# A (relatively unknown) page from Assia's mathematical life !

C. R. Acad. Sci. Paris, t. 305, Série I, p. 677-680, 1987 677

Équations aux dérivées partielles/*Partial Differential Equations*

## Problèmes aux limites pour des systèmes hyperboliques non linéaires de deux équations à une dimension d'espace

Assia BENABDALLAH et Denis SERRE

***Résumé*** — Pour un système hyperbolique de lois de conservation, on étudie deux types de conditions aux limites, l'une provenant de la méthode de viscosité, l'autre de la résolution du problème de Riemann.

## Boundary conditions for nonlinear $2 \times 2$ hyperbolic systems of conservation laws

***Abstract*** — *We study two types of boundary conditions for hyperbolic systems of conservation laws. The first one comes from the viscosity method, the second from the resolution of the Riemann problem.*

On s'intéresse au problème de Cauchy dans un intervalle borné, avec conditions aux limites :

(1.1) $$U_t + (F(U))_x = 0, \qquad x > 0, \quad t > 0$$

# Two results of controllability of hyperbolic conservation laws

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supported by ANR (project CoToCoLa)

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1. **Feedback source control of general scalar conservation law**
   - Source control for SCL: setting and results
   - (Feedback) source control: an abstract framework
   - Feedback source control for SCL: numerical approximation
   - Attainability for 1D scalar conservation law with source
2. **Attainability for a class of triangular systems of Keyfitz-Kranzer kind**
   - Keyfitz-Kranzer system and some generalizations
   - Attainability for convex 1D scalar conservation law
   - The non-resonant case: attainability by isentropic solutions
   - The resonant case: exact and approximate attainability
   - Numerics for backward resolution of KK kind triangular systems

# Feedback source control of SCL:
# setting and results

# Setting of source control for scalar conservation law

**Problem:** given initial and terminal data $u_0, u_T$ find a control $f$ such that

$$\begin{cases} u_t + \mathrm{div}_x F(u) = f(t,x) & u, f\text{: the unknowns} \\ u|_{t=0} = u_0, \quad u|_{t=T} = u_T & \text{(given data).} \end{cases}$$

$u$ is scalar ; PDE is defined on $Q_T := (0,T) \times \mathbb{R}^N$;
flux $F \in Lip_{loc}(\mathbb{R}; \mathbb{R}^N)$. Not necessarily $N = 1$; convexity not needed.

**Setting:** $L^\infty$ solutions in the Kruzhkov (entropy) sense.

**Space for** $f$**:** different options, e.g. $C(Q_T)$, $L^\infty(Q_T)$, $L^1(Q_T)$, $L^1(0,T; L^\infty(\mathbb{R}^N))$ ... we'll have something a bit more special.
Recent result: [Corghi, Marson '15] for continuous source.

**Questions:**

1. Which states $u_T$ are attainable with source (and from which $u_0$, with which $f$)?
2. Construction of $f$ knowing $u_0, u_T$ (including numerics) ?

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# Main results on source control for scalar conservation law

**Answers:**

2. Construct $f$ using a well-chosen feedback control to trajectories:

Assume $\exists v : [s,T] \mapsto L^\infty(R^N)\ \exists g$ $\begin{cases} v_t + \mathrm{div}_x F(v) = g(t,x) \\ v|_{t=s} = v_s, \quad v|_{t=T} = u_T. \end{cases}$

Then let $u$ evolve freely ($f \equiv 0$) on $[0,s]$,
take $u_s := u(s)$ and solve the Cauchy problem on $[s,T]$:

$$\begin{cases} u_t + \mathrm{div}_x F(u) = \text{singular “nudging” source} \\ u|_{t=s} = u_s, \end{cases}$$

$$\text{singular “nudging” source} := p(t)(v(t,x) - u) + g(t,x)$$

$$p \geq 0,\ p \nearrow +\infty \ \text{ as } t \nearrow T,\ \int^T p(t)\,dt = +\infty.$$

Here $p(t) = \frac{1}{T-t}$ is a typical (optimal?) example.
**Simplification:** $s = 0$ in the sequel.

**Analogies:** [Barbu'91] ; Lueneberger'66 observers (for CL: cf. [Auroux,Blum'05] , [Boulanger,Moireau,Perthame,Sainte-Marie'14] ).

