Quasi-periodic solutions with beating effects for the quintic NLS on the circle

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joint work with Emanuele Haus

Dynamics of Evolution Equations - Luminy

The problem

The equation

- We consider the defocusing Schrödinger equation on a torus

$$\begin{cases} -i\partial_t u + \Delta u = |u|^{2p} u, \quad x \in \mathbb{T}^d, \\ u(0,x) = u_0(x) \end{cases} \quad (NLS) \qquad (1)$$

- a very reasonable question is to study the time evolution of the Sobolev norms

$$\int_{\mathbb{T}^2} |\nabla^s u|^2$$

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  $p = 1$ is the cubic NLS, $p = 2$ the quintic...
  you can consider any analytic non-linearity
  we will work only on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ or $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$

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- Two conserved quantities are the Hamiltonian

$$H = \int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{p+1} \int_{\mathbb{T}^2} |u|^{2p+2}$$

and the mass (the L^2 norm)

$$L = \int_{\mathbb{T}^2} |u|^2$$

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Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

- The cubic one-dimensional NLS is completely integrable
  the Sobolev norms are almost preserved
- global well-posedness in $H^s(\mathbb{T}^2)$, $s \geq 1$
  smooth solution for all times from smooth initial data.
- For (defocusing) polynomial nonlinearities of the type $\partial_{\bar{u}} P(|u|^2)$, the Sobolev norms ($s > 1$) grow at most polynomially in time (Bourgain '96, Staffilani '97 and later improvements)

Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

If we pass to the Fourier modes:

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{in\cdot x}$$

the NLS can be seen as an infinite chain of harmonic oscillators coupled by the non-linearity.

$$H = \sum_{n\in\mathbb{Z}^2} |n|^2 |a_n|^2 + \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i = 0}} a_{n_1}\bar a_{n_2}\dots a_{n_{2p+1}}\bar a_{n_{2p+2}}$$

The linear NLS is of completely resonant (namely all the harmonic oscillators have the same linear frequency)

Basic properties: $-i\partial_t u + \Delta u = |u|^{2p} u$

the linear flow:

- preserves the actions $|a_n|^2$
- all the linear solutions are periodic

When we take into account the non-linearity the solutions should be much more complicated,

We EXPECT:

- Existence of periodic,quasi-periodic and almost-periodic solutions.
- Existence of solutions for which the linear actions $|a_n|^2$ are not approximately preserved.
- Solutions which are initially supported only on low Fourier modes and eventually transfer energy to arbitrarily high modes (weak turbulence).

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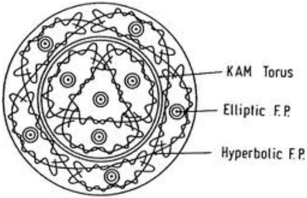
Finite dimensional systems

If we were working on a finite dimensional system there are some well accepted strategies prove existence of:

- Existence of a positive measure set of maximal tori.
- Existence of families of lower dimensional tori
- Existence of orbits which exhibit drift in the action variables



This program requires some non degeneracy conditions which for one single equation might not hold true!
We are not able to produce maximal or codimension one tori!

Our goal: existence of special types of small solutions

1. quasi-periodic solutions (existence and linear stability)

Definition

Quasi-periodic solution of frequency ω: a torus embedding $\mathbb{T}^d \ni \varphi \to u(\varphi, x)$ such that $u(\omega t, x)$ solves the equation

look for solutions where $\omega_i = |j_i|^2 + O(\varepsilon)$, i.e. with frequencies close to the linear ones.

Our goal: existence of special types of small solutions

1. quasi-periodic solutions (existence and linear stability) with frequencies close to the linear ones.

2. solutions whose Sobolev norms grow in time by a finite but arbitrary factor

Energy cascade solutions

Given $s > 1$, $K \gg 1$ and $\delta \ll 1$ we look for a solution $u(t) = u(t, \cdot)$ of (NLS) such that

$$\|u(0)\|_{H^s} \leq \delta \quad \text{and} \quad \|u(T)\|_{H^s} \geq K.$$

Our goal: existence of special types of small solutions

1. quasi-periodic solutions (existence and linear stability) with frequencies close to the linear ones.

