*

$$f(\tilde{\Omega}, s, \|\rho_0\|_{D(\mathcal{A}^s)}) \geq 0$$

such that any sequence of solutions $u_\epsilon(x, \omega, t)$ *of the transport equation with well prepared initial data* $u_\epsilon(x, \omega, 0) = \rho_0(x) \in D(\mathcal{A}^s)$ *satisfies the relation:*

$$\|\rho_0\|_{L^2(\Omega)} \leq f(\tilde{\Omega}, s, \|\rho_0\|_{D(\mathcal{A}^s)}) \lim_{\epsilon \to 0} \|u_\epsilon\|_{L^2(\tilde{\Omega} \times \mathbf{S}^{N-1})} \qquad (1)$$

Proof

If (1) would be wrong there would exists a sequence of $u_{\epsilon(n)}(x,\omega,t)$ of solutions of the transport equation with initial data $u_{\epsilon(n)}(x,\omega,0)=\rho_n(x,0)$ and $\|\rho(x,0)\|_{D(\mathcal{A}^s)})\leq C<\infty$ such that

$$\|\rho_n(x,0)\|_{L^2(\Omega)}\geq\alpha>0 \text{ and } \lim_{\epsilon\to 0}\|u_{\epsilon(n)}\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})}=0$$

Then with the point wise (in time) convergence of the diffusion approximation and the compact injection of $D(\mathcal{A}^s)$ in $\mathcal{H}$ the sequence $\rho_n(x,0)\rightarrow\rho(x,0)\neq 0$ which will be the initial data for a solution $e^{-t\mathcal{A}}(\rho(x,0))$ which vanishes on the set of uniqueness $\tilde{\Omega}\times\{T\}$ Hence the contradiction.

Improving the error estimate with refined properties for the diffusion equation.

Jerison, Lebeau, Le Rousseau , Robbiano, .. and for the equivalence J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Phung, G. Wang...

Theorem

With $U \subset \Omega$ an open subset of Ω and C denoting several constants depending only on Ω , U , and A the following statements are true and equivalent:

$$\int_\Omega |(e^{T\mathcal{A}}u_0)(x)|^2dx \le Ce^{\frac{C}{T}}\Big(\int_U |(e^{T\mathcal{A}}u_0)(x)|^2dx\Big)^{\frac12}\Big(\int_\Omega |u_0(x)|^2dx\Big)^{\frac12}$$

$$\forall\{a_j\ 1\le j\le\infty\}\qquad \sum_{1\le j\le N}|a_j|^2 \le Ce^{C\sqrt{\lambda_N}}\int_U |\sum_{1\le j\le N} a_je_j(x)|^2dx$$

$$u'(t) + \mathcal{A}u = 0 \quad \mathrm{f}_k(u) = \mathrm{f}_k(u_0) = \frac{(\mathcal{A}^{k+1}(u_0), u_0)}{(\mathcal{A}^k(u_0), u_0)}$$

$$N_k(t) = \frac{(\mathcal{A}^{k+1}(u)(t), u(t))}{(\mathcal{A}^k(u)(t), u(t))}, \quad \frac{dN^k, (t)}{dt} < 0 \Rightarrow N^k(t)(u) \leq f_k(u),$$

$$\frac{\frac{d}{dt}(\mathcal{A}^k u(t), u(t))}{(\mathcal{A}^k u(t), u(t))} = -2\frac{(\mathcal{A}^{k+1}u(t), u(t))}{(\mathcal{A}^k u(t), u(t))}(\mathcal{A}^k u(t), u(t))$$

$$(\mathcal{A}^k u(0), u(0))(\mathcal{A}^k u(0), u(0))$$

$$= (\mathcal{A}^k u(T), u(T))e^{2\int_0^T \frac{(\mathcal{A}^{k+1}u(t), u(t))}{(\mathcal{A}^k u(t), u(t))}dt}$$

Theorem

$$((\mathcal{A}^k u(0), u(0)) \leq e^{2T\mathrm{f}_{k+1}(u_0)}(\mathcal{A}^k u(T), u(T))$$

The error estimate for the diffusion approximation. Below estimations done for $\sigma(x) = 1$ with absorbing boundary condition.

Theorem.