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# Main results, cont$^{d}$

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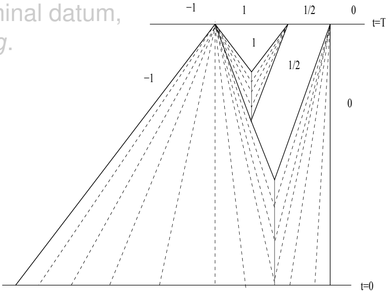


**(Forced) backward construction:** to be detailed in the sequel.

**Step II.** The above nudging feedback strategy

$$f(t,x) = g(t,x) + \frac{1}{T-t}(v(t,x) - u(t,x)),$$

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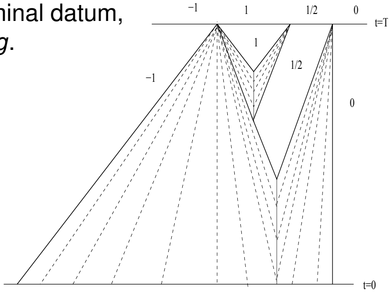


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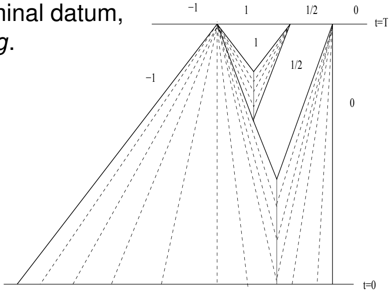


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# Idea of the argument

**Why singular nudging works:**
The result relies upon $L^1$ contraction for entropy solutions of SCL .
"Multiply PDE($u$)-PDE($v$) by sign($u-v$)", integrate on $[0,t]\times\mathbb{R}^N \Rightarrow$

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$$\Leftrightarrow\quad \|u-v\|_1(t)+\int_0^t p(\tau)\|u-v\|_1(\tau)\,d\tau\le\|u_0-v_0\|_1.$$

NB: analogously, one gets $\|u-v\|_1(t):=\|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R}^N)}\searrow$ with $t$.

Gronwall inequality and the assumption $\int^T p(\tau)\,d\tau=+\infty\Rightarrow$
$\|u-v\|_1\le\|u_0-v_0\|_1\exp\left(-\int_0^t p(\tau)\,d\tau\right)\ \Rightarrow\ \|u-v\|_1(t)\to 0,\ \ t\to T.$

Moreover: if $p(t)=\frac{1}{T-t}$ , then $\|u-v\|_1\le\frac{T-t}{T}\|u_0-v_0\|$ (optimality?)
and (assuming $g\equiv 0$ for simplicity)

$$\|f(t,\cdot)\|_1\le p(t)\|u-v\|_1(t)\le\frac{1}{T}\|u_0-v_0\|\ \Rightarrow f\in L^\infty(0,T;L^1(\mathbb{R}^N)).$$

Finally: maximum principle $\Rightarrow \sup_{\tau\in[0,t]}\|u\|_\infty(\tau)\le\sup_{\tau\in[0,t]}\|v\|_\infty(\tau)$
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# (Feedback) source control:
# an abstract framework

## $m$-accretive operators in Banach spaces and generation of contraction semigroups

**Abstract framework:** $(X, \|\cdot\|)$ Banach space.

**General source control problem:** find a control $f \in L^1(0,T;X)$ and a (mild) solution $u \in C([0,T];X)$ such that

$$\begin{cases} u' + Au \ni f \\ u(0) = u_0, \quad u(T) = u_T \end{cases}$$

where $\overline{A}$ is an $m$-accretive operator on $X$, i.e,

- the resolvent $(\mathrm{Id} + \lambda A)^{-1}$ is non-expansive, $\lambda > 0$
- the resolvent is densely defined in $X$.