2. solutions whose Sobolev norms grow in time by a finite but arbitrary factor

3. quasi-periodic solutions whose Sobolev norm grows by some finite factor with frequency not close to the linear ones (secondary tori)

Two dual results:

1. quasi periodic solutions

Theorem (Procesi-M.P. 15)

There exist frequencies $S = \{\mathtt{j}_1, \dots, \mathtt{j}_d\} \subset \mathbb{Z}^2$, and a compact set $\mathcal{C}_\varepsilon$ of initial data (contained in $\varepsilon/2 \le \xi_i \le \varepsilon\}$ of positive measure) parametrizing bijectively a set of analytic quasi–periodic solutions of NLS of the type:

$$u(\xi, x, t) = \sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}t(|\mathtt{j}_i|^2 + \omega_i(\xi))} e^{\mathrm{i}\mathtt{j}_i \cdot x} + O(\xi^2)$$

Moreover the quasi-periodic solutions for all $\xi \in \mathcal{C}_\varepsilon$ are reducible KAM tori.

Two dual results:

2. Energy cascade

Theorem (Guardia-Haus-M.P. 15)

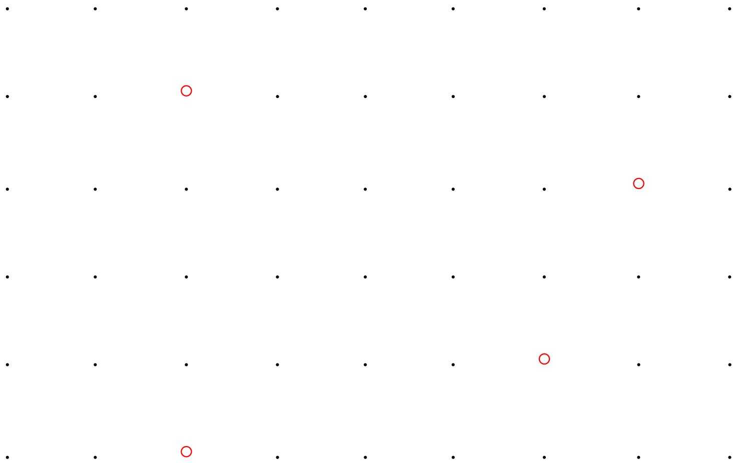
There exist choices of the frequencies $S_1 = \{\mathtt{j}_1, \dots, \mathtt{j}_N\} \subset \mathbb{Z}^2$, $S_2 = \{\mathtt{v}_1, \dots, \mathtt{v}_N\} \subset \mathbb{Z}^2$, there exist initial data ξ and a time T such that:
$u(0,x)$ is mostly localized at S_1
while $u(T,x)$ is mostly localized at S_2.
The sets and initial data can be constructed so that

$$|u(0)|_s \ll \delta\,, \quad |u(T)|_s \gg K$$

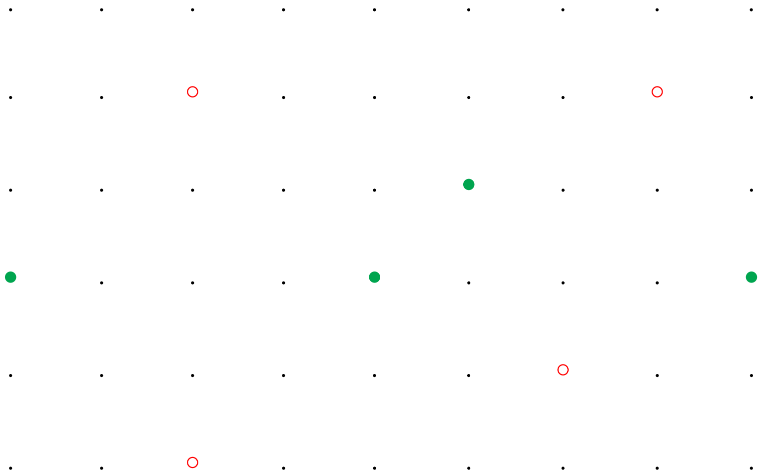
We can also give estimates on the time

$$T \leq \left(\frac{K}{\delta}\right)^{16p\frac{K}{\delta}\ln\left(\frac{K}{\delta}\right)}$$

