Perturbative techniques of the dynamics in the C^1-topology

Sylvain Crovisier and Nicolas Gourmelon

Dynamics of Evolution Equations

Luminy, March 21-25th, 2016

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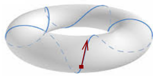
- geometric systems (geodesic flow on $M = T^1 N$).



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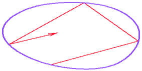

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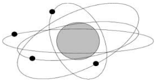

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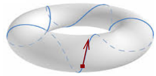

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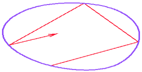

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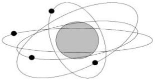

▶ *We will mainly discuss the case of diffeomorphisms.*

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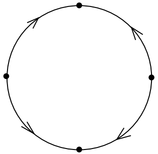


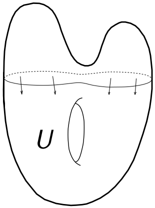
Example. (Peixoto) In dimension 1, there exists a dense open set of diffeomorphisms whose dynamics is Morse-Smale, i.e. supported on finitely many periodic orbits.

Qualitative dynamics

Topological properties:

– *Transitivity* : does there exists a dense orbit?

– *Attractors*: find open sets U such that $f(\overline{U}) \subset U$.



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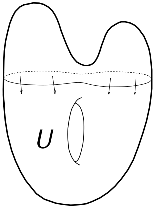

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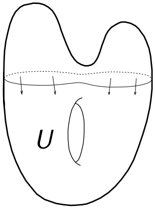

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- This lecture addresses the perturbations of the orbit structure.

- The second lecture will consider more quantitative properties: $Per(f)$ and the linear dynamics above this set will be a skeleton for study further dynamical properties.

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Proof. – One can "stabilizes" the periodic points
$\Rightarrow$ $Perf(f)$ can not shrink by perturbation (semi-continuity).

– Baire argument: $f \mapsto Per(f)$ is continuous on a dense G_δ set.

– If $Per(f)$ is smaller than the set of recurrent points (when $f \in \mathcal{G}$), the closing property allows to increase $Per(f)$ (contradicts the continuity).

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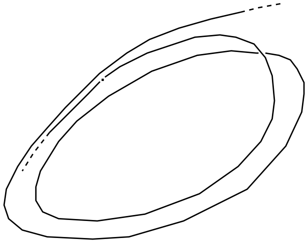

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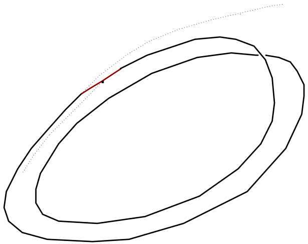

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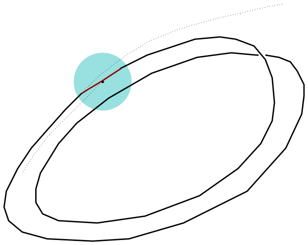
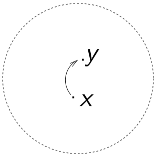


An elementary perturbation lemma

Elementary perturbation. For any x, y close and $\varepsilon > 0$, there exists a diffeomorphism φ such that $\varphi(x) = y$ and which:

– is ε-close to the identity for the C^k-topology,

– coincides with the identity outside a ball with radius



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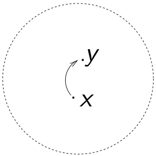


$$R \sim \varepsilon^{-1/k} d(x, y)^{1/k}.$$

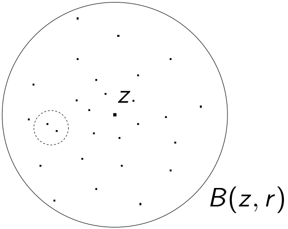
- The perturbation is localized at a point.
- The support is 'huge' for high topologies.
- For $k = 1$, the ratio $R/d(x, y)$ is fixed when $\varepsilon > 0$ is fixed.

The C^1-closing lemma: the case $\varepsilon > 4/3$

Selection lemma. *For any forward recurrent point z, and $r > 0$, there exist two iterates $p, q = f^n(p)$ in the forward orbit of p such that the ball*

$$\widehat{B} := B\left(\frac{p+q}{2}, \frac{3}{2}\frac{d(p,q)}{2}\right)$$

is contained in $B(z,r)$ and is disjoint from $f(p), \ldots, f^{n-1}(p)$.