2 Let $u_\epsilon(x, v, t)$ be the solution of the rescaled transport equation with absorbing boundary condition and well prepared initial data $u_\epsilon(x, v, 0) = \rho_0(x) \in H^1_0(\Omega) \cap H^2(\Omega)$ and let $\rho(x, t)$ be the solution of the diffusion equation:

$$\partial_t \rho - \frac{1}{N}\Delta\rho = 0\,, \text{ in } \Omega \times \mathbf{R}^+_t \quad \rho(x, t) = 0 \text{ for } (x, t) \in \partial\Omega \times \mathbf{R}^+_t$$

with initial data $\rho(x, 0) = \rho_0$. Then one has:

$$\|u_\epsilon(x, v, t) - \rho(x, t)\|_{L^2(\Omega\times \mathbf{S}^{N-1})} \le \epsilon^{\frac{1}{2}} D\|\rho_0\|_{H^1(\Omega)} \qquad (2)$$

Proof With $\sigma(x) = 1$ with correctors and well prepared initial data.

Start from an Hilbert ansatz:

$$\partial_t(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) + \frac{\omega\cdot\nabla_x(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))}{\epsilon}$$
$$+\frac{1}{\epsilon^2}((u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) - \overline{((u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))}$$
$$= \epsilon\partial_t(\omega\cdot\nabla_x\rho) + (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho) = \mathcal{R}_\epsilon(x,\omega,t)\,.$$

Next one estimates

$$\int_0^t\int_{\Omega\times S^{N-1}}\mathcal{R}_\epsilon(x,\omega,t)(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))dx\overline{d\omega}ds$$

With ρ solution of the limit equation and

$$(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho)) = (u_\epsilon - \overline{u_\epsilon}) + (\overline{u_\epsilon} - \rho + \epsilon\omega\cdot\nabla_x\rho)$$

one has:

$$\int_{\Omega\times\mathbf{S}^{N-1}} (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho) + \epsilon\partial_t(\omega\cdot\nabla_x\rho))$$
$$((u_\epsilon - \overline{u_\epsilon}) + (\overline{u_\epsilon} - \rho + \epsilon\omega\cdot\nabla_x\rho))dx\overline{d\omega}$$
$$= \int_{\Omega\times\mathbf{S}^{N-1}} (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)(u_\epsilon - \overline{u_\epsilon})dx\overline{d\omega}$$
$$+\,\epsilon\int_{\Omega\times\mathbf{S}^{N-1}} \partial_t(\omega\cdot\nabla_x\rho)(u_\epsilon - \overline{u_\epsilon} + \epsilon(\omega\cdot\nabla_x\rho))dx\overline{d\omega}$$

And eventually:

$$
\begin{aligned}
&|\int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)(u_\epsilon - \overline{u_\epsilon})dx\overline{d\omega}ds| \\
&\leq \epsilon(\int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} (-\partial_t\rho + (\omega\cdot\nabla_x)^{\otimes^2}\rho)^2 dx\overline{d\omega}ds)^{\frac{1}{2}} \\
&\qquad (\int_0^t \int_{\Omega\times\mathbf{S}^{N-1}} \frac{1}{\epsilon^2}(u_\epsilon - \overline{u_\epsilon})^2 dx\overline{d\omega}ds)^{\frac{1}{2}} \\
&\leq C\epsilon\|\rho_0\|_{H^1(\Omega)}\|\rho_0\|_{L^2(\Omega)}
\end{aligned}
$$

By the same token

$$\epsilon \int_{\Omega\times\mathbf{S}^{N-1}} \partial_t(\omega\cdot\nabla_x\rho)(u_\epsilon - \overline{u_\epsilon} + \epsilon(\omega\cdot\nabla_x\rho))dx\overline{d\omega} =$$

$$\left[\epsilon\int_{\Omega\times\mathbf{S}^{N-1}}(\omega\cdot\nabla_x\rho)(u_\epsilon-\overline{u_\epsilon})dx\overline{d\omega} + \epsilon^2\frac{1}{2}\int_{\Omega\times\mathbf{S}^{N-1}}(\omega\cdot\nabla_x\rho))^2dx\overline{d\omega}\right]_0^t$$

$$-\epsilon\int_0^t\int_{\Omega\times\mathbf{S}^{N-1}}(\omega\cdot\nabla_x\rho)\partial_t(u_\epsilon-\overline{u_\epsilon})dx\overline{d\omega}ds$$

The first term above is bounded by $C\epsilon\|\rho_0\|^2_{H^1_0(\Omega)}$.