EXAMPLE: S is given by 4 points marked ○



EXAMPLE: S_1 is the 4 points marked ○, S_2 is the 4 points marked ●



Summarizing

We choose a finite Fourier support S_1 (through a complicated procedure) then:

- For all initial data (essentially) supported on S_1 with amplitude in some Cantor set one has quasi-periodic solutions
- There exists initial data (essentially) supported on S_1 which drift over long but finite times to another set S_2.

Question:

Can we construct solutions which transfer energy from S_1 to S_2 quasi-periodically ?

The answer is yes for very simple choices of S_1, S_2.

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beating solutions for $\mathrm{i}u_t - u_{xx} + |u|^4 u$

Consider the quintic NLS on the circle. Fix $S_1 = \{-2, 1\}$ and $S_2 = \{-1, 2\}$.

Theorem (Haus,P.)

There exists a Cantor set $C_\varepsilon \subset [\varepsilon/2, \varepsilon]^4$ such that for each $\xi \in C_\varepsilon$ there exists a quasi-periodic solution $u(x,t)$ with frequency $\lambda(\xi) = (0,1,1,4) + \omega(\xi)$

- *$u(x,t)$ is approximately supported on $S_1 \cup S_2$ at all times*
- *$u(x,t)$ has a quasi-periodic transfer of the energy between S_1 and S_2.*

Summarizing

Some comments:

- This solutions are all very special.
- The result for the quasi-periodic solutions of first type is quite general (a similar approach can be used for other PDEs or for the NLS on other compact domains) the most model depending feature is the linear stability.
- The result for the energy cascade solutions is very model depending. Already passing from the cubic NLS to the quintic requires some very new strategies.
- We believe that the secondary tori are also a quite general phenomenon, but even generalizing to $\mathbb{T}^2$ is technically very challenging.

literature on quasi-periodic solutions

- *results on the circle* Kuksin, Wayne, Pöschel, Kuksin-Pöschel, Chierchia-You, Craig-Wayne, Bourgain.
- *Existence results on tori*: Bourgain '98,'05, Wang '10-'15, Berti-Bolle '10- '15
- *Existence and stability results on tori*: Geng-You, Eliasson-Kuksin '10, Geng-You-Xu '10, P.-Xu'11, Procesi-P. '11-'15, Eliasson-Grebert-Kuksin '15 .

Growth of Sobolev norms in NLS on $\mathbb{T}^d$

- NLS on $\mathbb{T}^2$:Colliander-Keel-Staffilani-Takaoka-Tao 2010: growth (of a finite but arbitrarily large factor) of Sobolev norms for the two-dimensional cubic equation
  Kaloshin-Guardia 2013: growth of Sobolev norms with control on the time.
  Haus- M.P. , Guardia,Haus,M.P. 2015 non-cubic NLS
- Hani-Pausader-Tzvetkov-Visciglia unbounded Sobolev orbits for the cubic NLS on $\mathbb{T}^2 \times \mathbb{R}$
- Gérard-Grellier on the half-wave equation and the cubic Szegő equation

beating effects

- Grébert and Thomann. and Haus and Thomann beating effects for the quintic NLS on the circle for finite long times.