**Well-posedness of the Cauchy problem:**
Crandall, Liggett, Bénilan in 1970ies

**Applications:** (schools of Brézis, Crandall, Bénilan,... )
– (fractional) heat eqn, porous medium and fast diffusion eqns, parabolic $p$-laplacian, Stefan and Hele-Shaw problems, total variation flow, scalar Hamilton-Jacobi equations,..., scalar conservation laws.
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## Control to trajectories of contractions semigroups

**Result:** "singular nudging" feedback source $\Rightarrow$ control to trajectories.

**Main Theorem (almost immediate from [Bénilan,Igbida'96] )**

*Let $A$ be an m-accretive operator, $\overline{D(A)} = X$.*
*Given $v$ a mild solution on $[0,T]$ with source $g$ and final datum $u_T$,*
*given $p \in L^1_{loc}([0,T))$, $p \geq 0$, $\int^T p(t)\,dt = +\infty$,*
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# Constructive approximation schemes for nudging control

**An example of semi-discretization for the nugding:**

Take $T=1$, $N\in\mathbb{N}$, set $\delta t=1/N$. Pick $(\alpha^n)_{n=1..N}\in(0,+\infty)$.

A mild solution $v$ (for simplicity, with source $g\equiv 0$) is approximated by

$$v^n+(\delta t)Av^n\ni v^{n-1} \text{ for } n=1..N, \text{ given } v^0.$$

Consider the following approximation of the nudging control problem:

$$u^n+(\delta t)Au^n\ni(1-\alpha^n)u^{n-1}+\alpha^n v^{n-1}=u^{n-1}+(\delta t)g^n, \text{ given } u^0.$$

where $g^n:=N\alpha^n(v^{n-1}-u^{n-1})$ is the associated discretized source.

**Lemma ("Discrete Gronwall")**

$$\|u^n-v^n\|\le\prod_{k=1}^{n}(1-\alpha^k)\|u^0-v^0\| \;\; and \;\; \|g^n\|\le N\alpha^n\prod_{k=1}^{n-1}(1-\alpha^k)\|u^0-v^0\|.$$

*For $\alpha^n:=((N+1)-n)^{-1}$, one has in particular*

$$\forall n=1..N \;\; \|u^n-v^n\|\le\frac{(N+1)-n}{N}\|u^0-v^0\| \;\; and \;\; \|g^n\|\le 1.$$

# Constructive approximation schemes for nudging control

**An example of semi-discretization for the nugding:**
Take $T=1$, $N\in\mathbb{N}$, set $\delta t=1/N$. Pick $(\alpha^n)_{n=1..N}\in(0,+\infty)$.
A mild solution $v$ (for simplicity, with source $g\equiv 0$) is approximated by

$$v^n+(\delta t)Av^n\ni v^{n-1} \text{ for } n=1..N, \text{ given } v^0.$$

Consider the following approximation of the nudging control problem:

$$u^n+(\delta t)Au^n\ni(1-\alpha^n)u^{n-1}+\alpha^n v^{n-1}=u^{n-1}+(\delta t)g^n, \text{ given } u^0.$$

where $g^n:=N\alpha^n(v^{n-1}-u^{n-1})$ is the associated discretized source.

## Lemma ("Discrete Gronwall")

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# Constructive approximation schemes for nudging control, cont$^{\mathrm{d}}$

**Another example of approximation:** using the semigroup $S(\cdot)$, set

$$v^n = S(\delta t)v^{n-1} \text{ and } u^n = (1-\alpha)S(\delta t)u^{n-1} + \alpha^n S(\delta t)v^{n-1}.$$

Same stability estimates hold as for the previous scheme !!
In all cases, we get a piecewise constant in time approximation:

$$u_N : t \mapsto \sum_{n=1}^{N} u^n \mathbb{1}_{[\frac{n-1}{N},\frac{n}{N})}(t), \quad g_N : t \mapsto \sum_{n=1}^{N} g^n \mathbb{1}_{[\frac{n-1}{N},\frac{n}{N})}(t).$$

**Convergence ? Compactness is needed...**
The thing to prove is that $(u_N)_N$ is pre-compact (in $C([0,1];X)$? weaker?).
Convergence would follow since $(g_N)_N$ would be pre-compact (e.g. locally, in $C([0,1);X)$), and since mild solutions are integral solutions [Bénilan'72]
$\Rightarrow$ easy passage to the limit in the formulation of integral solution.

