Quadratic Interactions in Dispersive Systems

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A system of Schrödinger equations with quadratic interaction

$$\begin{cases} i\partial_t u + \dfrac{1}{2m}\Delta u = \lambda v\bar{u}, \\ i\partial_t v + \dfrac{1}{2M}\Delta v = \mu u^2. \end{cases}$$

u, v : complex - valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\Delta = \partial_1^2 + \cdots + \partial_n^2$,

$\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $x = (x_1, \cdots, x_n)$.

m, $M > 0$: masses

$\lambda, \mu \in \mathbb{C}$: complex coupling constants

$\bar{u}(t, x) = \overline{u(t, x)}$: complex conjugate

Basic Results on the Cauchy Problem, Ground States, ・・・

N. Hayashi, T. O, and K. Tanaka, Ann IHP, AN, 2013.

Small Data Scattering for $n \geq 3$

N. Hayashi, C. Li, and T. O, Differ. Equ. Appl., 2011.

Basic Identities $(c \in \mathbb{R})$

$$\|u(t)\|_2^2 + c\|v(t)\|_2^2 = \|u(0)\|_2^2 + c\|v(0)\|_2^2 + 2\mathrm{Im}\int_0^t (\lambda - c\bar{\mu})(v(t')|u^2(t'))dt',$$

$$\frac{1}{2m}\|\nabla u(t)\|_2^2 + \frac{c}{4M}\|\nabla v(t)\|_2^2$$
$$= \frac{1}{2m}\|\nabla u(0)\|_2^2 + \frac{c}{4M}\|\nabla v(0)\|_2^2 - \mathrm{Re}\int_0^t [\lambda(v(t')|\partial_t(u^2)(t')) + c\bar{\mu}(\partial_t v(t')|u^2(t'))]dt'.$$

Conservation Laws for Charge and Energy $\Leftrightarrow \exists c_0 \in \mathbb{R} : \lambda = c_0\bar{\mu}$

$$Q(t) = \|u(t)\|_2^2 + c_0\|v(t)\|_2^2,$$

$$E(t) = \frac{1}{2m}\|\nabla u(t)\|_2^2 + \frac{c_0}{4M}\|\nabla v(t)\|_2^2 + \mathrm{Re}(\lambda(v(t)|u^2(t))).$$

Lagrangian Density under the Constraint of Coupling Constants $: \lambda = c_0\bar{\mu}$

$$\mathfrak{L} = i(\bar{u}\dot{u} - u\dot{\bar{u}}) + \frac{ic_0}{2}(\bar{v}\dot{v} - v\dot{\bar{v}}) - \frac{1}{m}\nabla u \cdot \overline{\nabla u} - \frac{c_0}{2M}\nabla v \cdot \overline{\nabla v} - V,$$

$$V = V(u, \bar{u}, v, \bar{v}) = \mu u^2 \bar{v} + \overline{\mu}\ \overline{u}^2 v.$$

Characterization of Nonlinear Potential V by the Gauge Structure

Theorem 1 (G. Hoshino, T.O)

Let $V : \mathbb{C} \ni (u, \bar{u}, v, \bar{v}) \mapsto V(u, \bar{u}, v, \bar{v}) \in \mathbb{R}$ be a cubic homogeneous polynomial with complex coefficients. Then :

$$(1)\ \ \exists \mu \in \mathbb{C} : V(u, \bar{u}, v, \bar{v}) = \mu u^2 \bar{v} + \bar{\mu} \bar{u}^2 v.$$

$$\Leftrightarrow\ \ (2)\ \forall \theta \in \mathbb{R},\ V(e^{i\theta}u, \overline{e^{i\theta}u}, e^{2i\theta}v, \overline{e^{2i\theta}v}) = V(u, \bar{u}, v, \overline{v}).$$

Proof. $(2) \Rightarrow (1)$:

$$V(u, \bar{u}, v, \bar{v}) = \sum_{|\alpha|=3} (C_\alpha u^{\alpha_1} \bar{u}^{\alpha_2} v^{\alpha_3} \bar{v}^{\alpha_4} + \overline{C_\alpha} \bar{u}^{\alpha_1} \bar{u}^{\alpha_2} \bar{v}^{\alpha_3} v^{\alpha_4}),$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4_{\geq 0}, \quad C_\alpha \in \mathbb{C}.$$

$$(2)\ \Rightarrow\ \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3.$$

$$\Leftrightarrow\ \alpha = (2, 0, 0, 1), (0, 2, 1, 0)\ \Rightarrow (1) \text{ with } \mu = C_{(2,0,0,1)} + \overline{C_{(0,2,1,0)}}.$$