The Birkhoff Normal Form

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{in\cdot x}$$

- The Hamiltonian in the Fourier coefficients a_n:

$$H = \sum_{n\in\mathbb{Z}^2} |n|^2|a_n|^2 + \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i=0}} a_{n_1}\bar{a}_{n_2}\dots a_{n_{2p+1}}\bar{a}_{n_{2p+2}}$$

- First step of Birkhoff normal form: degree $2p+2$ nonresonant terms are removed from the Hamiltonian

$$H = H^{(2)} + H^{(2p+2)} \longrightarrow H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

The Birkhoff Normal Form

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$$H = H^{(2)} + H^{(2p+2)} \longrightarrow H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

- Resonant terms are terms of degree $2p+2$ which Poisson commute with the quadratic term

$$\sum_{n\in\mathbb{Z}^2} |n|^2 |a_n|^2$$

The Birkhoff Normal Form

- First step of Birkhoff normal form: degree $2p+2$ nonresonant terms are removed from the Hamiltonian

$$H = H^{(2)} + H^{(2p+2)} \longrightarrow H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

- By taking u small enough we can ignore the term $H^{(4p+2)}$ (for finite but long time!) and study the dynamics of

$$H_{Birk} = H^{(2)} + H^{(2p+2)}_{Res}$$

The Birkhoff Normal Form

- First step of Birkhoff normal form: degree $2p+2$ nonresonant terms are removed from the Hamiltonian

$$H = H^{(2)} + H^{(2p+2)} \longrightarrow H^{(2)} + H^{(2p+2)}_{Res} + H^{(4p+2)}$$

- actually since $H^{(2)}$ is a constant of motion we just study

$$H_{Birk} = H^{(2p+2)}_{Res}$$

Resonances

$$H_{Birk} = \frac{1}{p+1} \sum_{\substack{n_1,\ldots,n_{2p+2}\in\mathbb{Z}^2 \\ \sum(-1)^i n_i = 0 \\ \sum(-1)^i |n_i|^2 = 0}} a_{n_1}\bar{a}_{n_2}\ldots a_{n_{2p+1}}\bar{a}_{n_{2p+2}}$$

The sum is restricted to resonances:

$$n_1 - n_2 + n_3 + \cdots + n_{2p+1} - n_{2p+2} = 0\,,$$

$$|n_1|^2 - |n_2|^2 + |n_3|^2 + \cdots + |n_{2p+1}|^2 - |n_{2p+2}|^2 = 0$$

Resonances: the cubic case

in the case of the cubic NLS resonances have a geometric interpretation:

$$n_1 - n_2 + n_3 - n_4 = 0\,, \qquad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$

are rectangles :

$\circ\, \mathtt{n}_1$ $\qquad\qquad$ $\circ\, \mathtt{n}_4$

$\circ\, \mathtt{n}_2$ $\qquad\qquad$ $\circ\, \mathtt{n}_3$

For $p > 1$ there is no simple interpretation, there are resonances also on the circle $S = \{-2, -1, 1, 2\} \subset \mathbb{Z}$

$$n_1 - n_2 + n_3 - n_3 + n_5 - n_6 = 0 \qquad n_1^2 - n_2^2 + n_3^2 - n_3^2 + n_5^2 - n_6^2 = 0$$

$$-2 - 2 + 1 - (-1) + 1 - (-1) = 0 \quad 4 - 4 + 1 - 1 + 1 - 1 = 0$$

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$$-2 \;-\; 2 \;+\; 1 - (-1) + 1 - (-1) = 0 \quad 4 \;-\; 4 + 1 - 1 + 1 - 1 = 0$$

Invariant subspaces

$$H_{Birk} = \frac{1}{p+1} \sum_{\substack{n_1,\dots,n_{2p+2} \in \mathbb{Z}^2 \\ \sum (-1)^i n_i = 0 \\ \sum (-1)^i |n_i|^2 = 0}} a_{n_1} \bar{a}_{n_2} \dots a_{n_{2p+1}} \bar{a}_{n_{2p+2}}$$

Has a complicated dynamics but many invariant subspaces...

We consider a set $\Lambda = \{j_1, \dots, j_m\}$, $j_i \in \mathbb{Z}^2$ and look for invariant subspaces of the form:

$$U_\Lambda := \{a_j = 0\,, \quad \forall j \notin \Lambda\}$$

Invariant subspaces

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Invariant subspaces

There are many invariant subsets U_Λ
if I choose a set $\Lambda = \{j_1, \dots, j_m\}$ generically then U_Λ is invariant and the dynamics on U_Λ preserves all the actions $|a_{j_i}|^2$.

