Local invariant manifolds for delay differential equations with states in the Fréchet space $C^1((-\infty, 0], \mathbb{R}^n)$

Hans-Otto Walther

Dynamics of Evolution Equations
Dedicated to the memory of Prof. Jack Hale
Luminy, March 21-25, 2016

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

$$x'(t) = f(x_t)$$

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

$$x'(t) = f(x_t)$$

$f : C^1 \supset U \to \mathbb{R}^n$, $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ (no norm)

$x_t(s) = x(t + s)$ for $s \leq 0$

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

$$x'(t) = f(x_t)$$

$f : C^1 \supset U \to \mathbb{R}^n$, $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ (no norm)

$x_t(s) = x(t + s)$ for $s \leq 0$

Solution on (t_0, t_1), $t_0 < t_1 \leq \infty$,

$x : (-\infty, t_1) \to \mathbb{R}^n$ cont diff, $x_t \in U$ for $t_0 < t < t_1$,

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

$$x'(t) = f(x_t)$$

$f : C^1 \supset U \to \mathbb{R}^n$, $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ (no norm)

$x_t(s) = x(t + s)$ for $s \leq 0$

Solution on (t_0, t_1), $t_0 < t_1 \leq \infty$,

$x : (-\infty, t_1) \to \mathbb{R}^n$ cont diff, $x_t \in U$ for $t_0 < t < t_1$,

$x'(t) = f(x_t)$ for $t_0 < t < t_1$.

Toy example, unbounded state-dependent delay:

$$x'(t) = g(x(t - d(x(t))))$$

$g : \mathbb{R} \to \mathbb{R}$ and $d : \mathbb{R} \to [0, \infty)$ cont diff

$$x'(t) = f(x_t)$$

$f : C^1 \supset U \to \mathbb{R}^n$, $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ (no norm)

$x_t(s) = x(t + s)$ for $s \leq 0$

Solution on (t_0, t_1), $t_0 < t_1 \leq \infty$,

$x : (-\infty, t_1) \to \mathbb{R}^n$ cont diff, $x_t \in U$ for $t_0 < t < t_1$,

$x'(t) = f(x_t)$ for $t_0 < t < t_1$.

IVP

$$x'(t) = f(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \tag{1}$$

The IVP (1) in a Banach space (W, 2012)

The IVP (1) in a Banach space (W, 2012)

Fix $a > 0$. $B_a \subset C$ given by

$$\lim_{t \to -\infty} \phi(t) e^{at} = 0$$

$B_a^1 \subset C^1$ given by $\phi \in B_a$ and $\phi' \in B_a$.

The IVP (1) in a Banach space (W, 2012)

Fix $a > 0$. $B_a \subset C$ given by

$$\lim_{t \to -\infty} \phi(t) e^{at} = 0$$

$B_a^1 \subset C^1$ given by $\phi \in B_a$ and $\phi' \in B_a$.

Hypotheses: $f : B_a^1 \supset U \to \mathbb{R}^n$ cont diff with

The IVP (1) in a Banach space (W, 2012)

Fix $a > 0$. $B_a \subset C$ given by

$$\lim_{t \to -\infty} \phi(t) e^{at} = 0$$

$B_a^1 \subset C^1$ given by $\phi \in B_a$ and $\phi' \in B_a$.

Hypotheses: $f : B_a^1 \supset U \to \mathbb{R}^n$ cont diff with

(e) linear extensions $D_e f(\phi) : B_a \to \mathbb{R}^n$ of $Df(\phi)$, $\phi \in U$, and

$$U \times B_a \ni (\phi, \chi) \mapsto D_e f(\phi, \chi) \in \mathbb{R}^n$$

continuous,

The IVP (1) in a Banach space (W, 2012)

Fix $a > 0$. $B_a \subset C$ given by

$$\lim_{t \to -\infty} \phi(t) e^{at} = 0$$

$B_a^1 \subset C^1$ given by $\phi \in B_a$ and $\phi' \in B_a$.

