Differential equations with queueing delays

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Joint work with István Balázs

Key elements of queueing systems:

User/Customer: refers to anything that arrives at a facility and requires service, e.g., people, machines, trucks, emails, packets, frames.

Server: refers to any resource that provides the requested service, e.g., repairpersons, machines, runways at airport, host, switch, router, disk drive, algorithm.

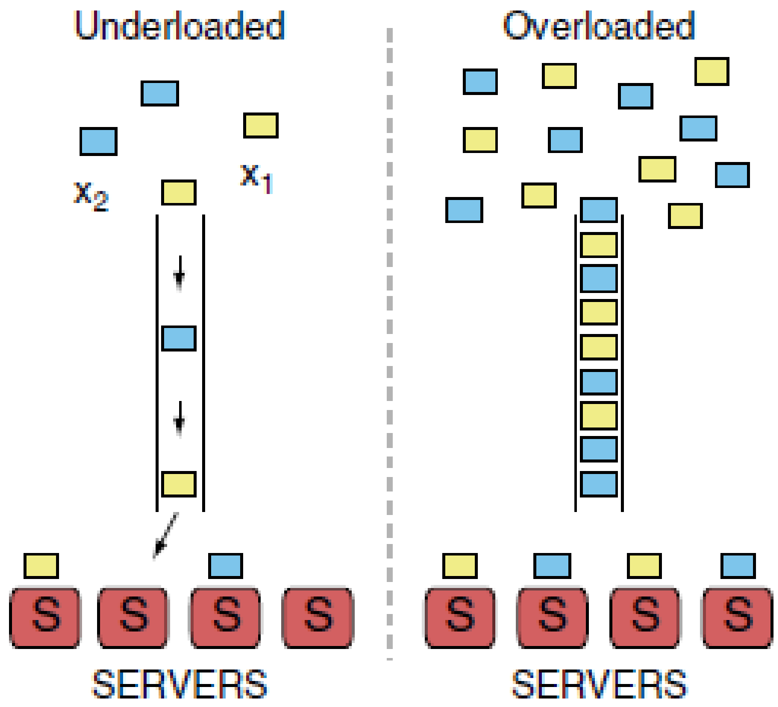
System	Customers	Server
Reception desk	People	Receptionist
Hospital	Patients	Nurses
Airport	Airplanes	Runway
Production line	Cases	Case-packer
Road network	Cars	Traffic light
Grocery	Shoppers	Checkout station
Computer	Jobs	CPU, disk, CD
Network	Packets	Router

Queueing up for enzymatic processing: correlated signaling through coupled degradation. Molecular Systems Biology 7, 2011

Natalie A Cookson, William H Mather, Tal Danino, Octavio Mondrago´n-Palomino, Ruth J Williams, Lev S Tsimring, Jeff Hasty