Very end of the proof

For the second one observes that $\partial_t u_\epsilon$ is a solution of the same transport equation with (well prepared initial data)

$$u_\epsilon(x,\omega,0) = \rho_0 \Rightarrow \partial_t u_\epsilon(x,\omega,0) = -\frac{1}{\epsilon}\omega \cdot \nabla_x \rho_0(x) \Rightarrow$$

$$|\int_0^t \int_{\Omega\times \mathbf{S}^{N-1}} (\omega\cdot\nabla_x\rho)(\partial_t(u_\epsilon - \overline{u_\epsilon}))dx\overline{d\omega}ds| \leq C\|\rho_0\|^2_{H^1(\Omega)}$$

$$|\int_0^t \int_{\Omega\times \mathbf{R}^{N-1}} \mathcal{R}_\epsilon(x,\omega,t)((u_\epsilon - \overline{u_\epsilon} + \epsilon\omega\cdot\nabla_x\rho)dx\overline{d\omega}| \leq C\epsilon\|\rho_0\|^2_{H^1(\Omega)}$$

Hence multiplying the equation

$$\partial_t(u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))+\frac{\omega\cdot\nabla_x(u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))}{\epsilon}$$
$$+\frac{1}{\epsilon^2}((u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))-\overline{((u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))}$$
$$=\epsilon\partial_t\omega\cdot\nabla_x\rho+(-\partial_t\rho+(\omega\cdot\nabla_x)^{\otimes^2}\rho)=\mathcal{R}_\epsilon(x,\omega,t)\,.$$

by $(u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))$ using the classical integrations by part gives:

$$|(u_\epsilon-(\rho-\epsilon\omega\cdot\nabla_x\rho))(x,\omega,t)|^2_{L^2(\Omega\times\mathbf{S}^{N-1})}$$
$$\le\epsilon^2\|(\omega\cdot\nabla_x\rho)(x,0)\|^2_{L^2(\Omega\times\mathbf{S}^{N-1})}$$
$$|\int_0^t\int_{\Omega\times\mathbf{R}^{N-1}}\mathcal{R}_\epsilon(x,\omega,t)((u_\epsilon-\rho+\epsilon\omega\cdot\nabla_x\rho)dx\overline{d\omega}|$$
$$+\epsilon\int_0^t\int_{(\partial\Omega\times\mathbf{S}^{N-1})\cap\omega\cdot n<0}|\omega\nabla_x\rho(x,s)|^2d\sigma\overline{d\omega}$$

And Eventually with

$$\left| \int_0^t \int_{\Omega\times\mathbf{R}^{N-1}} \mathcal{R}_\epsilon(x,\omega,t)((u_\epsilon - \rho + \epsilon\omega\cdot\nabla_x\rho) dx \overline{d\omega} \right| \le C\epsilon \|\rho_0\|^2_{H^1(\Omega)}$$

one has:

$$\|(u_\epsilon - \rho)(t)\|_{L^2(\Omega\times\mathbf{S}^{N-1})} \le \|(u_\epsilon - (\rho - \epsilon\omega\cdot\nabla_x\rho))(x,\omega,t)\|_{L^2(\Omega\times\mathbf{S}^{N-1})}$$
$$+ \epsilon\|(\omega\cdot\nabla_x\rho))(x,\omega,t)|_{L^2(\Omega\times\mathbf{S}^{N-1})} + C\sqrt{\epsilon}\|\rho_0\|_{H^1(\Omega)}$$
$$\le D\sqrt{\epsilon}\|\rho_0\|_{H^1(\Omega)}$$

Evaluation of the source $u_\epsilon(x,\omega,0) = \rho_0(x)$ in term of frequency and Observation.

Introduce $F(\rho_0, T) = \mathfrak{f}_0(\rho_0)e^{2\mathfrak{f}_0(\rho_0)T}$
In all what follows D is the constant related to the diffusion approximation cf (2) and C is given by the theorem 2

Theorem.

For any $\tilde\Omega \subset \Omega$ and $T > 0$ one has :

$$\|\rho_0\|_{H^1_0(\Omega)}(1-\sqrt{\epsilon}DF(\rho_0, T)Ce^{\frac{C}{T}}\|\rho_0\|_{H^1_0(\Omega)})$$
$$\leq F(\rho_0, T)Ce^{\frac{C}{T}}\|u_\epsilon(x,\omega, T)\|_{L^2(\tilde\Omega\times \mathbf{S}^{N-1})}$$

Proof

Corollary.