Hypotheses: $f : B_a^1 \supset U \to \mathbb{R}^n$ cont diff with

(e) linear extensions $D_e f(\phi) : B_a \to \mathbb{R}^n$ of $Df(\phi)$, $\phi \in U$, and

$$U \times B_a \ni (\phi, \chi) \mapsto D_e f(\phi, \chi) \in \mathbb{R}^n$$

continuous,

- compare *almost Fréchet differentiable*,

Mallet-Paret, Nussbaum, Paraskevopoulos (1994)

(lbd) for every $\phi \in U$

there are a neighbourhood $N \subset U$ and $r > 0$

such that for all ψ, χ in N,

$$\psi(s) = \chi(s) \quad \text{on} \quad [-r, 0]) \Longrightarrow f(\psi) = f(\chi).$$

(lbd) for every $\phi \in U$

there are a neighbourhood $N \subset U$ and $r > 0$

such that for all ψ, χ in N,

$$\psi(s) = \chi(s) \quad \text{on} \quad [-r, 0]) \Longrightarrow f(\psi) = f(\chi).$$

(no counterpart for constant delay)

(lbd) for every $\phi \in U$

there are a neighbourhood $N \subset U$ and $r > 0$

such that for all ψ, χ in N,

$$\psi(s) = \chi(s) \quad \text{on} \quad [-r, 0]) \Longrightarrow f(\psi) = f(\chi).$$

(no counterpart for constant delay)

Then

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

is a C^1-submanifold, $codim\, X_f = n$,

(lbd) for every $\phi \in U$

there are a neighbourhood $N \subset U$ and $r > 0$

such that for all ψ, χ in N,

$$\psi(s) = \chi(s) \quad \text{on} \quad [-r, 0]) \Longrightarrow f(\psi) = f(\chi).$$

(no counterpart for constant delay)

Then

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

is a C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a continuous semiflow on X_f, $(t, \phi) \mapsto x_t^{\phi}$,

cont diff solution operators $\phi \mapsto x_t^{\phi}$, $t \geq 0$

(lbd) for every $\phi \in U$

there are a neighbourhood $N \subset U$ and $r > 0$

such that for all ψ, χ in N,

$$\psi(s) = \chi(s) \quad \text{on} \quad [-r, 0]) \Longrightarrow f(\psi) = f(\chi).$$

(no counterpart for constant delay)

Then

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

is a C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a continuous semiflow on X_f, $(t, \phi) \mapsto x_t^\phi$,

cont diff solution operators $\phi \mapsto x_t^\phi$, $t \geq 0$

At stationary points $\sigma \in X_f$, W_a^s and W_a^u

Extension (W, 2013) to nonautonomous eqs $x'(t) = f(t, x_t)$,

$$x'(t) = \int_0^t \ldots, \quad \text{Volterra}$$

$$x'(t) = a\,x(\lambda t) + b\,x(t), \quad 0 < \lambda < 1, \quad \text{pantograph}$$

Extension (W, 2013) to nonautonomous eqs $x'(t) = f(t, x_t)$,

$$x'(t) = \int_0^t \ldots, \quad \text{Volterra}$$

$$x'(t) = a\,x(\lambda t) + b\,x(t), \quad 0 < \lambda < 1, \quad \text{pantograph}$$

Deficiency - B_a^1 misses many solution segments:

Extension (W, 2013) to nonautonomous eqs $x'(t) = f(t, x_t)$,

$$x'(t) = \int_0^t \ldots, \quad \text{Volterra}$$

$$x'(t) = a\,x(\lambda t) + b\,x(t), \quad 0 < \lambda < 1, \quad \text{pantograph}$$

Deficiency - B_a^1 misses many solution segments:

Almost all eigensolutions $t \mapsto \Re\, e^{zt}$ of

$$y'(t) = \alpha\, y(t - r)$$

grow too fast at $-\infty$

$C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ contains all x_t

of any continuously differentiable solution of $x'(t) = f(x_t)$

$C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ contains all x_t

of any continuously differentiable solution of $x'(t) = f(x_t)$

Topology on C^1: Uniform convergence of maps

and their derivatives on compact sets, Fréchet space

$C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ contains all x_t

of any continuously differentiable solution of $x'(t) = f(x_t)$

Topology on C^1: Uniform convergence of maps

and their derivatives on compact sets, Fréchet space

? IVP (1), cont diff solution operators on a manifold in C^1 ?

$C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ contains all x_t

of any continuously differentiable solution of $x'(t) = f(x_t)$

Topology on C^1: Uniform convergence of maps

and their derivatives on compact sets, Fréchet space

? IVP (1), cont diff solution operators on a manifold in C^1 ?

Calculus in Fréchet spaces : $g : F \supset U \to G$ cont diff (MB) iff all

$$Dg(u)v = \lim_{0 \neq h \to 0} \frac{1}{h}(g(u + hv) - g(u)) \in G$$

exist with

$$U \times F \ni (u, v) \mapsto Dg(u)v \in G \qquad \text{continuous}$$

$C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ contains all x_t

of any continuously differentiable solution of $x'(t) = f(x_t)$

Topology on C^1: Uniform convergence of maps

and their derivatives on compact sets, Fréchet space

? IVP (1), cont diff solution operators on a manifold in C^1 ?

Calculus in Fréchet spaces : $g : F \supset U \to G$ cont diff (MB) iff all

$$Dg(u)v = \lim_{0 \neq h \to 0} \frac{1}{h}(g(u + hv) - g(u)) \in G$$

exist with

$$U \times F \ni (u, v) \mapsto Dg(u)v \in G \qquad \text{continuous}$$

In infinite dim Banach spaces, weaker than cont diff (Fréchet)

Reference for calculus in Fréchet spaces : R. S. Hamilton, *The Inverse Function Theorem of Nash and Moser*, 1982

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Then (W, 2015)

$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$ C^1-submanifold, $codim\, X_f = n$,

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Then (W, 2015)

$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$ C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a cont semiflow on X_f, $\qquad S(t, \phi) = x_t^{\phi}$

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Then (W, 2015)

$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$ C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a cont semiflow on X_f, $\qquad S(t,\phi) = x_t^{\phi}$

with all $S(t,\cdot)$ cont diff

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Then (W, 2015)

$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$ C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a cont semiflow on X_f, $\quad S(t,\phi) = x_t^{\phi}$

with all $S(t,\cdot)$ cont diff

Remark: $D_1 S(t,\phi)$ only at $\phi \in C^2((-\infty,0],\mathbb{R}^n)$

Reference for calculus in Fréchet spaces : R. S. Hamilton,

The Inverse Function Theorem of Nash and Moser, 1982

Hypothesis

$f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB), (e) holds

Then (W, 2015)

$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$ C^1-submanifold, $codim\, X_f = n$,

IVP (1) defines a cont semiflow on X_f, $\qquad S(t, \phi) = x_t^{\phi}$

with all $S(t, \cdot)$ cont diff

Remark: $D_1 S(t, \phi)$ only at $\phi \in C^2((-\infty, 0], \mathbb{R}^n)$

Hypothesis (lbd) ??