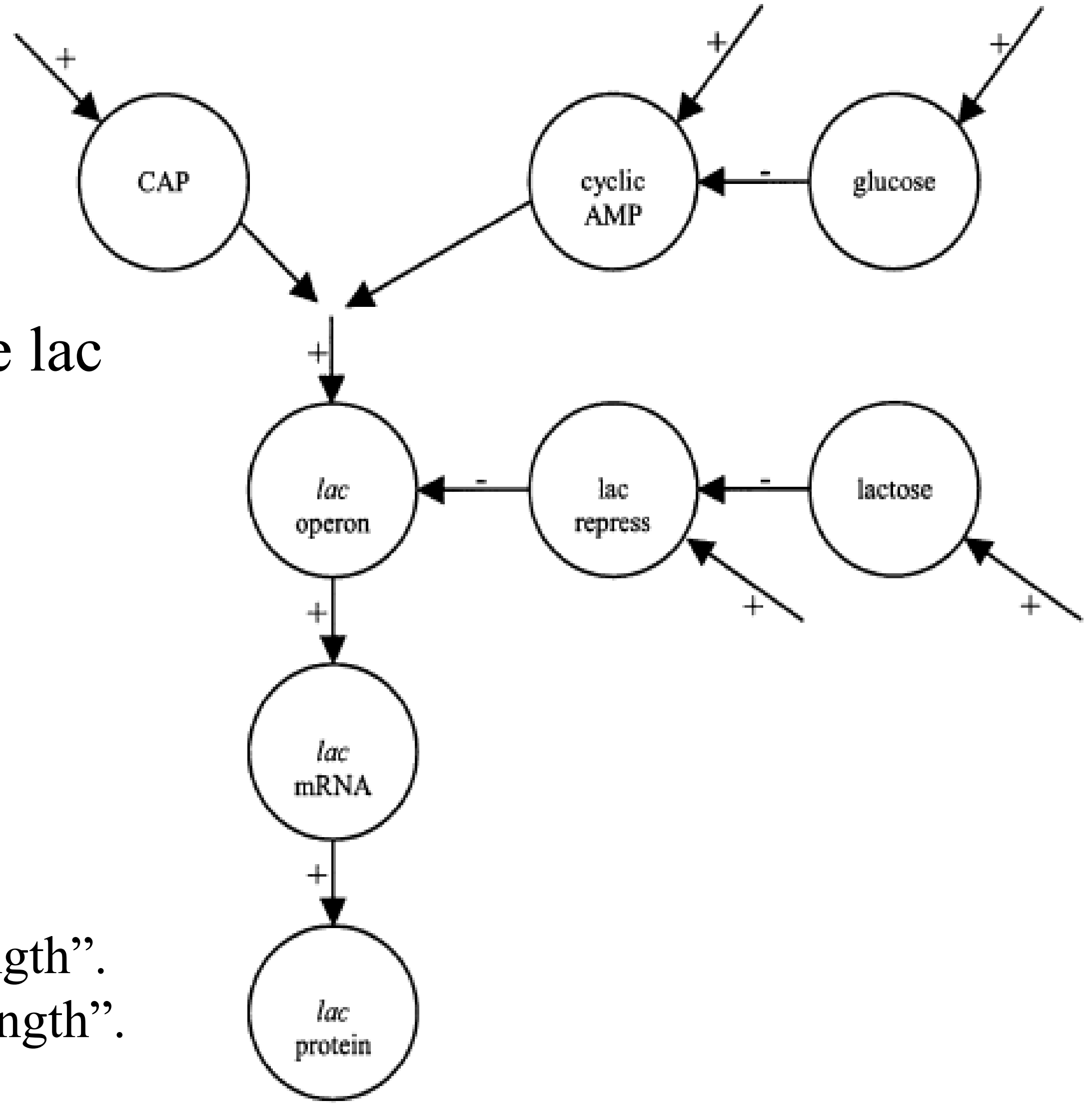
2 proteins: X1, X2

Enzymatic servers: S



Genetic networks and queueing

Arnon Arazia, Eshel Ben-Jacobb, Uri Yechialia, Bridging genetic networks and queueing theory. Physica A 332 (2004) 585 – 616



The regulatory circuit of the lac operon

Circles: represents a biological elements ("queues")

Arrows: possible transitions of "customers" between "queues"

+ sign: increase in the "queue length".
– sign: decrease in the "queue length".

For the sake of definiteness:
Consider a computer network with one user and one server

The user sends data to the server for procession

A waiting line (queue) is formed by the incoming data

$X(t)$ — rate of data coming to the server
$c > 0$ — capacity of the server

Suppose the unit of data, which leaves the server at time t, arrived at the waiting line at time $t - \tau(t)$

At time $t - \tau(t)$ there was a queue with length $y(t - \tau(t))$

The waiting time: $\nu(t) = (1/c)y(t - \tau(t))$

The procession time is 1

$\tau(t) = 1 + \nu(t)$ and $\tau(t) = 1 + (1/c)y(t - \tau(t))$

$q > 0$ — upper bound for the lenght of the queue

Equation for the length of the queue:

$$y'(t) = \begin{cases} X(t) - c, & \text{if } 0 < y(t) < q, \\ [X(t) - c]^+, & \text{if } y(t) = 0, \\ [X(t) - c]^-, & \text{if } y(t) = q \end{cases}$$

$$u^+ = \max\{0, u\},\ u^- = \min\{0, u\}$$

Protocol: defines how the data are exchanged in the network

$U(x)$ utility of rate x
$p(x)$ price for the x-th unit of rate

Maximize $V(x) = U(x) - \int_0^x p(y)\,dy$ Maximum at x_*

$x_* \in (0, c)$ — equilibrium rate
$x(t) := X(t) - x_*$

$f(x) = -\kappa x U'(x)$ and $g(x) = \kappa x p(x)$

The protocol equation:

$$x'(t) = -f(x(t)) - g(x(t-1-\nu(t))) \tag{1.1}$$

or

$$x'(t) = -f(x(t)) - h(\tau(t))$$

An implicit algebraic relation between ν and y:

$$\nu(t) = (1/c)y(t-1-\nu(t)), \tag{1.2}$$

Equation for the length of the queue:

$$y'(t) = \begin{cases} x(t) - d, & \text{if } 0 < y(t) < q, \\ [x(t) - d]^+, & \text{if } y(t) = 0, \\ [x(t) - d]^-, & \text{if } y(t) = q, \end{cases} \tag{1.3}$$

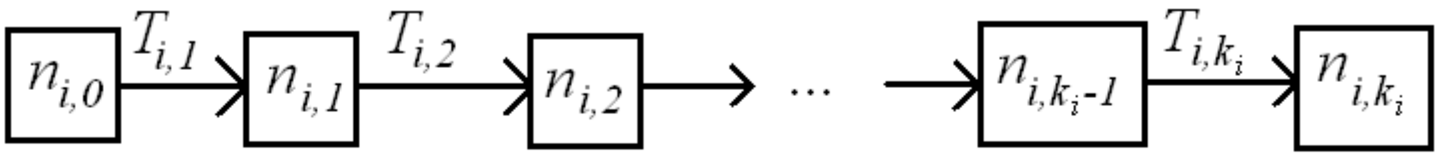
where $d = c - x_*$

The general model

A network (directed graph): nodes $\mathcal{N}$ and links $\mathcal{L}$

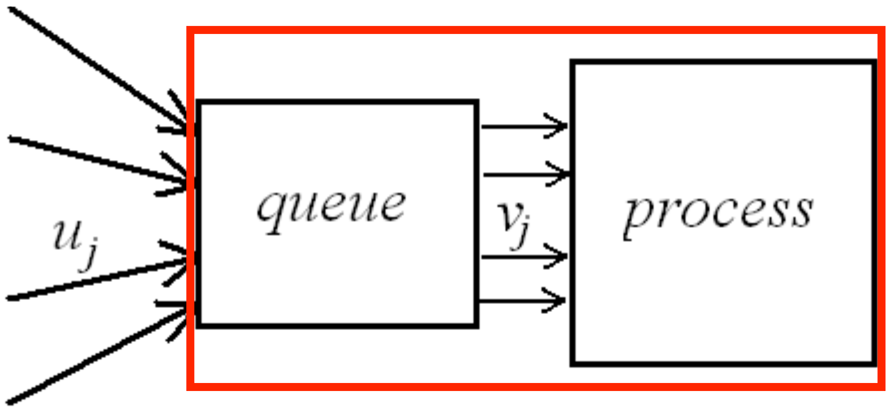
$\mathcal{M}$ is the set of users

User i sends data along the path $\quad R_i = \{n_{i,0}, n_{i,1}, \ldots, n_{i,k_i}\}$



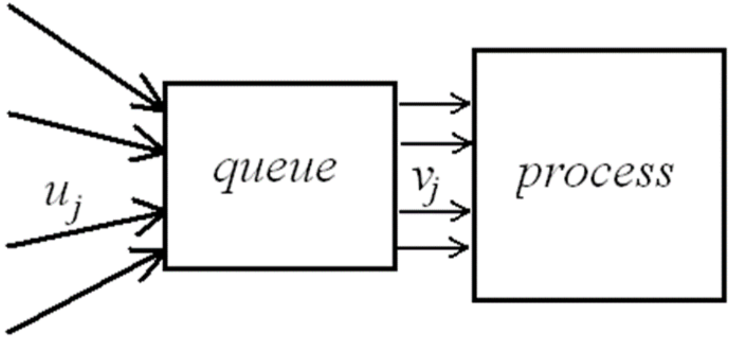
$T_{i,1}, \ldots, T_{i,k_i}$ transfer delays

Several users can send data to the same server



A given unit of data served at time t arrived at the server

$S + \nu_n(t)$ time earlier



$S_n > 0$ procession time (same for all users at a given node/server)