For ϵ small enough with respect to the frequency of the solution and the time T of observation:

$$\epsilon \leq (DF(\rho_0, T)Ce^{\frac{C}{T}})^{-1}$$

one has the following estimate on the source in term of observation on an arbitrarily small small open set $\tilde{\Omega} \subset \Omega$ at any time $T \geq \delta > 0$

$$\|\rho_0\|_{H^1(\Omega)} \leq \frac{DF(\rho_0, T)Ce^{\frac{C}{T}}\|u_\epsilon(x, \omega, T)\|_{L^2(\tilde{\Omega}\times \mathbf{S}^{N-1})}}{(1 - \sqrt{\epsilon}DF(\rho_0, T)Ce^{\frac{C}{T}}\|\rho_0\|_{H^1(\Omega)})}.$$

Proof

Introduce $F(\rho_0, T) = \mathrm{f}_0(\rho_0)e^{2\mathrm{f}(\rho_0)T}$ and start from:

$$
\begin{aligned}
&\|\rho_0\|^2_{H^1(\Omega)} \leq \mathrm{f}_0(\rho_0)e^{2\mathrm{f}_0(\rho_0)T}\|\rho(x, T)\|^2_{L^2(\Omega)} \\
&\leq F(\rho_0, T)Ce^{\frac{C}{T}}\|\rho(x, T)\|_{L^2(\tilde{\Omega})}\|\rho(x, T)\|_{H^1(\Omega)} \\
&\leq F(\rho_0, T)Ce^{\frac{C}{T}}\|\rho(x, T) - u_\epsilon(x, \omega, T)\|_{L^2(\tilde{\Omega}\times\mathbf{S}^{N-1})}\|\rho_0\|_{H^1(\Omega)} \\
&+ F(\rho_0, T)Ce^{\frac{C}{T}}\|u_\epsilon(x, \omega, T)\|_{L^2(\tilde{\Omega}\times\mathbf{S}^{N-1})}\|\rho_0\|_{H^1(\Omega)} \\
&\leq \sqrt{\epsilon}DF(\rho_0, T)Ce^{\frac{C}{T}}\|\rho_0\|^2_{H^1(\Omega)} \\
&+ F(\rho_0, T)Ce^{\frac{C}{T}}\|u_\epsilon(x, \omega, T)\|_{L^2(\tilde{\Omega}\times\mathbf{S}^{N-1})}\|\rho_0\|_{H^1(\Omega)}
\end{aligned}
$$

or finally:

$$
\begin{aligned}
&\|\rho_0\|_{H^1(\Omega)}(1 - \sqrt{\epsilon}DF(\rho_0, T)Ce^{\frac{C}{T}}\|\rho_0\|_{H^1(\Omega)}) \\
&\leq DF(\rho_0, T)Ce^{\frac{C}{T}}\|u_\epsilon(x, \omega, T)\|_{L^2(\tilde{\Omega}\times\mathbf{S}^{N-1})}
\end{aligned}
$$

Remark 1 Relation with Hyperbolic Approximation:

Start as above from the average:

$$\partial_t \overline{u_\epsilon} - \nabla_x (\nabla_x \overline{\omega \otimes \omega u_\epsilon}) = \epsilon \nabla_x (\partial_t \overline{\omega u_\epsilon})$$

$$\text{From} \quad \epsilon \partial_t u_\epsilon + \omega \cdot \nabla_x u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - \overline{u_\epsilon}) = 0$$

$$\text{deduces} \quad \overline{\omega \cdot \nabla_x u_\epsilon} = -\epsilon \overline{\partial_t u_\epsilon} \Rightarrow$$

$$\epsilon^2 \partial_t^2 \overline{u_\epsilon} + \partial_t \overline{u_\epsilon} - \frac{1}{N} \Delta \overline{u_\epsilon} = \overline{(\omega \cdot \nabla_x)^{\otimes^2} (u_\epsilon - \overline{u_\epsilon})}$$

Modulo the reminder (which goes to zero with ϵ) the transport equation generates a hyperbolic approximation of the diffusion equation and proofs can be obtained along the same line.
Cf; A Lopez, Zhang ,Zuazua version: J.Math. Pures Appli 2000.

Remark 2 Relation with Fokker Planck equation:

An approximation of Kinetic equation leads to a form of the Fokker Planck equation In the simplest form:

$$\partial_t u_\epsilon + \frac{1}{\epsilon}\mu\partial_x u_\epsilon - \frac{1}{\epsilon^2}\frac{1}{2}\partial_\mu((1-\mu^2)\partial_\mu u_\epsilon) = 0$$

in $]-a,a[\times]-1,1[$

with absorbing of specular boundary condition on $x = \pm a$

The diffusion approximation is also well established.

$$\partial_t\rho - \frac{1}{3}\partial_x^2\rho = 0$$

Then what about the limit estimates of the type Karin Beauchard and ...obtained for hypo elliptic operators..