This is the basis for constructing stable and unstable quasi-periodic solutions for the NLS with frequency close to the linear ones

Invariant subspaces: General NLS

Think of $\{j_1, \ldots, j_m\}$ as a point in $(\mathbb{Z}^2)^m \subset (\mathbb{C}^2)^m$

There exists a proper algebraic variety $\mathfrak{A}$ such that for all $\{j_1, \ldots, j_m\} \in (\mathbb{Z}^2)^m \setminus \mathfrak{A}$:

$a_n = 0$ for all $n \neq j_1, \ldots, j_m$ is an invariant subspace on which the actions $|a_j|^2$ are constants of motion.

Invariant subspaces: Cubic NLS

In the cubic case:

$$H_{Birk} = \frac{1}{2} \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 - n_4 = 0 \\ |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0}} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4}$$

first idea: take $\{j_1, \dots, j_m\}$, $j_i \in \mathbb{Z}^2$

$\circ\, j_1$ $\circ\, j_4$

$\circ\, j_2$ $\circ\, j_3$

any three points do not form a right angle

$a_n = 0$ for all $n \neq j_1, \dots, j_m$ is an invariant subspace on which the actions $|a_j|^2$ are constants of motion.

The quasi-periodic solutions of type 1. have support on generic sets

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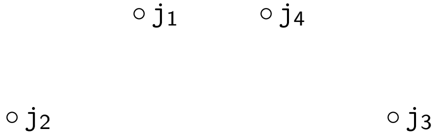

any three points do not form a right angle

$a_n = 0$ for all $n \neq j_1, \dots, j_m$ is an invariant subspace on which the actions $|a_j|^2$ are constants of motion.

the condition is $(j_1 - j_2, j_3 - j_2) = 0$ The quasi-periodic solutions of type 1. have support on generic sets

Invariant subspaces: Cubic NLS

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$\circ\, j_1$ $\quad$ $\circ\, j_4$

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The quasi-periodic solutions of type 1. have support on generic sets

Cubic NLS: Integrable dynamics on a rectangle

Consider a rectangle(this is a codimension 3 manifold in $(\mathbb{C}^2)^4$)

$$\begin{array}{cc} \circ\ \mathrm{j}_1 & \circ\ \mathrm{j}_4 \\ & \\ \circ\ \mathrm{j}_3 & \circ\ \mathrm{j}_2 \end{array}$$

set $a_n = 0$ for all $n \neq j_1, \dots, j_4$

This is an invariant subspace with Hamiltonian

$$h(a) = \frac{1}{4}\sum_{i=1}^{4} |a_{j_i}|^4 - \frac{1}{2}\left(a_{j_1} a_{j_2} \bar{a}_{j_3} \bar{a}_{j_4} + \bar{a}_{j_1} \bar{a}_{j_2} a_{j_3} a_{j_4}\right)$$

and constants of motion

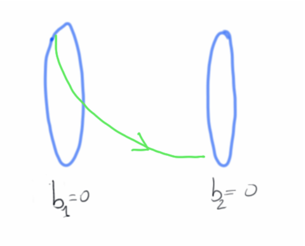
$$J = \sum_{i=1}^{4} |a_{j_i}|^2, \quad J_1 = |a_{j_1}|^2 - |a_{j_2}|^2, \qquad J_3 = |a_{j_3}|^2 - |a_{j_4}|^2$$

Integrable dynamics on a rectangle

$$h(a) = \frac{1}{4}\sum_{i=1}^{4} |a_{j_i}|^4 - \frac{1}{2}\left(a_{j_1} a_{j_2} \bar{a}_{j_3} \bar{a}_{j_4} + \bar{a}_{j_1} \bar{a}_{j_2} a_{j_3} a_{j_4}\right)$$

$$J = \sum_{i=1}^{4} |a_{j_i}|^2, \quad J_1 = |a_{j_1}|^2 - |a_{j_2}|^2, \qquad J_3 = |a_{j_3}|^2 - |a_{j_4}|^2$$