$\nu_n(t) \geq 0$ waiting time

$$\nu_n(t) = \frac{1}{c_n} y(t - \nu_n(t) - S_n)$$

$y_n(\cdot)$ length of queue at server n

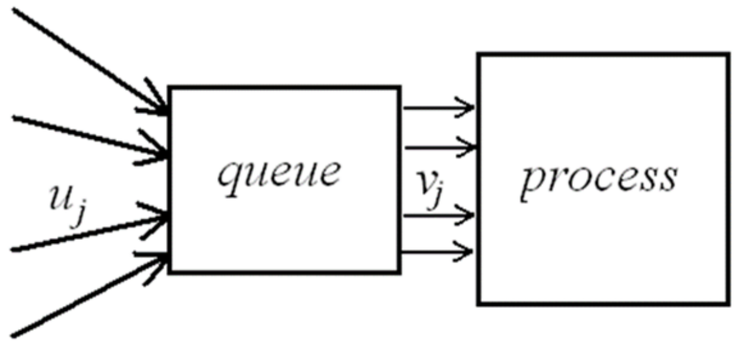
Assume $\nu_n(t) > 0$

The data arriving at server n form a queue at time $t - S_n - \nu_n(t)$

FIFO: first in, first out

The rate u_i of data sent by user i slows down by the rule

$$u_i(t) = c_n \frac{u_i(t - \nu_n(t) - S_n)}{\sum_{j=1}^{J} u_j(t - \nu_n(t) - S_n)}$$



The capacity c_n is shared among the users $\{1, 2, \ldots, J\}$ proportionally to their rates.

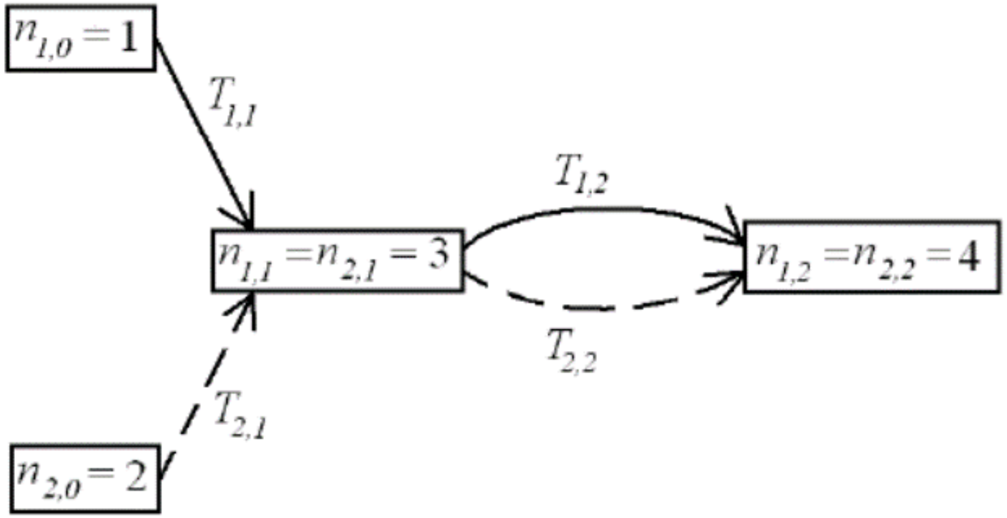
The length y_n of the queue changes as

$$\dot{y}_n(s) = \begin{cases} \sum_{j=1}^{J} u_j(s) - c_n, & \text{if } 0 < y_n(s) < q_n, \\ [\sum_{j=1}^{J} u_j(s) - c_n]^+, & \text{if } y_n(s) = 0, \\ [\sum_{j=1}^{J} u_j(s) - c_n]^-, & \text{if } y_n(s) = q_n \end{cases}$$

Protocol equation

$$\dot{x}_i(t) = F_i\left((x_i)_t\,, \tau_i(t)\right)$$

$\tau_i(t)$: data finished the whole process at time t was sent at time $t - \tau_i(t)$



The protocol equation:

$$x'(t) = -f(x(t)) - g(x(t-1-\nu(t))) \tag{1.1}$$

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Hypotheses:

(H1) a, b, c, d, q positive reals, $d < b$ and $a + d < c$;

(H2) $f : [-a, b] \to \mathbb{R}$ and $g : [-a, b] \to \mathbb{R}$ are C^1-smooth, $xf(x) > 0$ and $xg(x) > 0$ for $x \in [-a, b] \setminus \{0\}$;

(H3) $g'(0) > f'(0)$;

(H4) $g([-a, b]) \subset (-f(b), -f(-a))$.