Remark 3 About Spectra and Lebeau Jerison formula

It is known that the spectra of the transport operator

$$f \mapsto \mathcal{T}_\epsilon = -\frac{1}{\epsilon} v \cdot \nabla_x f - \frac{1}{\epsilon^2} \int_{\mathbf{R}^N} [k(x, v', v) f(v') - k(x, v, v') f(v)] dv$$

has a spectral gap property an isolated leading eigenvalue λ_ϵ with is real simple with a positive eigenvector $\phi_\epsilon(x, \omega)$. When $\epsilon \to 0$ $(\lambda_\epsilon, \phi_\epsilon)$ converge to the corresponding eigenvalue and eigenvector of the corresponding elliptic operator.

Remark 3 About Spectra and Lebeau Jerison formula

In the present example the spectra of

$$u \mapsto -\frac{1}{\epsilon}\omega \cdot \nabla_x u - \frac{1}{\epsilon^2}(u - \overline{u})$$

is composed of a sequence real eigenvalues

$$0 > -\lambda_1^\epsilon \geq \lambda_2^\epsilon \geq \ldots ...$$

It would be possible to analyze the convergence of the other eigenvalues.. Then what about an approximate Lebeau Jerison formula.

Conclusion

- One reason for the above analysis is the perspective of applications.
- *An other one would have been to contribute to the understanding of the effect of diffusion in conditional observation. Information comes from one ray and propagates (with a velocity of the order of* ϵ^{-1} *hence which goes to* ∞ *with* $\epsilon \to 0$ *) however with the effect of the opacity*

$$\sigma(x)\epsilon^{-2}(u_\epsilon - \overline{u_\epsilon})$$

this information is distributed in all direction and then can reach in a short time the domain of observation. Therefore for ϵ *small enough one expect the above estimates.*
However up to now such estimates are obtained with the properties of the diffusion equation.

Conclusion

In spite of the intuition this does not seems possible diffusion has to come first.
Eventually this is in agreement with the following observation:
It seems that in a hierarchy of equations starting from the more detail level and ending at the more coarse level, theorems concerning the behavior between level n and level $n+1$ may be reachable only when the corresponding results are already obtained at the level $n+1$.
Perfect illustration of this belief is given by the state of the art concerning hydrodynamical inviscid limit of Boltzmann equation compared with our knowledge of the Navier-Stokes equations themselves.

Au sujet d' A...Extrait d' AMAR EL KOLLI ET MARTIN ZERNER Mathématiques à Alger (1966-78)

"Dans les années 1970, à Alger une ambiance de recherche commence à prendre forme au département des mathématiques. C'était évidemment un des buts du projet, cela devient à ce moment-là un facteur clef de réussite. Les jeunes étudiants sortis du DEA participent activement à la création de cette ambiance en associant parfois des étudiants plus jeunes, de troisième année" A. s'est inscrite en magister qui devait pour la première fois être enseigné par des professeurs algériens (Benachour, Moussaoui). Le magister n'a pas pu ouvrir, sans doute pour des raisons administratives. Elle a passé à Paris un DEA (1978) et une thèse de troisième cycle sous la direction de Bardos (1979). Elle est ensuite revenue enseigner à Alger."

Thèse de 3 Cycle Université Paris XIII 1979 France."
"Relations entre les problèmes compressibles et incompressibles pour un fluide visqueux"
Lagha-Benabdallah, A. *Limites des équations d'un fluide compressible lorsque la compressibilité tend vers zéro.*
Fluid dynamics (Varenna, 1982), 139- 165, Lecture Notes in Math., 1047, Springer, Berlin, 1984.

d'Assia à Zerner et El Kolli

“J'avais été séduite par le fait de pouvoir faire des maths "utiles à mon pays", par la "beauté" de ces maths et la bonne ambiance qui régnait parmi les chercheurs.
Je m'étais dit que, même venant de ce petit pays, l'Algérie, je pourrais peut-être résoudre un problème de maths "ouvert". C'était vraiment glorifiant et encourageant. Pour moi ou ma "génération", je crois que l'activité de recherche que vous avez impulsée (même si elle a certainement des défauts que je ne voyais pas), a été déterminante dans notre projection dans le monde de la recherche. J'avais même envisagé de rester préparer ma thèse en Algérie (1977) "

Conclusion II

THANKS FOR INVITATION,

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