On the optimization of traffic flow at a junction

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Contrôle des EDP et Applications

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Joint research with A. Cesaroni, G.M. Coclite, M. Garavello

The LWR traffic flow model (1955, 1956)

> *A new problem, which has arisen in the twentieth century, is how to organize road traffic so that the full benefits of our increased mobility can be enjoyed at the lowest cost in human life and capital. The problem has many sides - constructional, legal, educational, administrative.* - M.J. Lighthill, G.B. Whitham (1955)

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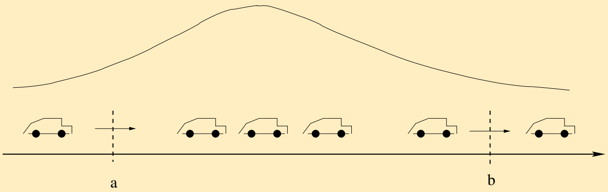

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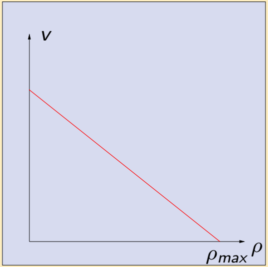

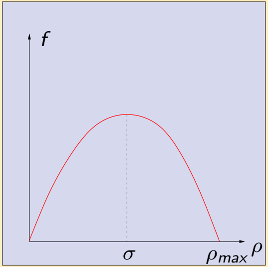

Traffic flow on a network of roads

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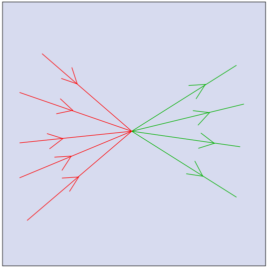

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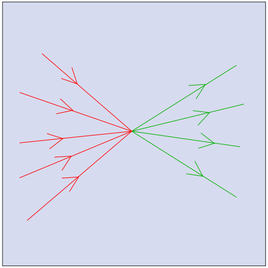

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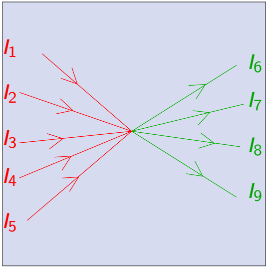

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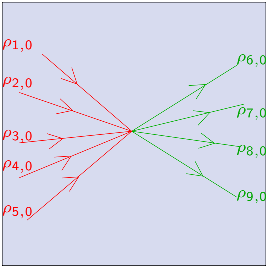

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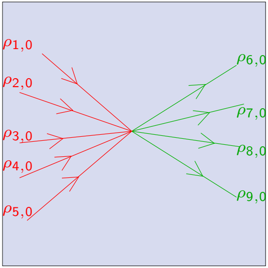

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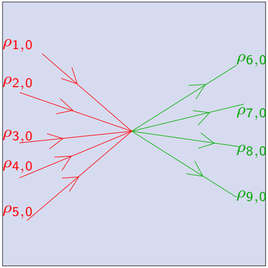

- Coupling transition condition at the node

$$\Psi(\rho_1(t,0),\dots,\rho_{m+n}(t,0)) = 0$$

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Achdou, Andreianov, Banda, Bastin, Bressan, Camilli, Canic, Coclite, Colombo, Coron, Costeseque, D'Apice, Donadello, Garavello, Gasser, Goatin, Göttlich, Gugat, Han, Herty, Holden, Imbert, Klar, Lattanzio, Lebacque, Leugering, Manzo, Marchi, Monneau, Moutari, Nguyen, Marigo, Piccoli, Rascle, Risebro, Rosini, Schleper, Shen, Tchou, Zidani, Ziegler...

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$$0 \le a_{ji}(t) \le 1 \quad \forall\ j, i \qquad\qquad \sum_{j=m+1}^{m+n} a_{ji}(t) = 1 \qquad \forall\ i$$

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(when $m > n$ and the total possible flux on the incoming roads is larger than the maximal flux that the outgoing roads can handle)

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Remark: the definition of JRS determines all features of possible solutions at the junction.