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- **Closing by perturbation of size $\sim 4/3$.**

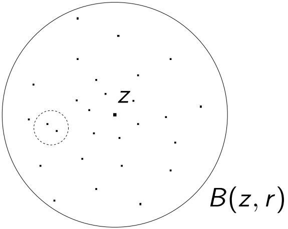
There exists g such that:

– g coincides with f outside $\widehat{B}$

– $d(g, f)_{C^1} \sim 4/3$

– $g(f^{-1}(q)) = p$.

$\Rightarrow$ p is periodic for g.



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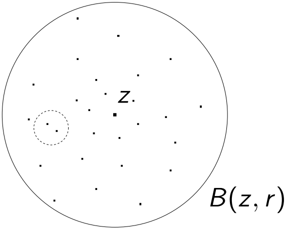
"Proof".

– Consider all the returns in $B(z,r)$ $p_1, \dots, p_s$ until a large time.

– Choose p_i, p_j minimizing $\frac{d(p_i,p_j)}{d\left(\frac{p_i+p_j}{2}, \partial B(z,r)\right)}$.



The C^1-closing lemma: the general case

Perturbation lemma.

For any f,

consider p, q and the ball $\widehat{B} := \left\{x, \quad d\left(x, \frac{p+q}{2}\right) < \frac{3}{4}d(p,q)\right\}$.
Then, there exists g such that

– $g \quad (x) = f \quad (y)$,

– $d_{C^1}(f, g) < 4/3$,

– f *and* g *coincide outside* $\widehat{B}$.

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Perturbation lemma. (Pugh)
For any f and $\varepsilon > 0$, there are $N \geq 0$ and a new metric $\tilde{d}$ such that "elementary perturbations in time N exist":

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If $\widehat{B}$ is disjoint from N first iterates, there exists g such that
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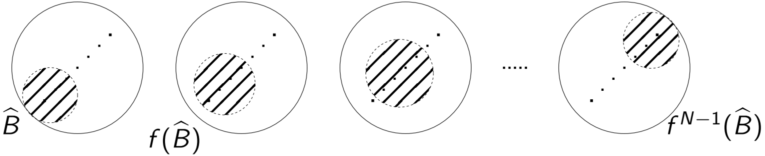
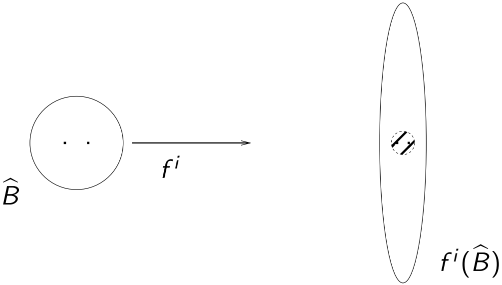


Non-conformality



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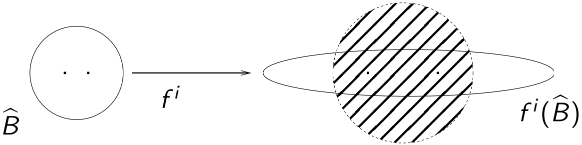

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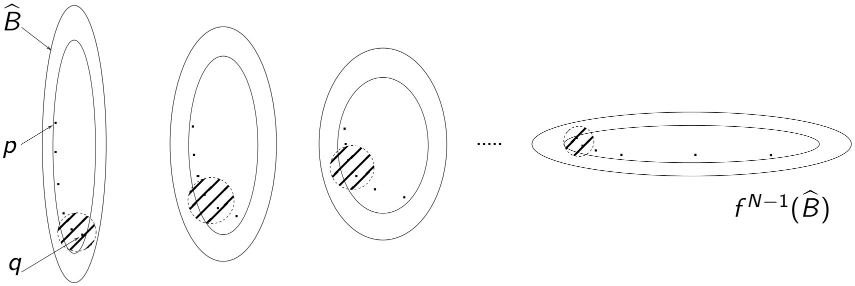

$\widehat{B}$ for the new metric $\widetilde{d}$.

Decomposition of the dynamics

Pseudo-orbits

Definitions. $(x_n)_{n\in\mathbb{Z}}$ is a *ε-pseudo-orbit* if $\forall n \in \mathbb{Z},\ d(x_n, f(x_n)) < \varepsilon$.

x is *chain-recurrent* if it belongs to a periodic ε-pseudo-orbit for any $\varepsilon > 0$.

$x \sim y$ if x, y belong to a same periodic ε-pseudo-orbit for any $\varepsilon > 0$.

