# Exact boundary controllability of a system of mixed order with essential spectrum

Contrôle des EDP et Applications

FARID AMMAR KHODJA

(Joint work with KARINE MAUFFREY and ARNAUD MÜNCH]

Laboratoire de Mathématiques de Besançon

CIRM: 9-13 Novembre 2015

# Problem formulation

$$\begin{cases} u_1'' = \Delta u_1 + b \cdot \nabla u_2 & \text{in } Q_T = \Omega \times (0,T) \\ u_2'' = -\nabla \cdot (b u_1) - a u_2 & \text{in } Q_T \\ u_1 = v\, 1_\Gamma & \text{on } \Sigma_T = \partial\Omega \times (0,T) \\ (u(\cdot,0), u'(\cdot,0)) = (u^0, u^1) & \text{in } \Omega. \end{cases}$$

- $\Omega \subset \mathbb{R}^n$ smooth bounded domain and $\Gamma \subset \partial\Omega$.
- $u = (u_1, u_2)$, $b \in C^\infty(\Omega; \mathbb{R}^n)$, $a \in C^\infty(\Omega; \mathbb{R})$ .
- **Exact controllability Issue**: Let $E$ be the energy space (to be defined):

$$\forall \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \begin{pmatrix} u_T^0 \\ u_T^1 \end{pmatrix} \in E,\ \exists v \in L^2(\Sigma_T) :\ \begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = \begin{pmatrix} u_T^0 \\ u_T^1 \end{pmatrix} ?$$

- We set:

$$H = \mathbb{L}^2(\Omega)\,;\;\; V = H_0^1(\Omega) \times L^2(\Omega)$$

- $V'$ denotes the dual of $V$ with respect to the pivot space $H$:

$$V \hookrightarrow H \hookrightarrow V'.$$

- The adjoint problem writes:

$$\begin{cases} \varphi'' = \Delta\varphi + b\cdot\nabla\psi & \text{in } Q_T \\ \psi'' = -\nabla\cdot(b\varphi) - a\psi & \text{in } Q_T \\ \varphi = 0 & \text{on } \Sigma_T \\ (\Phi(\cdot,0), \Phi'(\cdot,0)) = \left(\Phi^0, \Phi^1\right) & \text{in } \Omega. \end{cases}$$

for $\Phi = (\varphi, \psi)$.

- As usual, it can be proved that the controllability issue is equivalent to the observability inequality:

$$\left\| \left(\Phi^0, \Phi^1\right) \right\|^2_{H_{1/2}\times H} \leq C \int_0^T\!\!\int_\Gamma \left(\frac{\partial\varphi}{\partial n} + (b\cdot n)\,\psi\right)^2 d\sigma dt.$$

# Result statement

**Theorem**

*Let* $\Gamma = \partial\Omega$ *and* $T > 0$. *Assume that*

$$\exists \, (x, \xi) \in \Omega \times \{\xi \in \mathbb{R}^n : |\xi| = 1\}, \; a(x) - (b(x) \cdot \xi)^2 > 0.$$

*Then the system*

$$\begin{cases} u_1'' = \Delta u_1 + b \cdot \nabla u_2 & \text{in } Q_T = \Omega \times (0, T) \\ u_2'' = -\nabla \cdot (b u_1) - a u_2 & \text{in } Q_T \\ u_1 = v \, 1_\Gamma & \text{on } \Sigma_T = \partial\Omega \times (0, T) \\ (u(\cdot, 0), u'(\cdot, 0)) = (u^0, u^1) & \text{in } \Omega. \end{cases}$$

*is not exactly controllable in* $H \times V'$.

This result is a consequence of a noncontrollability result due to Geymonat-Valente (2000).