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A Riemann solver at the node is a map

$$\mathcal{RS} : [0, \rho_{max}]^{m+n} \to [0, \rho_{max}]^{m+n},$$

$$(\rho_{0,1}, \ldots, \rho_{0,m+n}) \mapsto (\widetilde{\rho}_1, \ldots, \widetilde{\rho}_{m+n})$$

that associates to an $(m+n)$-tuple of constant initial data an $(m+n)$-tuple of constant boundary data which are the traces at the node of the solution to the corresponding Riemann problem.

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Algorithm

- $\Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}$
- Maximize $E = \gamma_1 + \cdots + \gamma_n$ on Ω
- Find the corresponding densities
- Select the densities that satisfy the priority rules

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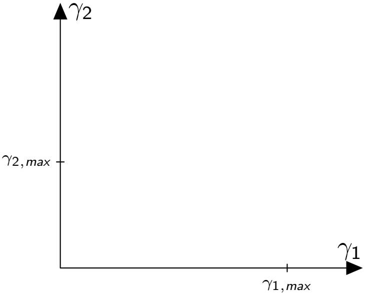

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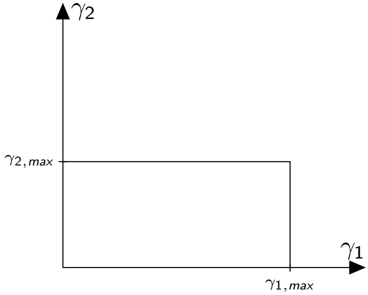

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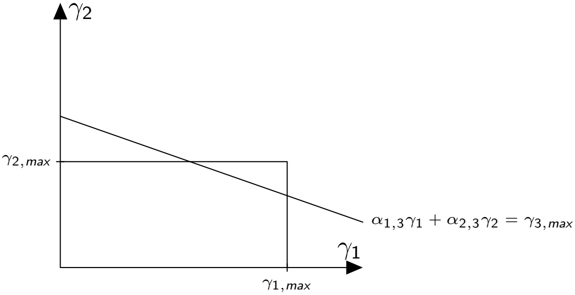

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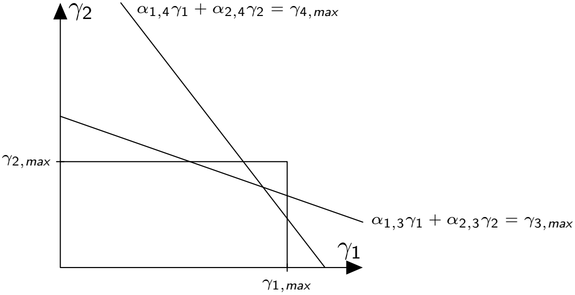

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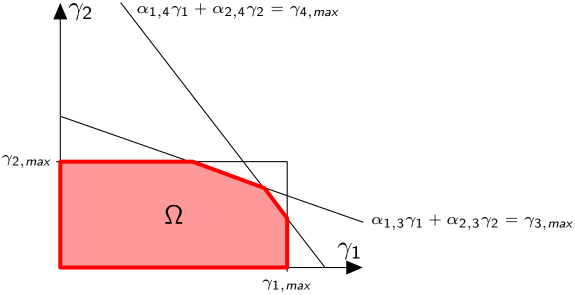

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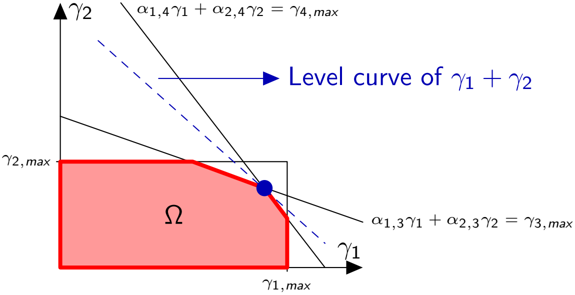

A different approach: a control theoretic point of view

Fix $T > 0$. Find the solution on $[0, T]$ to the Cauchy problem

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that satisfies the flux constraint