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

$C_d^1 = C^1([-d, 0], \mathbb{R}^n)$, Banach space

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

$C^1_d = C^1([-d, 0], \mathbb{R}^n)$, Banach space

$R : C^1 \ni \phi \mapsto \phi|_{[-d,0]} \in C^1_d$, restriction, linear, cont, onto

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

$C^1_d = C^1([-d, 0], \mathbb{R}^n)$, Banach space

$R : C^1 \ni \phi \mapsto \phi|_{[-d,0]} \in C^1_d$, restriction, linear, cont, onto

$\sigma_d =_{def} R\sigma$

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

$C^1_d = C^1([-d, 0], \mathbb{R}^n)$, Banach space

$R : C^1 \ni \phi \mapsto \phi|_{[-d,0]} \in C^1_d$, restriction, linear, cont, onto

$\sigma_d =_{def} R\sigma$

$P : C^1_d \to C^1$ prolongation,

$P\chi(t) = \chi(-d) + \chi'(-d)(t + d)$ for $t < -d$, linear, cont

Proposition $f : C^1 \supset U \to \mathbb{R}^n$ cont diff (MB) $\Longrightarrow$ (lbd)

Local invariant manifolds for S ? (W, 2016)

$\sigma \in X$ stationary point of S, $\sigma : (-\infty, 0] \to \mathbb{R}^n$ constant

A further hypothesis - preparations: From (lbd),

open nbhd N of σ in U and $d > 0$

$C^1_d = C^1([-d, 0], \mathbb{R}^n)$, Banach space

$R : C^1 \ni \phi \mapsto \phi|_{[-d,0]} \in C^1_d$, restriction, linear, cont, onto

$\sigma_d =_{def} R\sigma$

$P : C^1_d \to C^1$ prolongation,

$P\chi(t) = \chi(-d) + \chi'(-d)(t + d)$ for $t < -d$, linear, cont

$P\sigma_d = \sigma$

Open nbhd U_d of σ_d with $PU_d \subset N$

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C^1_d \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C_d^1 \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Hypothesis : f_d cont diff (F)

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C_d^1 \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Hypothesis : f_d cont diff (F)

- is mild, see toy example

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C_d^1 \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Hypothesis : f_d cont diff (F)

- is mild, see toy example

Consequence: f_d has prop (e) and (W, 2003)

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C^1_d \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Hypothesis : f_d cont diff (F)

- is mild, see toy example

Consequence: f_d has prop (e) and (W, 2003)

IVP

$$x'(t) = f_d(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \chi$$

(segments on $[-d, 0]$) defines semiflow S_d on

$$X_d = \{\chi \in U_d : \chi'(0) = f_d(\chi)\}$$

Open nbhd U_d of σ_d with $PU_d \subset N$

$f_d : C^1_d \supset U_d \to \mathbb{R}^n$, $f_d(\chi) = f(P\chi)$

Hypothesis : f_d cont diff (F)

- is mild, see toy example

Consequence: f_d has prop (e) and (W, 2003)

IVP

$$x'(t) = f_d(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \chi$$

(segments on $[-d, 0]$) defines semiflow S_d on

$$X_d = \{\chi \in U_d : \chi'(0) = f_d(\chi)\}$$

Stationary point σ_d

W^s_d (and W^u_d, W^c_d) ((Krishnan, 1998); Krisztin, W ($\leq$ 2006))

codim $W^s_d = n + \dim\ W^u_d + \dim\ W^c_d$

A stable manifold W^s for S at $\sigma \in X$:

A stable manifold W^s for S at $\sigma \in X$:

For some open neighbourhood $V \subset N$ of σ,

$$W^s =_{def} V \cap R^{-1} W_d^s$$

A stable manifold W^s for S at $\sigma \in X$:

For some open neighbourhood $V \subset N$ of σ,

$$W^s =_{def} V \cap R^{-1} W^s_d$$

is a C^1-submanifold with

$$T_\sigma W^s = R^{-1} T_{\sigma_d} W^s_d$$

A stable manifold W^s for S at $\sigma \in X$:

For some open neighbourhood $V \subset N$ of σ,

$$W^s =_{def} V \cap R^{-1} W^s_d$$

is a C^1-submanifold with

$$T_\sigma W^s = R^{-1} T_{\sigma_d} W^s_d$$

because of transversality $RC^1 + T_{\sigma_d} W^s_d = C^1_d$ and codim $W^s_d < \infty$