Gauge Structure and Mass Resonance

$$\begin{cases} \frac{1}{2mc^2}\partial_t^2 u - \frac{1}{2m}\Delta u + \frac{mc^2}{2}u = -\lambda v\bar{u}, \\ \frac{1}{2Mc^2}\partial_t^2 u - \frac{1}{2M}\Delta v + \frac{Mc^2}{2}v = -\mu u^2. \end{cases}$$

(Bachelot, Georgiev, D'Ancona, ･･･)

$\Rightarrow \ (u_c, v_c) = (e^{itmc^2}u, \ e^{itMc^2}v)$ satisfy

$$\begin{cases} \frac{1}{2mc^2}\partial_t^2 u_c - i\partial_t u_c - \frac{1}{2m}\Delta u_c = -e^{it(2m-M)c^2}\lambda v_c\overline{u_c}, \\ \frac{1}{2Mc^2}\partial_t^2 v_c - i\partial_t v_c - \frac{1}{2M}\Delta v_c = -e^{it(M-2m)c^2}\mu u_c^2. \end{cases}$$

(M. Tsutsumi, Machihara, Nakanishi, O, ･･･)

$\boxed{M = 2m}$ **Mass resonance Condition** $\Rightarrow$ **Nonrelativistic Limit**

$\Downarrow$

Galilei Invariance

$$J_m = J_m(t) = U_m(t)xU_m(-t) = x + i\frac{t}{m}\nabla = \mathcal{M}_m(t)i\frac{t}{m}\nabla\mathcal{M}_m(-t),$$

$$U_m(t) = \exp(i\frac{t}{2m}\Delta), \ \mathcal{M}_m(t) = \exp(i\frac{m}{2t}|x|^2).$$

$$J_m(v\bar{u}) = \mathcal{M}_m i\frac{t}{m}\nabla(\mathcal{M}_{2m}^{-1}v\overline{\mathcal{M}_m^{-1}u})$$

$$= (\mathcal{M}_{2m}i\frac{t}{m}\nabla\mathcal{M}_{2m}^{-1}v)\overline{u} - v\overline{(\mathcal{M}_m i\frac{t}{m}\nabla\mathcal{M}_m^{-1}u)} = 2(J_{2m}v)\bar{u} - v\overline{J_m u}.$$

$$J_{2m}(u^2) = \mathcal{M}_{2m}i\frac{t}{2m}\nabla(\mathcal{M}_m^{-1}u)^2 = u\mathcal{M}_m i\frac{t}{m}\nabla\mathcal{M}_m^{-1}u = uJ_m u.$$

Large Mass Behavior for Semi-Relativistic System (SRS)

$$\begin{cases} i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda v \bar{u}, \\ i\partial_t u + (M^2 - \Delta)^{1/2} v = \mu u^2. \end{cases}$$

$\Rightarrow (u_m, v_M) = (e^{itm}u, e^{itM}v)$ satisfy

$$\begin{cases} i\partial_t u_m + ((m^2 - \Delta)^{1/2} - m)u_m = e^{it(2m-M)}\lambda v_M \overline{u_m}, \\ i\partial_t v_M + ((M^2 - \Delta)^{1/2} - M)v_M = e^{it(M-2m)}\mu u_m^2. \end{cases}$$

$\boxed{M = 2m}$ **Mass Resonance Condition $\Longrightarrow$ Infinite Mass Limit**

$$(m^2 - \Delta)^{1/2} - m \overset{\mathcal{F}}{\leftrightarrow} (m^2 + |\xi|^2)^{1/2} - m = \frac{1}{2m}|\xi|^2 + O(\frac{1}{m^3})$$

I.E. Segal, Adv. in Math., 1976 ; Y. Cho and T. O, SIAM J. Math. Anal., 2006

SRS as a Hamilton System (under $\lambda = c_0\bar{\mu}$)

$$E(u,v) = \|(m^2-\Delta)^{1/4}u\|_2^2 + \frac{c_0}{2}\|(M^2-\Delta)^{1/4}v\|_2^2 - \mathrm{Re}(\lambda(v|u^2)).$$

$$E : H^{1/2}\times H^{1/2} \ni (u,v) \mapsto E(u,v) \in \mathbb{R}$$

$$(H^{1/2}\times H^{1/2})' \overset{\text{Riesz}}{\longleftrightarrow} H^{1/2}\times H^{1/2} \quad \text{via} \quad ((u,v)|(\varphi,\psi)) = 2\mathrm{Re}(u|\varphi) + c_0\mathrm{Re}(v|\psi).$$