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and maximizes an integral cost

$$\int_0^T \sum_{l=1}^{n} f\left(\rho_l(t,0)\right)\, dt$$

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$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0, & x \in I_l, l \in \{1, \ldots, n+m\}, t > 0 \\ \rho_l(0, x) = \rho_{0,l}(x), & x \in I_l, l \in \{1, \ldots, n+m\}, \end{cases}$$

that satisfies the flux constraint

$$A \cdot (f(\rho_1(t,0)), \ldots, f(\rho_m(t,0)))^T = (f(\rho_{m+1}(t,0)), \ldots, f(\rho_{m+n}(t,0)))^T$$

and maximizes an integral cost

$$\int_0^T \mathcal{J}\left(f\left(\rho_1(t,0)\right), \ldots, f\left(\rho_n(t,0)\right)\right) dt$$

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- The maximization is global on $[0, T]$ among all admissible solutions (fulfilling linear flux constraints) and not pointwise in time among all admissible self-similar solutions of the Riemann problems

Existence of solution

Given initial data $\overline{\rho}_l : \mathbb{R} \to [0, \rho_{max}]$, $l \in \{1, \ldots, m+n\}$,

- .

Existence of solution

- For $l \in \{1, \dots, m\}$, $I_l = (-\infty, 0]$ (incoming arcs)

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0 & x < 0, t > 0 \\ \rho_l(0, x) = \bar{\rho}_l(x) & x < 0 \\ \rho_l(t, 0) = \tilde{\rho}_l(x) & t > 0 \end{cases}$$

$$\mathcal{F}_l = \mathcal{F}_l(\overline{\rho}_l) \doteq \left\{ f(\rho_l(\cdot, 0)) \mid \rho_l \text{ sol on } [0, T] \times I_l, \ \tilde{\rho}_l(t) \in [0, \rho_{max}] \right\}$$

$$\mathcal{F}_l^M = \mathcal{F}_l^M(\overline{\rho}_l) \doteq \left\{ f(\rho_l(\cdot, 0)) \in \mathcal{F}_l \mid \text{T.V.}(\rho_l(\cdot, 0)) \leq M \right\}$$

- .

Existence of solution

- For $l \in \{m+1, \dots, m+n\}$, $I_l = [0, +\infty)$ (outgoing arcs)

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0 & x > 0, t > 0 \\ \rho_l(0,x) = \bar{\rho}_l(x) & x > 0 \\ \rho_l(t,0) = \tilde{\rho}_l(x) & t > 0 \end{cases}$$

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Existence of solution

- . Admissible flux traces

$$\mathcal{G} = \left\{ (g_1, \ldots, g_m) \in \prod_{i=1}^{m} \mathcal{F}_i : \sum_{i=1}^{m} \alpha_{ji} g_i \in \mathcal{F}_j,\, j = m+1, \ldots, m+n \right\}$$

$$\mathcal{G}^M = \left\{ (g_1, \ldots, g_m) \in \prod_{i=1}^{m} \mathcal{F}_i^M : \sum_{i=1}^{m} \alpha_{ji} g_i \in \mathcal{F}_j,\, j = m+1, \ldots, m+n \right\}$$

Existence of optimal solutions

Theorem (A.,Cesaroni, Coclite, Garavello)

Let $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}$ be a continuous map. For every $M > 0$, there exists $\widehat{g} \in \mathcal{G}^M$ such that

$$\int_0^T \mathcal{J}\left(\widehat{g}(t)\right) dt = \sup_{g \in \mathcal{G}^M} \int_0^T \mathcal{J}\left(g(t)\right) dt$$

Ex: $\quad \mathcal{J}(g_1, \ldots, g_m) = \sum_i g_i, \qquad \mathcal{J}(g_1, \ldots, g_m) = \prod_i g_i,$

Proof:

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- Uniform BV bounds and Helly's compactness theorem $\Longrightarrow$ convergence of subsequence of flux traces

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Proof:

- Uniform BV bounds and Helly's compactness theorem $\Longrightarrow$ convergence of subsequence of flux traces
- Divergence theorem on $[-\max|f'(\rho)| \cdot T, 0] \times [0, T]$ $\Longrightarrow$ convergence of flux traces of solutions to flux trace of limit solution

Remark on boundary data for conservation laws

$$\begin{cases} \partial_t \rho + \partial_x \rho = 0 & x \in (-\infty, 0), \ t > 0, \\ \rho(0,x) = \bar{\rho}(x) & x \in (-\infty, 0) \\ \rho(t,0) = \tilde{\rho}(t) & t > 0 \end{cases}$$

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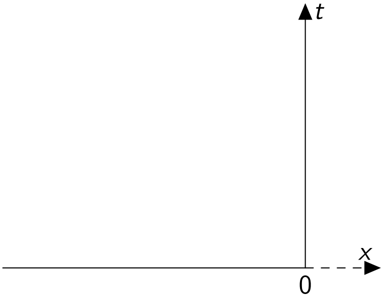

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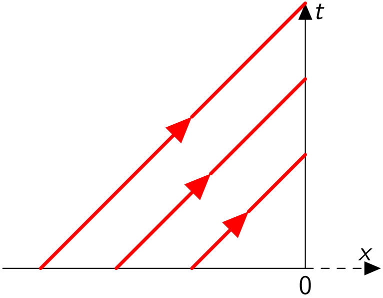

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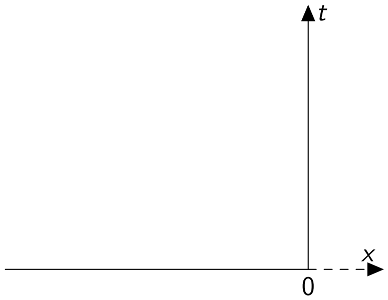

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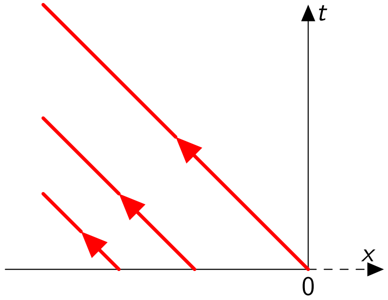

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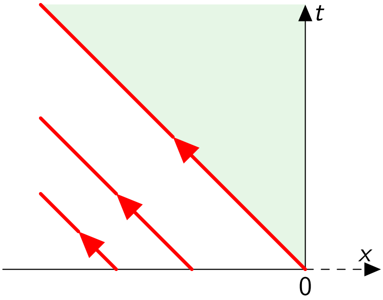

Case $n = m = 1$: comparison with entropy solution of CP

$$\begin{cases} \partial_t \rho_1 + \partial_x f(\rho_1) = 0 & x < 0, t > 0, \\ \rho_1(0,x) = \bar{\rho}_1(x) & x < 0, \end{cases} \qquad \begin{cases} \partial_t \rho_2 + \partial_x f(\rho_2) = 0 & x > 0, t > 0 \\ \rho_2(0,x) = \bar{\rho}_2(x) & x > 0, \end{cases}$$

$$\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2 \qquad \mathcal{G}^M = \mathcal{F}_1^M \cap \mathcal{F}_2$$

$$\sup_{g \in \mathcal{G}^M} \int_0^T \mathcal{J}(g(t))\, dt \qquad (max)_M$$

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$$\mathcal{F}_1 \doteq \Big\{ f(\rho_1(\cdot,0)) \mid \rho_1 \text{ sol on } [0,T] \times (-\infty, 0],\ \tilde{\rho}_1(t) \in [0, \rho_{max}] \Big\},$$

$$\mathcal{F}_2 \doteq \Big\{ f(\rho_2(\cdot,0)) \mid \rho_2 \text{ sol on } [0,T] \times [0, +\infty),\ \tilde{\rho}_2(t) \in [0, \rho_{max}] \Big\},$$

$$\mathcal{F}_l^M \doteq \Big\{ f(\rho_l(\cdot,0)) \in \mathcal{F}_l \mid \text{T.V.}(\rho_l(\cdot,0)) \leq M \Big\}, \quad l = 1, 2.$$