- The question is then: does there exist a subspace of $H \times V'$ where the exact controllability holds true?
- We answer the question in the following special situation:

$$
\begin{aligned}
n &= 2,\ \Omega = (0,1)^2,\ \Gamma = \left\{(x,y) \in \overline{\Omega} : xy = 0\right\} \\
b &= (\alpha, 0) \in \mathbb{R}^2,\ a \in \left]0, \infty\right[ .
\end{aligned}
$$

In this case, the adjoint system writes

$$
\begin{cases}
\varphi'' = \Delta\varphi + \alpha\partial_x\psi & \text{in } Q_T \\
\psi'' = -\alpha\partial_x\varphi - a\psi & \text{in } Q_T \\
\varphi = 0 & \text{on } \Sigma_T \\
\left(\Phi(\cdot,0), \Phi'(\cdot,0)\right) = \left(\Phi^0, \Phi^1\right) & \text{in } \Omega.
\end{cases}
.
$$

- The operator

$$A = \begin{pmatrix} -\Delta & -\alpha\partial_x \\ \alpha\partial_x & a \end{pmatrix}, \; D(A) = \left(H^2 \cap H_0^1\right) \times H_x^1$$

where $H_x^1 = \left\{\varphi \in L^2(\Omega) : \alpha\partial_x\varphi \in L^2(\Omega)\right\}$, is symmetric but not closed in $\mathbb{L}^2(\Omega)$.

- The closure of $A$ (again denoted by $A$) in $\mathbb{L}^2(\Omega)$ can be defined by:

$$A\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -\Delta\left(\varphi + \alpha\Delta^{-1}\partial_x\psi\right) \\ \alpha\partial_x\varphi + a\psi \end{pmatrix},$$

$$D(A) = \left\{(\varphi, \psi)^T \in H_0^1(\Omega) \times L^2(\Omega) : \varphi + \alpha\Delta^{-1}\partial_x\psi \in H^2(\Omega)\right\},$$

- The injection $D(A) \hookrightarrow \mathbb{L}^2(\Omega)$ is not compact: this gives rise to essential spectrum.

- For $p, q \geq 1$, let

$$\mu_{pq} = \left(p^2 + q^2\right) \pi^2$$

and

$$\varphi_{pq}(x, y) = 2 \sin(p\pi x) \sin(q\pi y) \,, \quad (x, y) \in \Omega = (0, 1)^2$$

- The operator admits the sequence of eigenvalues $\{\lambda_{p,q}^{\pm}\} \cup \{a\}$ defined by:

$$\lambda_{p,q}^{\pm} = \frac{1}{2} \left( \mu_{pq} + a \pm \sqrt{\left(\mu_{pq} - a\right)^2 + 4\alpha^2 p^2 \pi^2} \right).$$

With $\lambda_{p,q}^{\pm}$ is associated the eigenvector

$$e_{p,q}^{\pm} = \left( \frac{(\lambda_{p,q}^{\pm} - a)}{\sqrt{(\lambda_{p,q}^{\pm} - a)^2 + \alpha^2 p^2 \pi^2}} \varphi_{pq}, \frac{\alpha}{\sqrt{(\lambda_{p,q}^{\pm} - a)^2 + \alpha^2 p^2 \pi^2}} \frac{\partial \varphi_{pq}}{\partial x} \right)$$

- For the eigenvalue $a$, we have

$$\mathrm{Ker}\,(A - aI_d) = \left\{ (0, \theta(y)) : \theta \in L^2(0,1) \right\}.$$

  Thus $a \in \sigma_{ess}(A)$.

- Indeed, by direct computations or as a consequence of a result of Grubb-Geymonat (1977):

$$\sigma_{ess}(A) = [a - \alpha^2, a].$$

- Notice that:
  - $\lambda^+_{p,q} \underset{\|(p,q)\| \to +\infty}{\sim} \mu_{pq}$
  - $\delta \in \sigma_{ess}(A) \Leftrightarrow \exists\,(p_k, q_k) \in \mathbb{N} \times \mathbb{N} : \lambda^-_{p_k,q_k} \to \delta$ as $k \to \infty$.

- The last notations we will need is the following:

$$H^{\pm} = \operatorname{span}\left(\left\{e^{\pm}_{p,q},\ p,q \geq 1\right\}\right),\ H^{a} = \operatorname{Ker}\left(A - aI_d\right)$$

and for $\delta \in \mathbb{R}$,

$$H_\delta = D\left(A^\delta\right),\ H^{\pm}_\delta = H_\delta \cap H^{\pm},\ H^{a}_\delta = H_\delta \cap H^{a}.$$

- We are now ready to set our main result:

### Theorem

*For every $N \in \mathbb{N}^*$, let us denote by $H^{N^-}$ (resp. $H^{N^-}_{-1/2}$) the Hilbert subspace of $H$ (resp. $H_{-1/2}$) spanned by the $e^-_{p,q}$ for $1 \leq p,q \leq N$. If $a \leq 2\pi^2$, then there exists $T_0 = T_0(N,a)$ s. t. for any $T > T_0$, the system is exactly controllable in*

$$\left(H^{a} \oplus H^{+} \oplus H^{N^-}\right) \times \left(H^{a}_{-\frac{1}{2}} \oplus H^{+}_{-\frac{1}{2}} \oplus H^{N^-}_{-1/2}\right).$$

# Noncontrollability: sketch of the proof (Valente-Geymonat)

- **First step**: As a consequence of a result of Grubb-Geymonat:

**Lemma**

*For the selfadjoint operator defined on* $\mathbb{L}^2(\Omega)$ *by*

$$A\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix} -\Delta\left(\varphi + \Delta^{-1} b\cdot\nabla\psi\right) \\ \nabla\cdot(b\varphi) + a\psi\end{pmatrix},$$

$$D(A) = \left\{(\varphi,\psi)\in H_0^1(\Omega)\times L^2(\Omega) : \varphi + \Delta^{-1} b\cdot\nabla\psi \in H^2(\Omega)\right\},$$

*we have*

$$\sigma_{ess}(A) = \left\{\lambda(x,\xi) = a(x) - (b(x)\cdot\xi)^2,\ \ (x,\xi)\in\Omega\times S_{n-1}\right\}.$$

- **Second step**:

## Lemma

*Let* $(x^*, \xi^*) \in \Omega \times S_{n-1}$ *and* $\lambda^* = \lambda^*(x^*, \xi^*) \in \sigma_{ess}(A)$ . *Then there is a singular sequence* $\{\Lambda_k = (\varphi_k, \psi_k)\}$ *of* $A - \lambda^* I_d$ *such that:*

1. $\lim_{k\to\infty} \langle A\Lambda_k, \Lambda_k \rangle_{\mathbb{L}^2(\Omega)} = \lambda^*$;
2. $C\Lambda_k := \dfrac{\partial \varphi_k}{\partial \nu} + (b \cdot \nu)\, \psi_k \to 0$ *in* $L^2(\Gamma)$ *strongly.*

- This singular sequence is constructed in the following way: let $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\|\theta\|_{L^2} = 1$ and set

$$\rho_k(x) = k^{n/2} e^{i(x-x^*)\cdot\xi^*} \theta(k(x-x^*)), \; k \geq 1.$$

  Then set

$$\Lambda_k = \frac{\widetilde{\Phi}_k}{\left\|\widetilde{\Phi}_k\right\|_{\mathbb{L}^2}}; \quad \widetilde{\Phi}_k = \begin{pmatrix} P(b\cdot\nabla)\rho_k \\ \rho_k \end{pmatrix}$$

  where $P$ is a suitable parametrix of $-\Delta$, chosen in such a way that *the support of $P(b\cdot\nabla)\rho_k$ is close to the support of $\rho_k$.*
- Note that the support of $\Phi_k$ is some neighborhood of $x^*$ for sufficiently large $k$.

- **Third step**:
Consider the system satisfied by $\{\Phi_k = (\varphi_k, \psi_k)\}$ :

$$\begin{cases} \partial_t^2 \Phi_k + A\Phi_k = 0, & Q_T \\ \varphi_k = 0, & \Sigma_T \\ \Phi_k(0) = \Lambda_k,\ \partial_t \Phi_k(0) = 0, & \Omega \end{cases}$$

If $\underline{\lambda^* > 0}$, it can be proved that

$$\int_{\Sigma_T} |C\Phi_k|^2 = \int_{\Sigma_T} \left| \frac{\partial \varphi_k}{\partial \nu} + (b \cdot \nu)\, \psi_k \right|^2 \to 0 \text{ as } k \to \infty,$$

while

$$\|(\Lambda_k,\ 0)\| = 1,$$

contradicting the observability inequality.