(SRS)
$$\begin{cases} i\partial_t u + (m^2-\Delta)^{1/2}u = \lambda v\bar{u}, \\ i\partial_t u + (M^2-\Delta)^{1/2}v = \mu u^2. \end{cases}$$

$$\Leftrightarrow \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = JE'\begin{pmatrix} u \\ v \end{pmatrix} \text{ as a Hamilton system in } H^{1/2}\times H^{1/2}, \text{ with } J = i.$$

Conserved Quantities (under $\lambda = c_0\bar{\mu}$)

$Q(u,v) = \|u\|_2^2 + c_0\|v\|_2^2$: charge $\leftrightarrow L^2$

$E(u,v) = \|(m^2-\Delta)^{1/4}u\|_2^2 + \frac{c_0}{2}\|(M^2-\Delta)^{1/4}v\|_2^2 - \mathrm{Re}(\lambda(v|u^2))$: energy $\leftrightarrow H^{1/2}$

Global Cauchy Problem for a Semi-Relativistic System in 1D

$$
\text{(SRS)} \quad \begin{cases} i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda \bar{u} v, \\ i\partial_t v + (M^2 - \Delta)^{1/2} v = \dfrac{\bar{\lambda}}{c_0} u^2, \\ u(0) = u_0, v(0) = v_0. \end{cases}
$$

$m, M \geq 0;\ c_0 > 0.$

Difficulties : $(m^2 - \Delta)^{1/2}$ and $(M^2 - \Delta)^{1/2}$ are nonlocal operators.

Massless Dirac system admits a special transformation $(t, x) \mapsto (t + x, t - x)$ and has a special null structure. (Machihara, Nakanishi, Tsugawa, Bournaveous, …)

Associated propagators exp $(\pm it(m^2 - \Delta)^{1/2})$ have Strichartz type estimates without extra regularity as the Klein-Gordon equation.
(gain of one derivative in Duhamel term)

The same as Dirac system (Escobedo, Vega, Machihara, Nakanishi, TO, D'Ancona , …

Related Results on Global Well-posedness in 1D

$\cdot\ i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda |u|^2 u.$

H^s with $s > 1/2 \cdots$ Borgna, Rial, JMP, 2012 ($H^s \hookrightarrow L^\infty$)

$H^{1/2} \cdots$ Krieger, Lenzmann, Raphaël, ARMA, 2013 (Compactness, Yudovitch Argument)

$$\cdot \begin{cases} i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda \bar{u} v, \\ i\partial_t v - (M^2 - \Delta)^{1/2} v = \dfrac{\bar{\lambda}}{c_0} u^2. \end{cases}$$

H^s with $s \geq 0 \ \cdots$ Fujiwara, Machihara, O, CMP, 2015(Bourgain Method)

Candidates for Maximal Function Spaces of Global Well-Posedness

- From a scaling point of view in the massless case ･･･ $H^{-1/2} \times H^{-1/2}$.
- From a point of view from conserved quantities：

- $L^2 \times L^2$ ･･･charge $Q(u,v) = \|u\|_2^2 + c_0\|v\|_2^2$.
- $H^{1/2} \times H^{1/2}$ ･･･energy $E(u,v) = \|(m^2-\Delta)^{1/4}u\|_2^2 + \frac{c_0}{2}\|(M^2-\Delta)^{1/2}v\|_2^2 - \mathrm{Re}(\lambda(v|u^2))$.

Perturbation Approach (Picard Iteration Scheme) Fails in $H^s \times H^s$ **with** $s < 1/2$

There exist $u_0, v_0 \in H^s$ such that the first iterate

$$\int_0^t U_m(t-t')\left(\overline{U_m(t')u_0}\cdot U_M(t')v_0\right)dt' \notin H^s,$$

where $U_m(t) = \exp(it(m^2-\Delta)^{1/2})$.

A reminiscent of the result of Molinet, Saut, and Tzvetkov on the possibility of the perturbation approach to the Benjamin-Ono equation (SIAM J. Math. Anal., 2001).