A stable manifold W^s for S at $\sigma \in X$:

For some open neighbourhood $V \subset N$ of σ,

$$W^s =_{def} V \cap R^{-1} W_d^s$$

is a C^1-submanifold with

$$T_\sigma W^s = R^{-1} T_{\sigma_d} W_d^s$$

because of transversality $RC^1 + T_{\sigma_d} W_d^s = C_d^1$ and codim $W_d^s < \infty$

Proposition: $g : F \supset \mathcal{O} \to G$ cont diff (MB),

$M \subset G$ C^1-submanifold, codim $M < \infty$,

$Dg(x)F + T_{g(x)}M = G \qquad \Longrightarrow$

for some neighbourhood $\mathcal{N}$ of x,

$\mathcal{N} \cap g^{-1}(M)$ C^1-submanifold, same codimension, ...

Properties of W^s : $W^s \subset X$,

locally positively invariant,

Properties of W^s : $W^s \subset X$,

locally positively invariant,

$T_\sigma W^s =$ stable space ?! of linearization of S at σ

Properties of W^s : $W^s \subset X$,

locally positively invariant,

$T_\sigma W^s =$ stable space ?! of linearization of S at σ

For solutions of IVP (1) with $x_0 = \phi \in W^s$,

$x(t) \to \sigma(0)$ exponentially as $t \to \infty$

Properties of W^s : $W^s \subset X$,

locally positively invariant,

$T_\sigma W^s =$ stable space ?! of linearization of S at σ

For solutions of IVP (1) with $x_0 = \phi \in W^s$,

$x(t) \to \sigma(0)$ exponentially as $t \to \infty$

Segments $x_t \approx \sigma$ of solutions of $x'(t) = f(x_t)$ with

$x(t) \to \sigma(0) \quad (t \to \infty)$ exponentially belong to W^s

An unstable manifold for S at $\sigma \in X$:

Preparations: $\sigma \in B_a^1$

An unstable manifold for S at $\sigma \in X$:

Preparations: $\sigma \in B_a^1$

Inclusion $J : B_a^1 \to C^1$ linear, cont

$B_a^1 \stackrel{R}{\to} C_d^1$ linear, cont

An unstable manifold for S at $\sigma \in X$:

Preparations: $\sigma \in B^1_a$

Inclusion $J : B^1_a \to C^1$ linear, cont

$B^1_a \stackrel{R}{\to} C^1_d$ linear, cont

Open neighbourhood U_a of σ in B^1_a with $JU_a \subset N$, $RU_a \subset U_d$.

An unstable manifold for S at $\sigma \in X$:

Preparations: $\sigma \in B_a^1$

Inclusion $J : B_a^1 \to C^1$ linear, cont

$B_a^1 \stackrel{R}{\to} C_d^1$ linear, cont

Open neighbourhood U_a of σ in B_a^1 with $JU_a \subset N$, $RU_a \subset U_d$.

$f_a : B_a^1 \supset U_a \to \mathbb{R}^n$,

$$f_a(\psi) = f(J\psi) = f(PR\psi) = f_d(R\psi) \qquad \text{(by (lbd))}$$

An unstable manifold for S at $\sigma \in X$:

Preparations: $\sigma \in B_a^1$

Inclusion $J : B_a^1 \to C^1$ linear, cont

$B_a^1 \stackrel{R}{\to} C_d^1$ linear, cont

Open neighbourhood U_a of σ in B_a^1 with $JU_a \subset N$, $RU_a \subset U_d$.