# Proof of the controllability result

It consists in proving:

**Theorem**

*Let* $\gamma = \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+|\alpha|}}$ *and* $T_0 = \frac{2\pi}{\gamma}\sqrt{1+2\frac{\left(\lambda_{1,1}^{+}-a+\alpha^2\right)^2}{\left(\lambda_{1,1}^{+}-a\right)^2}}$. *If* $a \leq 2\pi^2$, *then for any* $T > T_0$ *there exists a positive constant* $C^{+}(T)$ *such that for all initial data* $\left(\Phi^0,\Phi^1\right)^T$ *in* $\left(H_{1/2}^{+}\times H^{+}\right)$ *the solution of the adjoint system satisfies the observability inequality:*

$$\left\|\left(\Phi^0,\Phi^1\right)\right\|_{X_1}^2 \leq C^{+}(T)\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu}+\alpha\psi\nu_1\right)^2 d\sigma dt.$$

The previous result is based on an adaptation to our case of the following Ingham inequality:

## Theorem (Mehrenberger 2009)

*There exist* $\gamma_1 > 0$ *and* $\gamma_2 > 0$ *such that for every* $p$, $p'$, $q$ *and* $q'$ *in* $\mathbb{N}^*$

$$\begin{aligned} p \leq \max(q, q') \Rightarrow \left|\sqrt{\mu_{pq}} \pm \sqrt{\mu_{pq'}}\right| \geq \gamma_1 \left|q \pm q'\right| \\ q \leq \max(p, p') \Rightarrow \left|\sqrt{\mu_{pq}} \pm \sqrt{\mu_{p'q}}\right| \geq \gamma_2 \left|p \pm p'\right|. \end{aligned}$$

*Moreover for any* $T > 2\pi\sqrt{\frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2}}$, *there exists a positive constant* $C(T)$ *such that*

$$T) \sum_{p,q\geq 1} (p^2+q^2) \left|z_{p,q}\right|^2 \leq \sum_{q\in\mathbb{N}^*} \int_0^T \left| \sum_{p\in\mathbb{N}^*} p\left(z_{p,q} e^{i\sqrt{\mu_{pq}}t} + \overline{z_{p,q}} e^{-i\sqrt{\mu_{pq}}t}\right)\right|^2 dt + \sum_{p\in\mathbb{N}^*} \int_0^T \left| \sum_{q\in\mathbb{N}^*} q\left(z_{p,q} e^{i\sqrt{\mu_{pq}}t} + \overline{z_{p,q}} e^{-i\sqrt{\mu_{pq}}t}\right)\right|^2 dt$$

# Comments and further results

- Using exactly the same singular sequence previously constructed following Valente-Geymonat, it could be proved:

**Theorem**

*Let $\overline{\omega} \subset \Omega$ and $T > 0$. Assume that*

$$\exists\, (x, \xi) \in \Omega \backslash \overline{\omega} \times \{\xi \in \mathbb{R}^n : |\xi| = 1\}\,,\; a\,(x) - (b\,(x) \cdot \xi)^2 > 0.$$

*Then the system*

$$\begin{cases} u_1'' = \Delta u_1 + b \cdot \nabla u_2 + v\, 1_\omega & \text{in } Q_T = \Omega \times (0, T) \\ u_2'' = -\nabla \cdot (b u_1) - a u_2 & \text{in } Q_T \\ u_1 = 0 & \text{on } \Sigma_T = \partial\Omega \times (0, T) \\ (u(\cdot, 0), u'(\cdot, 0)) = \left(u^0, u^1\right) & \text{in } \Omega. \end{cases}$$

*is not exactly controllable in $V \times H$ (even if $\omega$ satisfies the Bardos-Lebeau-Rauch condition).*

- A similar positive result should hold on a subspace of the energy space for the distributed control.
- The noncontrollability result remains true for any other "reasonable" boundary condition (obtained from Green's formula).
- The essential spectrum does not depend on (reasonable) boundary conditions.

# The associated parabolic control problem

$$\begin{cases} u_1' = \Delta u_1 + b \cdot \nabla u_2 + w\, 1_\omega & \text{in } Q_T = \Omega \times (0, T) \\ u_2' = -\nabla \cdot (b u_1) - a u_2 & \text{in } Q_T \\ u_1 = v\, 1_\Gamma & \text{on } \Sigma_T = \partial\Omega \times (0, T) \\ u(\cdot, 0) = \left(u^0, u^1\right) & \text{in } \Omega. \end{cases}$$

- The general null-controllability problem is widely open: only some special cases have been solved.
- Conjecture: the boundary or distributed control problem should not be null-controllable on $\mathbb{L}^2(\Omega)$.
- The proof of this negative result should work by contradicting the observability inequality using a singular sequence.
- A result of Guerrero-Imanuvilov (COCV:2013) goes in this direction.