Case $n = m = 1$: comparison with entropy solution of CP

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Cauchy problem on $\mathbb{R}$

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \begin{cases} \bar{\rho}_1(x) & \text{if } x < 0 \\ \bar{\rho}_2(x) & \text{if } x > 0 \end{cases} \end{cases} \qquad \text{(CP)}$$

$$\left.\begin{array}{l} \rho_e : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ entropy admissible sol. to (CP)} \\ \text{T.V.}(\rho_e(\cdot,0)) \leq M \end{array}\right\} \implies \rho_e(\cdot,0) \in \mathcal{G}^M.$$

Case $n = m = 1$: comparison with entropy solution of CP

Consider $\mathcal{J}(g) = g$

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For every $T > 0$, the entropy admissible solution ρ_e to (CP) solves the maximization problem $(max)_M$, i.e.

$$\int_0^T f\left(\rho_e(t,0)\right) dt = \sup_{g \in \mathcal{G}^M} \int_0^T g(t) dt,$$

for every $M \geq \mathrm{T.V.}(\rho_e(\cdot, 0))$.

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for every $M \geq \mathrm{T.V.}(\rho_e(\cdot, 0))$.

The proof relies on Hopf-Lax formula for explicit representation of viscosity solutions to IBV for Hamilton-Jacobi equation

$$\rho(t,x) \text{ entropy weak sol'n of } \partial_t \rho + \partial_x (f(\rho) = 0$$

$$\Downarrow$$

$$v(t,x) \doteq \text{viscosity sol'n of } \partial_t v + f(\partial_x v) = 0$$

Case $n = m = 1$: non entropic optimal solutions

$$f(\rho) = \rho(1-\rho)$$

$$\begin{cases} \partial_t \rho_1 + \partial_x \left(\rho_1(1-\rho_1)\right) = 0 \\ \rho_1(0,x) = \frac{1}{4} \end{cases} \qquad \begin{cases} \partial_t \rho_2 + \partial_x \left(\rho_2(1-\rho_2)\right) = 0 \\ \rho_2(0,x) = \frac{1}{4} \end{cases}$$

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The functions

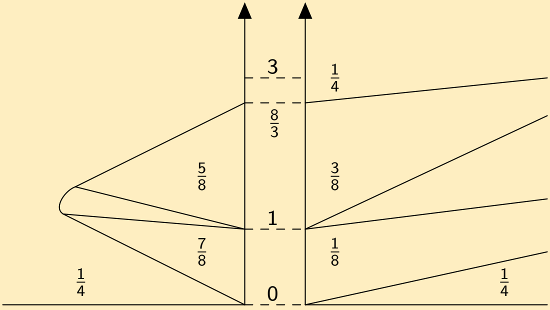
$$\rho_1(t,x) \equiv \frac{1}{4} \qquad \rho_2(t,x) \equiv \frac{1}{4}$$

provide an optimal solution, i.e.

$$\frac{3}{16}T = \int_0^T f\left(\rho_1(t,0)\right)dt = \sup_{g \in \mathcal{G}^M} \int_0^T g(t)dt$$

Case $n = m = 1$: non entropic optimal solutions

The functions



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Case $n = m = 1$: non entropic optimal solutions

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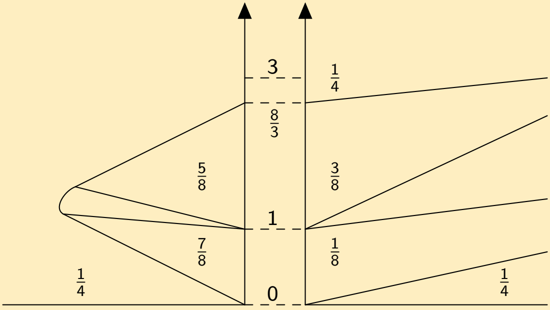
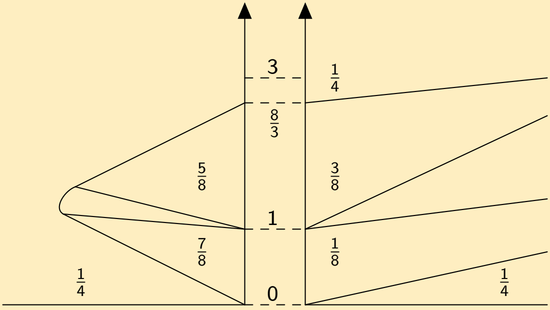