- general semigroup theory should apply for compactness...?
- if $(\mathrm{Id}+\lambda A)^{-1}$ (resp., $S(.)$) is compact, compactness of $(u_N)_N$ is easier...
- in applications to parabolic equations/to SCL, the PDE formulations possess their own (weaker) compactness frameworks
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# Feedback source control of SCL: fully discrete approximation

# Fully discrete Finite Volume approximation of the nudging for SCL

**Standard (explicit in operator) FV scheme for SCL** $u_t + \mathrm{div} F(v) = source$:
Given $\mathcal{F} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ consistent with $F$, monotone, Lipschitz numerical flux, with $\frac{\delta t}{\delta x} \leq CFL$, discretize the trajectory $v$ (with $g \equiv 0$) and the "nudged" $u$ by

$$\begin{aligned}
\frac{v_i^n - v_i^{n-1}}{\delta t} + \frac{\mathcal{F}(v_i^{n-1}, v_{i+1}^{n-1}) - \mathcal{F}(v_{i-1}^{n-1}, v_i^{n-1})}{\delta x} &= 0, \\
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\end{aligned}$$

**Properties of these FV discretizations:**

- maximum principle, control of $L^1$, $L^\infty$ and $BV$ norms (decrease in time)
- discrete contraction (and discrete entropy inequalities) [Crandall,Tartar]

| Implicit nudging | Explicit nudging |
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| Like for the continuous case, $\|u^n - v^n\| + \sum_{k=1}^{n} \alpha^k \|u^k - v^k\| \leq \|u^0 - v^0\|$ | Like for above semi-discretizations, $\|u^n - v^n\| \leq \frac{(N+1)-n}{N} \|u^0 - v^0\|$, $\|source^n\| \leq 1$, for $\alpha^n := \frac{1}{(N+1)-n}$. |

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# Attainability
# for 1D conservation law with source

# Typical solutions (building blocks) of SCL and their gluing

**Typical solutions of SCL:** (building blocks)

- shock waves (a mess: admissibility issues, irreversibility)
- rarefactions $U(t,x) = [F']^{-1}(\frac{x}{t})$, reversible
- in fact, constants are also solutions :-)

**NB:** irreversibility is very bad for backward constructions...

**Modification and gluing of the blocks:**

- solutions can be glued continuously e.g across lines
- one can modify the block using symmetries: translation invariance , scaling, change $(t,x,F) \to (T-t, \mp x, \pm F)$,...

**The obvious solution with source:** (case $a < b$)

$$U(t,x) = \max\{a, \min\{b, \tfrac{x}{t}\}\} \text{ fulfills } U_t + F(U)_x = g,$$

$$g := \left(-\tfrac{x}{t} + F'(\tfrac{x}{t})\right) \tfrac{1}{t} 1\!\!1_{[a,b]}(\tfrac{x}{t}),$$

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moreover, $\|U(t,\cdot)\|_{L^\infty(\mathbb{R})} = \max\{|b|,|a|\}$ and $|U(t,\cdot)|_{BV(\mathbb{R})} = b - a$.

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**Gluing, for piecewise constant terminal datum $u_T$:**

Juxtapose reversed profiles $U(T-t, x-x_0)$ at jump points $x_0$ of $u_T$.
Overlapping $\rightsquigarrow$ cancellation $\rightsquigarrow$ continuous gluing
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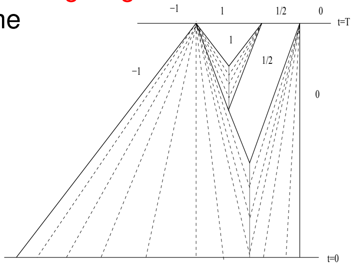


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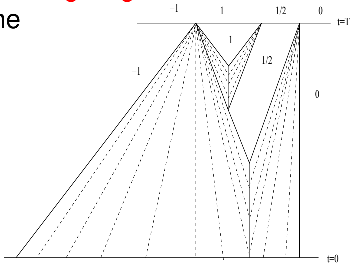


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# Attainability
# for a class of triangular systems
# of Keyfitz-Kranzer kind

# Keyfitz-Kranzer system

**Hyperbolic systems of conservation laws:**
Beyond the scalar case, few is known even about well-posedness.
Particular (classes of) systems may possess special structure.