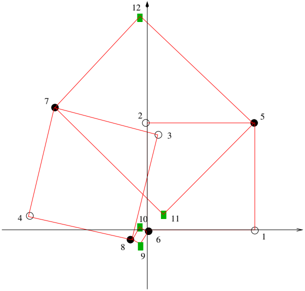
this is an integrable system study it at $J = 1, J_1 = J_3 = 0$



two periodic orbits connected by a heteroclinic

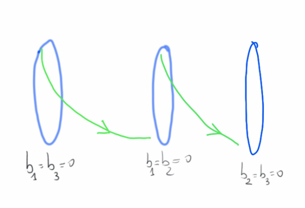
A "genealogical tree" of rectangles: three-generation set Λ

For ONE rectangle we have a growth of Sobolev norm of $2^{(s-1)/2}$ in order to obtain more growth we need more rectangles...



12 points, growth of Sobolev norm by $(2^{(s-1)/2})^2$

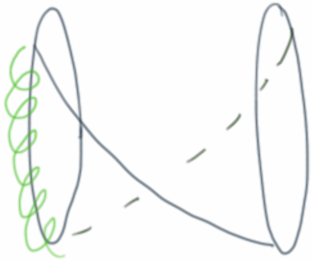
3 generations



three periodic orbits connected by two heteroclinic

Our beating solution: back to the simple rectangle resonance

remember that our system has 4 degrees of freedom. I describe a secondary 4-torus



first the solution moves around $a_{j_1} = 0$,

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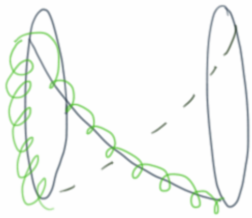

first the solution moves around $a_{j_1} = 0$ then it goes close to $a_{j_3} = 0$

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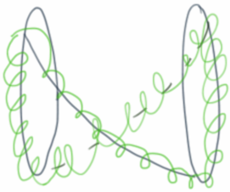
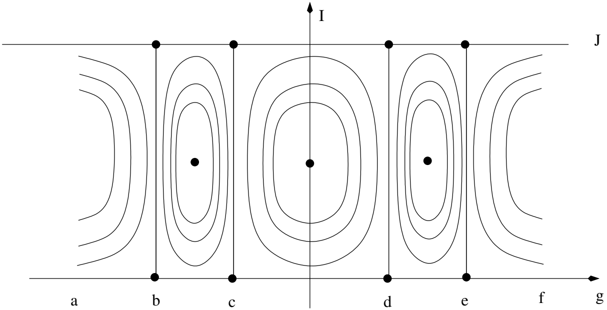


then it moves around $a_{j_3} = 0$, and finally comes back again.

Integrable dynamics

This is an integrable system, reducing w.r.t the constants on motion



KAM theory

Once we have an approximate solution (i.e. a solution of the Birkhoff system):
Linearize the NLS equation at the approximate solution, if:

- We can reduce the linearized equation to a time independent block diagonal form
- We can prove the Melnikov non-degeneracy conditions these are non-resonance conditions on the eigenvalues of the linearized operator

then we can apply KAM theory in order to deduce existence of quasi-periodic solutions for the NLS equation.
unfortunately this is a rather complicated program and becomes harder with the dimension.

Back to the circle: a simple resonance

Fix the simplest possible resonance for the quintic NLS on the circle

$$S = \{-2, -1, 1, 2\}$$

the Hamiltonian is

$$6\left(\sum_{j\in S}|a_j|^2\right)^3 - 9\left(\sum_{j\in S}|a_j|^2\right)\left(\sum_{j\in S}|a_j|^4\right) + 4\left(\sum_{j\in S}|a_j|^6\right) +$$

$$9(a_1^2 a_{-2}\bar{a}_{-1}^2\bar{a}_2 + \bar{a}_1^2\bar{a}_{-2}a_{-1}^2 a_2.)$$

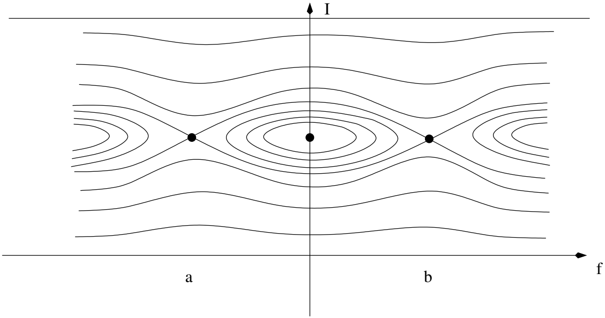
again this system is integrable!