$r = 1 + q/c$

$C = C([-r, 0], \mathbb{R}),\ ||\phi|| = \max_{-r \leq s \leq 0} |\phi(s)|,\ \phi \in C$

$R := \max_{x,y \in [-a,b]} (|f(x)| + |g(y)|)$

$$C_R = \left\{ \phi \in C([-r, 0], [-a, b]) : \sup_{-r - \leq s < t \leq 0} \frac{|\phi(t) - \phi(s)|}{t - s} \leq R \right\}$$

C_R is a compact subset of C

If $u : (t_0, t_1) \to \mathbb{R}$ continuous, $t - r, t \in (t_0, t_1)$, then

$$u_t \in C, \qquad u_t(s) = u(t + s),\ (-r \leq s \leq 0)$$

Phase space of system (1.1),(1.2),(1.3):

$$X := C_R \times [0, q/c] \subset C \times \mathbb{R}$$

Initial value problem and solutions for (1.1),(1.2),(1.3):

For $(\phi, \delta) \in X$, $\omega \in (0, \infty]$, the pair of functions

$$x : [-r, \omega) \to \mathbb{R} \text{ and } \nu : [0, \omega) \to \mathbb{R}$$

is called a *solution* of (1.1),(1.2),(1.3) *with initial condition*

$$x_0 = \phi, \quad \nu(0) = \delta$$

if

(i) x and ν are continuous, x is differentiable on $(0, \omega)$;

(ii) $x([-r, \omega)) \subset [-a, b]$, $\nu([0, \omega)) \subset [0, q/c]$;

(iii) equation (1.1) holds for all $t \in (0, \omega)$;

(iv) there exists an absolutely continuous function $y : [-1-\delta, \omega) \to [0, q]$ such that equation (1.3) holds almost everywhere in $[-1-\delta, \omega)$;

(v) equation (1.2) holds for all $t \in [0, \omega)$.

How to solve (1.1),(1.2),(1.3)?

$(\phi, \delta) \in X$ given

Step 1: $\nu(0) = \delta \Rightarrow y(-1-\delta) = c\nu(0) = c\delta$

Step 2: Solve the problem

$$\begin{aligned} y'(t) &= \begin{cases} \phi(t) - d, & \text{if } 0 < y(t) < q, \\ [\phi(t) - d]^+, & \text{if } y(t) = 0, \\ [\phi(t) - d]^-, & \text{if } y(t) = q \end{cases} \\ y(-1-\delta) &= c\delta \end{aligned}$$

on the interval $[-1-\delta, 0]$

Step 3: $\exists$ a unique $\nu : [0,1] \to [0, r-1]$ so that

$$\nu(t) = \frac{1}{c} y(t - 1 - \nu(t)), \qquad t \in [0,1]$$

and $\quad t \mapsto t - 1 - \nu(t) \quad$ strictly increases

Step 4: Solve

$$\begin{aligned} x'(t) &= -f(x(t)) - g(x(t-1-\nu(t))) \\ &= -f(x(t)) - g(\phi(t-1-\nu(t))) \\ &= -f(x(t)) - k(t), \quad t \in [0,1] \\ x(0) &= \phi(0) \end{aligned}$$

This gives a solution on $[0, 1]$.

Repeat the process to get solution on $[1, 2]$, $[2, 3]$, ...