**[Keyfitz-Kranzer'79] system:**

$$U_t + (\phi(|U|)U)_x = 0$$

can be rewritten with $r = |U|$, $w = U/|U|$ as

$$\begin{cases} r_t + (r\phi(r))_x = 0 \\ (r\,w)_t + (r\phi(r)\,w)_x = 0 \end{cases} \Leftrightarrow \begin{cases} \text{(convex) scalar cons. law} \\ \text{continuity eqn. } (Aw)_t + (Bw)_x = 0 \end{cases}$$

where $(A, B) = (r, r\phi(r))$ is a divergence-free in $(t, x)$ field.

**Well-posedness:**
True for "strong entropy solutions" **[Freistuhler'94],[Panov'00]**.
The idea is that $r$ is a Kruzhkov entropy solution of the SCL ,
and $w$ is a renormalized solution of the continuity equation.
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# Triangular systems. Attainability and backward construction.

**Our framework:** (motivated by KK, though it does not include KK)

$$\begin{cases} u_t + f(u)_x = 0, \ f \text{ strictly convex} \\ v_t + (g(u)v)_x = 0 \end{cases}$$

Structure: (non-strictly) hyperbolic: eigenvalues $f'(u)$ and $g(u)$.

**Objectives:**

- Describe the set of attainable states at time $t = T$.
- Provide a backward construction (analytical, then numerical).

**Notion(s) of solution:** entropy solutions ??

- non-resonant case ($\forall u \ f'(u) \neq g(u)$): $\exists$ non-trivial entropies.
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# Attainability for convex conservation law

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## Proposition

*Let $T > 0$. Consider a scalar conservation law $u_t + f(u)_x = 0$ with strictly convex $C^1$ flux $f$. Then*

**(i)** *Set of states attainable by entropy solutions at time $T$:*

$$\mathcal{A}_T = \Big\{ u \in L^\infty(\mathbb{R}) \ :\ \exists \rho : \mathbb{R} \to \mathbb{R} \ \text{ nondecreasing} \\ \text{such that } f'(u) = \frac{x - \rho(x)}{T} \Big\}.$$

**(ii)** *For every $u_T \in \mathcal{A}_T(\mathbb{R})$ there exists a unique isentropic solution $u$ on $[0, T] \times \mathbb{R}$ that verifies $u(T, \cdot) = u_T$.*

**Proofs:**

- [Ancona,Marson'98] , with Dafermos' generalized characteristics
- [Adimurthi,Ghoshal,V.Gowda'14] , with Hopf-Lax-Oleinik formula

**Numerical counterpart:**

Discretize $z_t + (-f(z))_x = 0$, $z|_{t=0} = u_T$ by a monotone FV scheme.
Attainability of $u_T \Leftrightarrow z$ is reversible. Take $u(t, \cdot) = z(T - t, \cdot)$.

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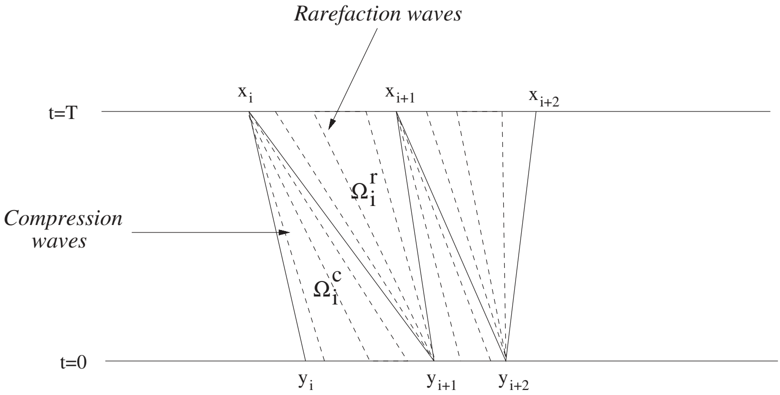
# (Isentropic version of) the construction of Adimurthi, Ghoshal, Veerappa Gowda:

**For $u_T$ corresponding to a piecewise constant $\rho$:**




**Extension by density:** (use uniform $BV$ estimates)
discretizing $\rho := x - Tf'(u_T(x))$, at the limit one reaches any $u_T \in \mathcal{A}_T$.