- An equivalence relation which defines the *chain-recurrence classes.*

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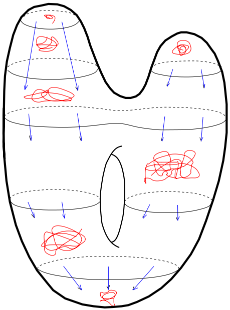
Other characterization. (Conley)
The *chain-recurrence classes are the largest invariant compact sets that cannot be split by attractors.*



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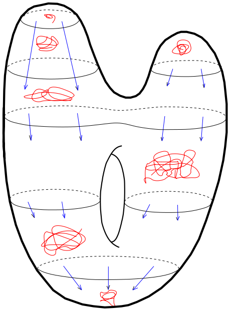
Questions.
– *How many classes?*
– *Which are the isolated classes?*
– *What is the dynamics inside each class?*



Connecting lemma for pseudo-orbits

Theorem. (Bonatti-C.) f with non-degenerated periodic points.

x chain-recurrent $\Rightarrow$ there is g C^1-close to f such that x is periodic.

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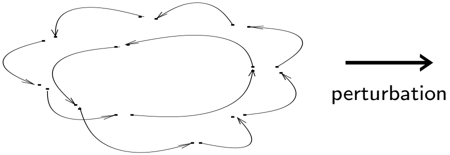

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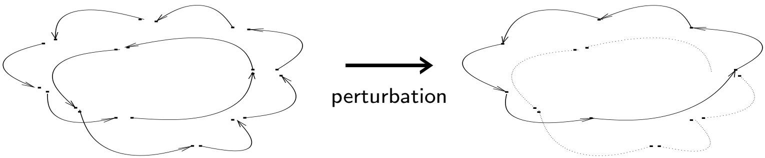

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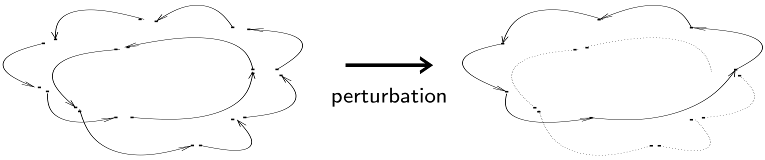

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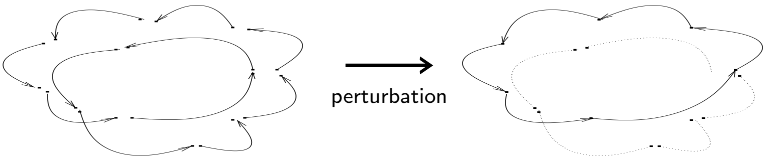

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Consequence 2. *In the space of* **conservative** *C^1-diffeomorphisms, the generic f is transitive.*

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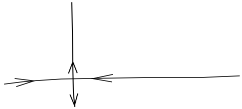
Definition. *A periodic orbit* $O = \{p, f(p), \dots, f^\ell(p) = p\}$ *is* *hyperbolic* *if the eigenvalues of* Df^ℓ *have moduli different from* 1.

- This property is generic.
- There exists stable and unstable manifolds $W^s(O)$, $W^u(O)$.



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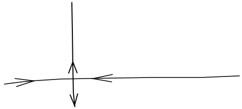

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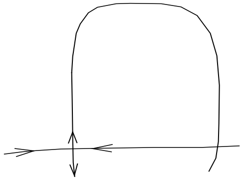

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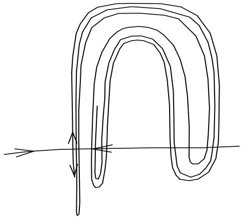

Consequence 3. *For* $f \in \mathrm{Diff}^1(M)$ *generic and for any (hyperbolic) periodic orbit* O*, the transverse intersection points between* $W^s(O)$ *and* $W^u(O)$ *are dense in the chain-recurrence class containing* O.

Connecting lemma for pseudo-orbits

Definition. *A periodic orbit* $O = \{p, f(p), \ldots, f^{\ell}(p) = p\}$ *is* *hyperbolic* *if the eigenvalues of* Df^{ℓ} *have moduli different from* 1.