Theorem 1. *If hypotheses (H1)–(H4) hold then for each* $(\phi, \delta) \in X$ *there exists a unique pair of functions*

$$x^{\phi,\delta} : [-r, \infty) \to \mathbb{R} \text{ and } \nu^{\phi,\delta} : [0, \infty) \to \mathbb{R}$$

such that x *and* ν *is a solution of (1.1),(1.2),(1.3) satisfying the initial condition*

$$x_0^{\phi,\delta} = \phi, \quad \nu^{\phi,\delta}(0) = \delta.$$

The mapping

$$\Phi : [0, \infty) \times X \times (0, \infty) \ni (t, \phi, \delta,) \mapsto (x_t^{\phi,\delta}, \nu^{\phi,\delta}(t)) \in X$$

defines a Lipschitz continuous semiflow on X*.*

Slowly oscillating periodic solutions

J. Mallet-Paret and R. Nussbaum, Arch. Rational Mech. Anal. 120 (1992), 99-146.

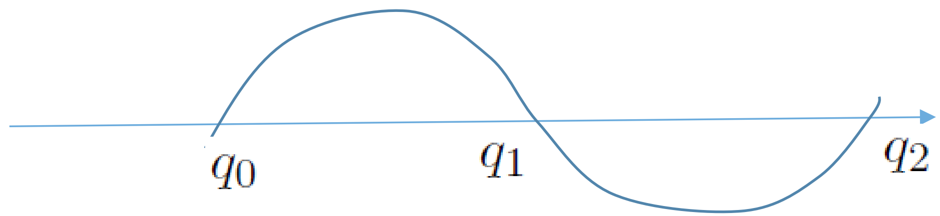
O. Arino, K. P. Hadeler, and M. L. Hbid, J. Differential Equations 144 (1998), 263-301.

P. Magal, O. Arino, J. Differential Equations 165 (2000), 61–95.

H.-O. Walther, J. Differential Equations 244 (2008), 1910–1945.

x SOP (slowly oscillatory periodic) if there exist $q_0 < q_1 < q_2$ so that $q_1 > q_0 + 1$, $q_2 > q_1 + 1$, $x(q_0) = x(q_1) = x(q_2) = 0$, $x(t) <$ on (q_0, q_1), $x(t) > 0$ on (q_1, q_2), and

$$x(t + q_2 - q_0) = x(t).$$



$$X_\alpha = \Big\{(\phi, \delta) \in C_R \times [0, r-1] : \phi(0) = 0,\ 0 \leq \phi(s) \leq b,\ -r \leq s \leq 0,$$
$$\int_{-1}^{0} \phi(s)\, ds \geq \alpha(\delta)\Big\}$$

$$\alpha : [0, r-1] \to \mathbb{R} \text{ convex, } \alpha(0) = 0$$

$$U_\alpha = \{(\phi, \delta) \in X_\alpha : \phi(s_0) > 0 \text{ for some } s_0 \in [-1, 0]\}$$

$(\phi, \delta) \in U_\alpha$, the corresponding solution is $(x_t, \nu(t))$

Define

$$\zeta_0 = \sup\{t : \; x(s) = 0, \text{ for } 0 \leq s \leq t\}.$$

Let

$$\zeta_1 = \inf\{t > \zeta_0 : \; x(t) = 0\}$$

and $\zeta_1 = \infty$ if $x(t) < 0$ for all $t > \zeta_0$.

If ζ_k is finite, define

$$\zeta_{k+1} = \inf\{t > \zeta_k : \; x(t) = 0\}$$

and $\zeta_{k+1} = \infty$ if x has no zeros on $(\zeta_k, \infty))$.

Define the Poincaré return map

$$P : X_\alpha \to C_R \times [0, r-1]$$

as follows:

$$P(\phi,\delta) = \begin{cases} (0,0), & \text{if } \phi|_{[-1-\delta,0]} \equiv 0 \text{ or } \zeta_2 = \infty \\ (\psi, \nu(\zeta_2)) & \text{if } (\phi,\delta) \in U_\alpha \text{ and } \zeta_2(\phi,\delta,\lambda) < \infty \end{cases}$$

where $\psi(t) = x(\zeta_2 + t)$, $t \in [-r, 0]$, in case $\zeta_2 - \zeta_1 \geq r$, and

$$\psi(t) = \begin{cases} x(\zeta_2 + t), & \text{for } t \in [\zeta_1 - \zeta_2, 0] \\ 0 & \text{for } t \in [-r, \zeta_1 - \zeta_2] \end{cases}$$

in case $\zeta_2 - \zeta_1 < r$.

If $(\phi,\delta) \in U_\alpha$ and $P(\phi,\delta) = (\phi,\delta)$, then the corresponding solution $x = x^{\phi,\delta}$ extends to an SOP solution of period $\zeta_2(\phi,\delta)$.