# Attainability for KK kind systems in the non-resonant case

## Non-resonant case: entropies and attainability

**Main Theorem (Attainability: only the component $u_T$ matters)**

*For non-resonant KK-like triangular system, any $(u_T, v_T) \in \mathcal{A}_T \times L^\infty$ is attainable at time $T$ by a (unique) isentropic solution.*

**Lemma (description of entropies of KK-like triangular systems)**

*If $f'(u) \neq g(u)$ for all $u \in [a, b]$, all smooth entropies have the form*

$$\mathcal{E}(u, v) = \eta(u) + e^{-H(u)}\mu(ve^{H(u)}),$$

*here $H$ is a primitive of $u \mapsto \frac{-g'(u)}{f'(u)-g(u)}$ , $\eta$, $\mu$ are any smooth functions.*

**Change of unknown:** $v = A(u)w$, $A(\cdot) := e^{-H(\cdot)}$ .
General entropies re-write as $\mathcal{E} = \eta(u) + A(u)\mu(w)$.

**Hint:** for isentropic solutions of SCL, $A(u)_t + (A(u)g(u))_x = 0$.

**Proof of Th.:** analogously to KK, solve the SCL (here, isentropically ); then for $(A(u)w)_t + (A(u)g(u)w)_x = 0$, $\exists!$ renormalized sol. [Panov'08]
It follows that the entropies $\mathcal{E} = \eta(u) + A(u)\mu(w)$ are conserved.

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*For non-resonant KK-like triangular system, any $(u_T, v_T) \in \mathcal{A}_T \times L^\infty$ is attainable at time $T$ by a (unique) isentropic solution.*

**Lemma (description of entropies of KK-like triangular systems)**

*If $f'(u) \neq g(u)$ for all $u \in [a, b]$, all smooth entropies have the form*

$$\mathcal{E}(u, v) = \eta(u) + e^{-H(u)}\mu(ve^{H(u)}),$$

*here $H$ is a primitive of $u \mapsto \frac{-g'(u)}{f'(u)-g(u)}$ , $\eta$, $\mu$ are any smooth functions.*

**Change of unknown:** $v = A(u)w$, $A(\cdot) := e^{-H(\cdot)}$ .
General entropies re-write as $\mathcal{E} = \eta(u) + A(u)\mu(w)$.

**Hint:** for isentropic solutions of SCL, $A(u)_t + (A(u)g(u))_x = 0$.

**Proof of Th.:** analogously to KK, solve the SCL (here, isentropically ); then for $(A(u)w)_t + (A(u)g(u)w)_x = 0$, $\exists!$ renormalized sol. [Panov'08]
It follows that the entropies $\mathcal{E} = \eta(u) + A(u)\mu(w)$ are conserved.

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# Attainability for KK kind systems in the resonant case

# Resonant case. Forward and backward non-uniqueness.

**Counterexamples:** Resonant (at $u_* = 1$) system

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ v_t + \left(u^2 v\right)_x = 0. \end{cases} \tag{1}$$

Given any $C^1$, compactly supported in $(0,1)$ function $K$ ,

$$u(t,x) = \min\left\{1, \max\left\{0, \frac{x}{t}\right\}\right\} \text{ with } v(t,x) = \frac{1}{x^2} K\left(\frac{1}{x} - \frac{1}{t}\right)$$

are admissible (from any reasonable standpoint?) solutions.

All the corresponding family of terminal states should be seen (?) as attainable from the same initial state $U_0 = (\text{sign}^+(u), 0)$.

Forward non-uniqueness of $v$, if $u$ has rarefactions focusing at $t = 0$
Backward non-uniqueness, if $u$ has compressions focusing at $t = T$

$\Rightarrow$ problems, if $u_T$ is "on the border" of the set $\mathcal{A}_T$

**Numerics:** it may be impossible to capture some solutions !

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## Regularized states and the main result.

**Regularized states of $\mathcal{A}_T$:**
The subset of all states in $\mathcal{A}_T$ without focusing waves:

$$\mathbb{A}_T := W^{1,\infty}(\mathbb{R}) \bigcap \left( \cup_{\delta>0} \mathcal{A}_{T+\delta} \right)$$

- $\mathbb{A}_T$ is dense in $\mathcal{A}_T$ w.r.t. $L^1_{loc}$ topology
- If $S(\cdot)$ is the evolution semigroup, $u_T \in \mathcal{A}_T \Rightarrow S(\delta)A_T \in A_{T+\delta}$.
  Solving SCL forward on $[0, \delta]$ rules out focusing at $t = 0$