- This property is generic.
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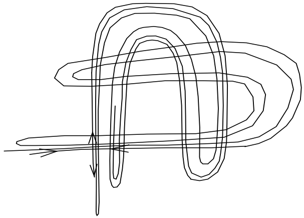

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Homoclinic classes

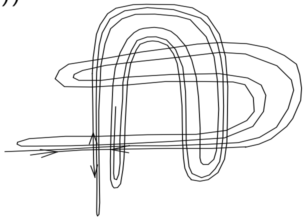
Definition. Two hyperbolic periodic orbits O, O' are *homoclinically related* if $W^u(O) \pitchfork W^s(O') \neq \emptyset$ and $W^u(O') \pitchfork W^s(O) \neq \emptyset$.

Definition. The homoclinic class of O:

$$H(O) := closure(\bigcup\{O' \text{ homoclinically related to } O\})$$

$$= closure(W^s(O) \pitchfork W^u(O)).$$

- The dynamics on $H(O)$ is transitive.
- The homoclinic orbits allow to "mix" the periodic behavior.



The dynamics on the periodic set

Theorem.(B.-C.) *For* $f \in \mathrm{Diff}^1(M)$ *generic, and* O *periodic orbit, the chain-recurrence class and the homoclinic class of* O *coincide.*

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Theorem.(C.) *For* $f \in \mathrm{Diff}^1(M)$ *generic, any chain-recurrence class is limit of a sequence of homoclinic classes (in Hausdorff topology).*

- The union of the homoclinic classes is a "skeleton" for the global dynamics.

Perturbation inside homoclinic classes

Franks lemma. *Fix* $\varepsilon > 0$*, any* f*, and* $O = \{p, \dots, f^\ell(p) = p\}$*.*
For any sequence of linear maps (A_i) *with* $\|A_i - Df(f^i(p))\| < \varepsilon$*,*
there exists g *that is* 2ε*-close to* f *in* $\mathrm{Diff}^1(M)$ *such that*

$$g^i(p) = f^i(p) \text{ and } Dg(g^i(p)) = A_i.$$

- Reduction to linear algebra (see the second lecture)

Beyond the C^1-diffeomorphisms

Other classes of systems

Our perturbative approach was based on:

– the elementary perturbation lemma,

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- the elementary perturbation lemma,
- the Franks lemma.

Huge difficulties to generalize:

- in higher regularity,

- when an additional structure is preserved.

The geodesic flow

N: a Riemannian manifold.
$M = T^1N$ supports the geodesic flow.

C^k-perturbation of the flow on T^1N $\longleftrightarrow$ C^{k+1}-perturbation of the metric on N

Difficulty: *a local perturbation of the metric does not induces a local perturbation of the geodesic flow!*

State of the art in C^1-topology:

– Franks lemma is known (Contreras, Vissher, Lazrag-Rifford-Ruggiero),

– the closing lemma is unknown.

A recent result...

Reeb flows: manifolds with a contact form have a natural vector field.

Example. N Riemannian $\Rightarrow T^1N$ is contact; geodesic flow $=$ Reeb flow.

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Corollary. *N: a surface.*
For the C^∞-generic Riemannian metric, the geodesic flow admits a set of periodic orbits which is dense in N.

- Still unknown in the phase space T^1N...
- Uses spectral invariant for the Embedded Contact Homology.
- No control on the support of the periodic orbits!

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A panorama of the space of C^1-diffeomorphisms

A bifurcating approach

Obstructions to hyperbolicity

Hyperbolic systems: stability, good description.

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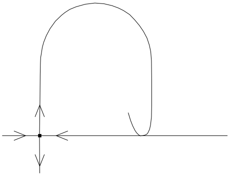
Obstruction 1

Lack of bundle splitting:

homoclinic tangency



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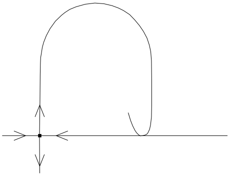
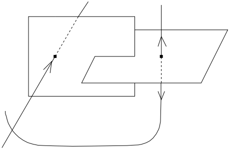


Obstruction 2

Lack of contraction/expansion:

heterodimensional cycle



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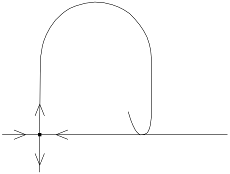

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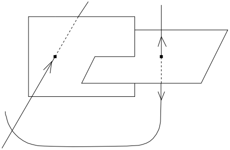

Hyperbolicity conjecture. (Palis)

Any diffeomorphism can be approximated by a diffeomorphism which

– either has a heterodimensional cycle or a homoclinic tangency,

– or is hyperbolic.

Conjectured panorama of $diff^1(M)$