The continuity of $P : X_\alpha \to C_R \times [0, r-1]$

(ϕ, δ) and $(\phi_k, \delta_k)_{k=0}^{\infty}$ in X_α with $(\phi_k, \delta_k) \to (\phi, \delta)$

Case 1: $(\phi, \delta) \in U_\alpha$ and $\zeta_2(\phi, \delta) < \infty$

The continuity of the semiflow Φ implies

$$\zeta_1(\phi_k, \delta_k) \to \zeta_1(\phi, \delta), \ \zeta_2(\phi_k, \delta_k) \to \zeta_2(\phi, \delta)$$

and

$$P(\phi_k, \delta_k) \to P(\phi, \delta)$$

Case 2: $(\phi, \delta) \in U_\alpha$ and $\zeta_2(\phi, \delta) = \infty$

Then $\zeta_2(\phi_k, \delta_k) \to \infty$

Lemma. $\forall \, \epsilon > 0 \ \exists \ T > 0$: $\quad \zeta_2(\psi, \gamma) > T \Longrightarrow ||P(\psi, \gamma)|| < \epsilon.$

Hence $P(\phi_k, \delta_k) \to (0, 0) = P(\phi, \delta)$

Case 3: $(\phi, \delta) \in X_\alpha \setminus U_\alpha$

Then $\delta = 0$ and $x(t) = 0$ for $t \in [-1, \infty)$.

Lemma and continuity of Φ imply $P(\phi_k, \delta_k) \to (0,0) = P(\phi, \delta)$.

Nontrivial that P maps X_α into X_α.

Conditions on $\alpha : [0, r-1] \to [0, \infty)$:

$\alpha(0) = 0$

$\alpha'(t) > 0, \;\; \alpha''(t) > 0$ for $t \in (0, r-1)$

For given $l_0 > 0$, $l_1 > 0$ and $r_0 \in (0, r-1)$

$$\alpha(r-1) \leq l_0, \;\; l_1 \alpha^2(t) \geq \alpha(t - r_0) \quad \text{for } t \in [r_0, r-1]$$

$$\alpha(t) = k_0 \exp\left(-k_1 e^{k_2/t}\right) \qquad \text{for suitable } k_j > 0.$$

Fixed point index

An algebraic count of the number of fixed points of P in X_α

Properties: additivity, homotopy, normalization

$ind_X(P, X_\alpha) = 1 \Rightarrow$ there is a fixed pont

$ind_X(P, X_\alpha) = 1$ from the normalization property

There is a fixed point. $(0,0)$ is a fixed point.

We want nontrivial fixed points.

Assume that there is an open nbhd of $(0,0)$ such that

$$ind_X(P, U) = 0$$

By the addidivity of the index $\quad ind_X(P, X_\alpha \setminus \overline{U}) = 1$

and there is a fixed point in $X_\alpha \setminus \overline{U}$.

Construction of U with $ind_X(P,U)=0$

It suffices to show that $(0,0)$ is ejective:
$(\phi,\delta)\in U\setminus\{(0,0)\}\Rightarrow\exists$ an integer $k\geq 1$ with $P^k((\phi,\delta))\notin U$

Homotopy to the constant delay case and to an equation with monotone $\hat{f}$ and $\hat{g}$.

$$x'(t)=-sf(x(t))-(1-s)\hat{f}(x(t))-sg(x(t-1-s\nu(t)))-(1-s)\hat{g}(x(t-1-s\nu(t)))$$

$s\in[0,1]$ homotopy parameter

Lemma. $\exists\ \rho>0$ *independently of* $s\in[0,1]$ *so that there is NO* $(\phi,\delta)\in U_\alpha\cap B_\rho$ *with*

$$P(\phi,\delta)=(\phi,\delta).$$

The homotopy property implies

$$ind_X(P_1, B_\rho) = ind_X(P_0, B_\rho)$$

Ejectivity of the fixed point $(0,0)$ of the Poincare map P_0 follows by the Poincare–Bendixson type result of Mallet-Paret and Sell (1996).