**Theorem (Attainability, the resonant case)**

*Under the local uniform convexity assumption of $f$,*

- *every state $U_T \in \mathbb{A}_T \times L^\infty(\mathbb{R})$ is attainable at time $t = T$*
- *every state in $U_T \in \mathcal{A}_T \times L^\infty(\mathbb{R})$ is approximatively attainable at time $t = T$, with respect to $L^1_{loc}(\mathbb{R})$ topology.*

*In the backward solutions for $U_T = (u_T, v_T) \in \mathbb{A}_T$, $u$ can be chosen to be the unique isentropic solution of the SCL with $u(T) = u_T$; then $v$ is the unique weak solution (and also the unique DiPerna-Lions renormalized solution) of the continuity eqn. with $v(T) = v_T$.*

# Numerics for backward-forward resolution

# The algorithm for backward and forward resolution

**Scheme:** (isentropicity $\Rightarrow$ same scheme forward and backward!)

- $u$ computed by explicit FV with Godunov numerical flux
- $v$ computed by the explicit FV scheme of [Gosse,James'00]

**Protocol of experiments:** $U_T \xrightarrow{\text{backward num.}} U_0 \xrightarrow{\text{forward num.}} \approx U_T$ ?

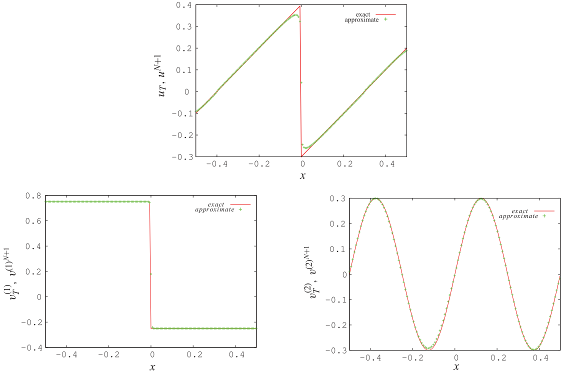
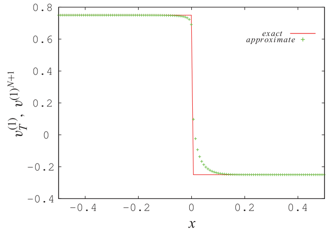
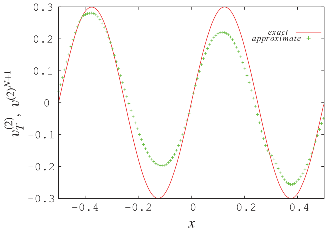



**Figure:** Test cases (non-resonant): one datum $u_T$, two different data $v_T^{1,2}$

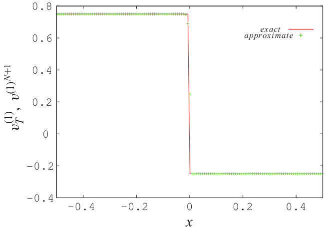
# Resonant case: inexact attainability and the interest of regularization.

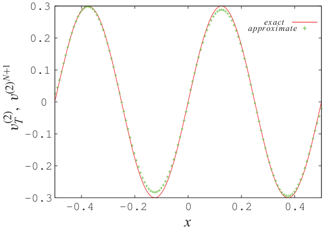
**Crude numerics:** bad backward-forward matching for $u_T \in \mathcal{A}_T \setminus \mathbb{A}_T$




**Regularization + numerics:** $u_T \in \mathcal{A}_T$ replaced by $S(\delta)u_T \in \mathbb{A}_T$



## Conclusions and perspectives

**Conclusions:**

- attainability with a source: trivial for scalar conservation laws and even for general contractive evolution equations, due to the singular nudging strategy [Adimurthi,A.,Ghoshal, in prep.]
- convex scalar case: attainability by isentropic solutions helps a lot in backward resolution of Keyfitz-Kranzer and similar systems [A.,Donadello,Ghoshal,Razafison'15]

**In progress:**
some necessary / some sufficient conditions for attainability for SCL with non-convex (cubic) flux [Donadello, Marson, A., in prep.]
...here, one cannot rely only upon reversible solutions !

**Open:**
e.g., extension of the nudging to hyperbolic systems using "dissipative source" [Dafermos,Hsiao'82] ?

Merci !