$f_a : B_a^1 \supset U_a \to \mathbb{R}^n$,

$$f_a(\psi) = f(J\psi) = f(PR\psi) = f_d(R\psi) \qquad \text{(by (lbd))}$$

f_a cont diff (F), (lbd), (e)

IVP

$$x'(t) = f_a(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi$$

defines semiflow S_a on $X_a = \{\psi \in U_a : \psi'(0) = f_a(\psi)\}$

IVP

$$x'(t) = f_a(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi$$

defines semiflow S_a on $X_a = \{\psi \in U_a : \psi'(0) = f_a(\psi)\}$

Stationary point $\sigma \in X_a$

(W_a^s and) W_a^u of S_a at $\sigma \in X_a$

IVP

$$x'(t) = f_a(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi$$

defines semiflow S_a on $X_a = \{\psi \in U_a : \psi'(0) = f_a(\psi)\}$

Stationary point $\sigma \in X_a$

(W_a^s and) $\qquad W_a^u$ of S_a at $\sigma \in X_a$

$W_a^u = w(Y_{a,u,\epsilon})$

$w : Y_{a,u} \supset Y_{a,u,\epsilon} \to B_a^1$ cont diff (F), parametrization

$Y_{a,u,\epsilon}$ small ball, center 0, in $Y_{a,u} = T_\sigma W_a^u$,

IVP

$$x'(t) = f_a(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi$$

defines semiflow S_a on $X_a = \{\psi \in U_a : \psi'(0) = f_a(\psi)\}$

Stationary point $\sigma \in X_a$

(W_a^s and) W_a^u of S_a at $\sigma \in X_a$

$W_a^u = w(Y_{a,u,\epsilon})$

$w : Y_{a,u} \supset Y_{a,u,\epsilon} \to B_a^1$ cont diff (F), parametrization

$Y_{a,u,\epsilon}$ small ball, center 0, in $Y_{a,u} = T_\sigma W_a^u$,

$$W^u =_{def} JW_a^u = Jw(Y_{a,u,\epsilon})$$

IVP

$$x'(t) = f_a(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi$$

defines semiflow S_a on $X_a = \{\psi \in U_a : \psi'(0) = f_a(\psi)\}$

Stationary point $\sigma \in X_a$

(W_a^s and) W_a^u of S_a at $\sigma \in X_a$

$W_a^u = w(Y_{a,u,\epsilon})$

$w : Y_{a,u} \supset Y_{a,u,\epsilon} \to B_a^1$ cont diff (F), parametrization

$Y_{a,u,\epsilon}$ small ball, center 0, in $Y_{a,u} = T_\sigma W_a^u$,

$$W^u =_{def} JW_a^u = Jw(Y_{a,u,\epsilon})$$

C^1-submanifold of C^1 because of

$J \circ Dw(0)$ injective and dim $Y_{u,a} < \infty$.

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\Longrightarrow$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\implies$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Properties of W^u :

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\Longrightarrow$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Properties of W^u :

$W^u \subset X$

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\Longrightarrow$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Properties of W^u :

$W^u \subset X$

$T_\sigma W^u =$ the unstable space (?!) of linearization of S at $\sigma \in X$

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\Longrightarrow$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Properties of W^u :

$W^u \subset X$

$T_\sigma W^u =$ the unstable space (?!) of linearization of S at $\sigma \in X$

Each $\phi \in W^u$ solution of $x'(t) = f(x_t)$ on $(-\infty, 0]$ with

$\phi(t) \to \sigma(0) \quad (t \to -\infty)$ exponentially

Proposition: $j : E \supset \mathcal{O} \to G$ cont diff (MB), dim $E < \infty$,

$x \in \mathcal{O}$, $Dj(x)$ injective

$\Longrightarrow$ for some open neighbourhood $\mathcal{O}_x$ of x in E,

$j(\mathcal{O}_x)$ C^1-submanifold of G, same dimension, ...