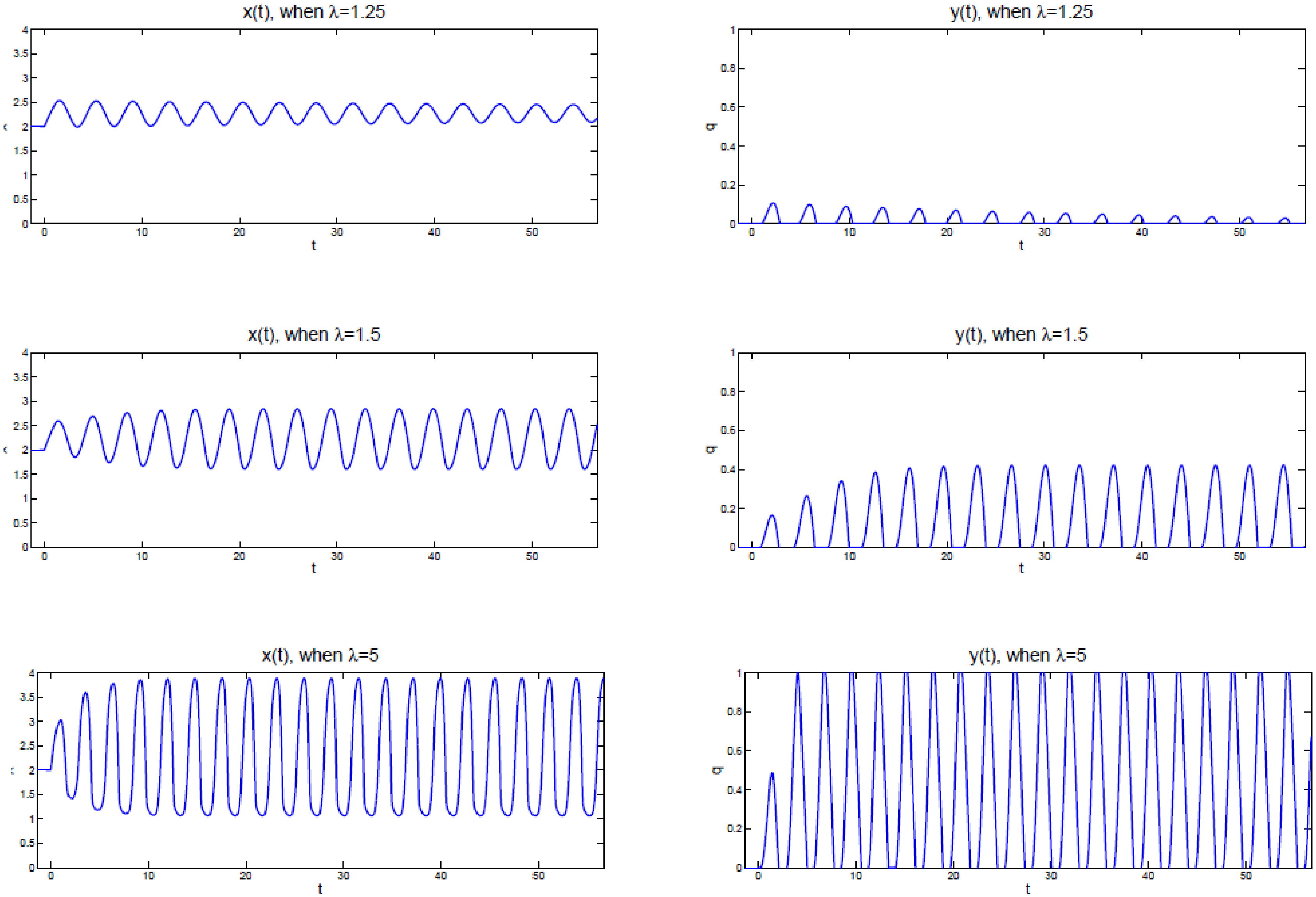
$ind_X(P_0, B_\rho) = 0$ by the ejectivity of $(0,0)$ and

by a result of Nussbaum (1975)

Theorem 2. *If hypotheses (H1)-(H4) hold, and the zero solution of*

$$u'(t) = -f'(0)u(t) - g'(0)u(t-1)$$

is unstable, then system (1.1),(1.2),(1.3) has a slowly oscillatory periodic solution.



x(t), when λ=1.25
y(t), when λ=1.25
x(t), when λ=1.5
y(t), when λ=1.5
x(t), when λ=5
y(t), when λ=5
t

Thank you!