Main theorem (with Fujiwara and Machihara) $\lambda \in \mathbb{C},\ c_0 > 0.$

$\forall (u_0, v_0) \in H^{1/2} \times H^{1/2}\ \exists 1 (u, v) \in C(\mathbb{R}; H^{1/2} \times H^{1/2}) \cap C^1(\mathbb{R}; H^{-1/2} \times H^{-1/2}):$

$$\text{(SRS)} \quad \begin{cases} i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda \bar{u} v, \\ i\partial_t v + (M^2 - \Delta)^{1/2} v = \dfrac{\bar{\lambda}}{c_0} u^2, \end{cases}$$

$$(u(0), v(0)) = (u_0, v_0).$$

Moreover, continuous dependence holds, i.e.

$$(u_{n,0}, v_{n,0}) \to (u_0, v_0) \text{ in } H^{1/2} \times H^{1/2} \text{ as } n \to \infty$$

$$\Rightarrow (u_n, v_n) \to (u, v) \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}; H^{1/2} \times H^{1/2}) \text{ as } n \to \infty.$$

Method of Proof

・Approximate solutions (u_ρ, v_ρ) with $\rho > 0$:

$$\begin{cases} u_\rho(t) = U_m(t)J_\rho u_0 - i\lambda \int_0^t U_m(t-t')J_\rho(\overline{J_\rho u(t')}J_\rho v(t'))dt', \\ v_\rho(t) = U_M(t)J_\rho v_0 - i\frac{\bar{\lambda}}{c_0}\int_0^t U_M(t-t')J_\rho(J_\rho u(t'))^2 dt', \end{cases}$$

where $J_\rho = \rho(\rho + (-\Delta)^{1/2})^{-1}$ (Yosida approximation).

(Ginibre, Velo, JFA, 1979 ; Segal, Bull. SMF, 1963)

$\cdot\{(u_\rho, v_\rho)\}$: bounded in $L^\infty(\mathbb{R}; H^{1/2} \times H^{1/2})$,

Cauchy in $L^\infty([-T,T]; L^2 \times L^2), \forall T > 0.$

・Yudovitch (Vladimirov) type argument.

・Independent of compactness argument.

Lemma $\exists C > 0 : \forall \varphi, \psi \in H^{1/2}(\mathbb{R}),\ \forall p \geq 1, \forall \rho > 0$

$$\|(1 - J_\rho)(\varphi\psi)\|_2 \leq C p^{1/2} \|(\varphi, \psi); H^{1/2} \times H^{1/2}\|^{1+1/p}$$
$$\cdot \|((1 - J_\rho)\varphi, (1 - J_\rho)\psi); L^2 \times L^2\|^{1-1/p}.$$

(Proof) $\tilde{\varphi} \equiv \mathcal{F}^{-1}|\mathcal{F}\varphi|,\ \tilde{\psi} \equiv \mathcal{F}^{-1}|\mathcal{F}\psi|.$

$$\begin{aligned}
\|(1 - J_\rho)(\varphi\psi)\|_2 &= C\| \int \frac{|\xi|}{\rho + |\xi|} \hat{\varphi}(\xi - \eta)\hat{\psi}(\eta) d\eta; L^2_\xi \| \\
&\leq C\| \int \frac{|\xi - \eta| + |\eta|}{\rho + |\xi - \eta| + |\eta|} |\hat{\varphi}(\xi - \eta)||\hat{\psi}(\eta)| d\eta; L^2_\xi \| \\
&\leq C\| \int \frac{|\xi - \eta|}{\rho + |\xi - \eta|} |\hat{\varphi}(\xi - \eta)||\hat{\psi}(\eta)| d\eta; L^2_\xi \| \\
&+ C\| \int \frac{|\eta|}{\rho + |\eta|} |\hat{\varphi}(\xi - \eta)||\hat{\psi}(\eta)| d\eta; L^2_\xi \| \\
&\leq C\|((1 - J_\rho)\tilde{\varphi})\tilde{\psi}\|_2 + C\|\tilde{\varphi}((1 - J_\rho)\tilde{\psi})\|_2.
\end{aligned}$$

Lemma $p > 1, \quad a, b > 0, \ f : [0, \infty) \to [0, \infty)$ s.t.

$$f(t) \le a + b \int_0^t f(s)^{1-1/p} ds, \quad t \ge 0.$$

$$\Rightarrow \quad f(t) \le (a^{1/p} + \frac{b}{p} t)^p, \quad t \ge 0.$$

Corollary $a = 0, \ b = M(p)$ with $M(p)/p \to 0 \ (p \to \infty), \ f : [0, \infty) \to [0, \infty)$ s.t.

$$f(t) \le M(p) \int_0^t f(s)^{1-1/p} ds, \quad t \ge 0.$$

$\Rightarrow \ f(t) \le \left(\frac{M(p)}{p} t \right)^p \to 0 \ (p \to \infty)$ on compact intervals.