Properties of W^u :

$W^u \subset X$

$T_\sigma W^u =$ the unstable space (?!) of linearization of S at $\sigma \in X$

Each $\phi \in W^u$ solution of $x'(t) = f(x_t)$ on $(-\infty, 0]$ with

$\phi(t) \to \sigma(0) \quad (t \to -\infty)$ exponentially

Solutions of $x'(t) = f(x_t)$ on $(-\infty, 0]$ with

$x(t) \to \sigma(0) \quad (t \to -\infty)$ exponentially

have $x_t \in W^u$ for t sufficiently close to $-\infty$

Center manifolds for S at $\sigma \in X$ — by a third approach

Center manifolds for S at $\sigma \in X$ — by a third approach

Thank you !

Bastiani, A., *Applications différentiables et varietés de dimension infinie*. J. Anal. Math. 13 (1964), 1-114.

Diekmann, O., van Gils, S. A., Verduyn Lunel, S. M., and H. O. Walther, *Delay Equations: Functional-, Complex- and Nonlinear Analysis*. Springer, New York, 1995.

Glöckner, H., *Implicit functions from topological vector spaces to Banach spaces*. Israel J. Math. 155 (2006), 205-252.

Glöckner, H., *Finite order differentiability properties, fixed points and implicit functions over valued fields*. arxiv.org/pdf/math/0511218, preprint, 59 pp, 2007.

Hale, J. K., and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*. Springer, New York 1993.

Hamilton, R. S., *The inverse function theorem of Nash and Moser*. Bull. (New Series) A. M. S. 7 (1982), 65-222.

Hartung, F., Krisztin, T., Walther, H. O., and J. Wu, *Functional differential equations with state-dependent delays: Theory and applications*. In *HANDBOOK OF DIFFERENTIAL EQUATIONS,* *Ordinary Differential Equations*, vol. 3, Canada, A., Drabek., P. and A. Fonda eds., pp. 435-545, Elsevier Science B. V., North Holland, 2006.

Hino, Y., Murakami, S., and T. Naito, *Functional Differential Equations with Infinite Delay*. Lecture Notes in Math., vol. 1473, Springer, Berlin, 1991.

Mallet-Paret, J., Nussbaum, R. D., and P. Paraskevopoulos, *Periodic solutions for functional differential equations with multiple state-dependent time lags.* Topological Methods Nonlinear Anal. 3 (1994) 101-162.

Michel, A.D., *Differential calculus in linear topological spaces*. Proc. Nat. Acad. Sci. 24 (1938), 340-342.

Sengadir, T., *Semigroups on Fréchet spaces and equations with infinite delay.* Proc. Indian. Acad. Sci (Math. Sci.) 117 (2007), 71-84.

Walther, H. O., *The solution manifold and* C^1*-smoothness of solution operators for differential equations with state dependent delay*. J. Diff. Eqs. 195 (2003), 46-65.

Walther, H. O., *Smoothness properties of semiflows for differential equations with state dependent delay*. Russian, in *Proceedings of the International Conference on Differential and Functional Differential Equations, Moscow, 2002*, vol. 1, pp. 40-55, Moscow State Aviation Institute (MAI), Moscow 2003. English version: J. Math. Sciences 124 (2004), 5193-5207.

Walther, H. O., *Differential equations with locally bounded delay.* J. Diff. Eqs. 252 (2012), 3001-3039.

Walther, H. O., *Evolution systems for differential equations with variable time lags.* (Russian), in *Proceedings of the Sixth International Conference on Differential and Functional Differential Equations, Moscow, August 14-21, 2011*, Part 4, Contemporary Mathematics. Fundamental Directions 48 (2013), 5-26. English version: DOI 10.1007/s10958-014-2086-6, J. Math. Sciences 202 (2014), 911-933.

Walther, H. O., *Semiflows for differential equations with locally bounded delay on solution manifolds in the space* $C^1((-\infty,0],\mathbb{R}^n)$. Topological Methods Nonlinear Anal., to appear.

Walther, H. O., *Local invariant manifolds for delay differential equations with state space in* $C^1((-\infty,0],\mathbb{R}^n)$. Preprint, 2016.