Integrable dynamics

We have the constants of motion

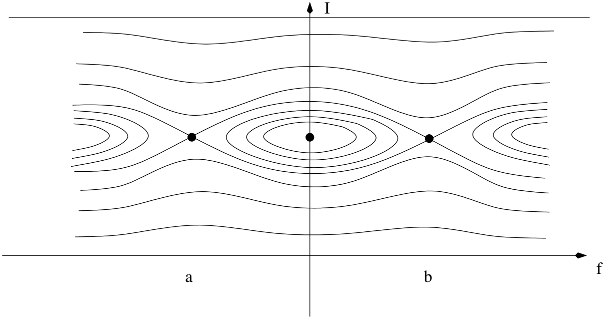
$$J = \sum_j |a_j|^2 , \quad J_1 = |a_2|^2 - 2|a_{-1}|^2 , \quad J_3 = |a_{-2}|^2 - 2|a_1|^2$$

so this is an integrable system, reducing w.r.t the constants on motion



Integrable dynamics

We study the system inside the eye.



We know that action-angle variables exist (call them $K, \varphi \in \mathbb{R}^4 \times \mathbb{T}^4$).

NLS Hamiltonian

Now we look at the full NLS Hamiltonian in the neighborhood of an approximate solution by setting $K = \xi + y$ and a_j small for $j \neq \pm 1, \pm 2$.

$$H_{\mathtt{NLS}} = \lambda(\xi) \cdot y + \sum_{j=3,4} \mathcal{H}_j(\xi, \varphi, a_{\pm j}) + \sum_{j \neq \pm 1, \pm 2} (j^2 + f(\xi, \varphi))|a_j|^2 + \mathcal{R}$$

$$\lambda(\xi) = (0, 1, 1, 4) + \omega(\xi)$$

$$\mathcal{H}_j = \mathcal{U}_j(\xi, \varphi)\mathrm{Re}(a_j \bar{a}_{-j}) + \varepsilon \mathcal{V}_j(\xi, \varphi)|a_{-j}|^2$$

We have little control on the functions $\omega(\xi), f(\xi, \varphi), \mathcal{U}_j, \mathcal{V}_j$

reduction to constant coefficients

$$H_{\mathtt{NLS}} = \lambda(\xi)\cdot y + \sum_{j=3,4}\mathcal{H}_j(\xi,\varphi,a_{\pm j}) + \sum_{j\neq\pm1,\pm2}(j^2+f(\xi,\varphi))|a_j|^2 + \mathcal{R}$$

There exists a symplectic change of variables which removes the angle dependence in the leading terms:

$$\lambda(\xi)\cdot y + \sum_{j=3,4}\mathcal{Q}_j(\xi,w_j,w_{-j}) + \sum_{j\neq\pm1,\pm2}(j^2+f_0(\xi))|w_j|^2 + \mathcal{R}$$

Then we can diagonalize the Hamiltonian

$$\lambda(\xi)\cdot y + \sum_{j\neq\pm1,\pm2,}\Omega_j|w_j|^2 + \mathcal{R}\,, \quad \Omega_j = (j^2+\varepsilon\Theta_j(\xi))\,, \qquad (2)$$

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Mel'nikov conditions

If we can prove that the analytic functions

$$\lambda(\xi) \cdot \ell\,, \quad \lambda(\xi) \cdot \ell + \Omega_j(\xi)\,, \quad \lambda(\xi) \cdot \ell + \Omega_j(\xi) \pm \Omega_i(\xi)$$

are never identically zero
then we may apply KAM theory

Thanks for your attention!