provide another optimal solution.

$$\int_0^T f\left(\rho(t,0)\right) dt = f\left(\frac{7}{8}\right) \cdot 1 + f\left(\frac{5}{8}\right) \cdot \frac{5}{3} + f\left(\frac{1}{4}\right) \cdot \left(T - \frac{8}{3}\right)$$

The functions



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$$\int_0^T f(\rho(t,0))\,dt = \frac{7}{64} + \frac{15}{64}\cdot\frac{5}{3} + \frac{3}{16}\cdot\left(T - \frac{8}{3}\right)$$

Case $n = m = 1$: non entropic optimal solutions

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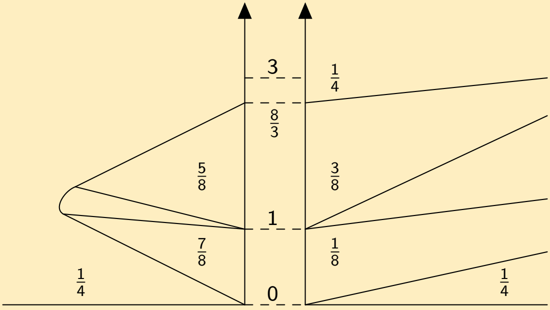

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Additional optimization criterium

Fix $T, M > 0$. In connection with the nodal problem

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0, & l \in \{1, \ldots, m+n\} \\ \rho_l(0,x) = \rho_{0,l}(x), \end{cases}$$

with flux constraints

$$A \cdot (f(\rho_1(t,0)), \ldots, f(\rho_m(t,0)))^T = (f(\rho_{n+1}(t,0)), \ldots, f(\rho_{m+n}(t,0)))^T$$

and $\mathcal{G}^M$ set of admissible flux traces as above, let $\mathcal{D}_M$ denote the set of optimal solutions of

$$\sup_{g \in \mathcal{G}^M} \int_0^T \mathcal{J}\,(g(t))\, dt \qquad (max)_M$$

Then minimizes

$$\min_{g \in \mathcal{D}^M} \sum_{i=1}^{m} \mathrm{T.V.}_{(0,T)} g_i(\cdot). \qquad (min)_M$$

Additional criterium: existence of solution

Theorem (A.,Cesaroni, Coclite, Garavello)

Let $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}$ be a continuous map. For every $M > 0$, there exists $\widehat{g} \in \mathcal{G}^M$ such that

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and minimizes $(min)_M$.

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The proof relies on the compactness of the set $\mathcal{G}^M$ with respect to the $\mathbb{L}^1$-topology and the lower semicontinuity of the total variation

Notice: solutions of the min-max problems fulfills the requirement of maximize the total flux through the junction keeping the oscillation the smallest possible

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Consider $\mathcal{J}(g) = g$

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Consider $\mathcal{J}(g) = g$

Theorem (A.,Cesaroni, Coclite, Garavello)

For every $T > 0$, let ρ_e be the entropy admissible solution to (CP) and assume that $\rho_e(\cdot, 0)$ is monotone. Then $\rho_e \rho_e(\cdot, 0)$ solves the minmax problem $(min)_M$, i.e.

$$\mathrm{T.V.}_{(0,T)} = \min_{g \in \mathcal{D}^M} \mathrm{T.V.}_{(0,T)} g(\cdot)$$

for every $M \geq \mathrm{T.V.}(\rho_e(\cdot, 0))$.

Merci de votre attention!