Regularity of Solutions of Delay Differential Equations: C^∞ versus Analytic

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Joint work with Roger Nussbaum

Delay Differential Equations

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Many other more complicated possibilities: distributed delays, implicitly defined delays, . . .

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Phase space $X = C[-R, 0]$ works well for constant delays.
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For non-constant delays much remains to be done (e.g., smoothness of stable manifold).

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There exists a solution for $t \in \mathbf{R}$ with $x(-\infty) = 1$. It is C^∞, but we don't know whether or not it is analytic.

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Replace $x(t-1)$ with $x(t-r)$ above, where $r = r(x(t))$ for appropriate $r(\cdot)$, for similar results.

$$\sigma \dot{x}(t) = -x(t) - kx(t-r),$$

$$\sigma > 0, \quad k > 1, \quad r(x(t)) = 1 + x(t).$$

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For a given C^∞ solution $x(t)$ we distinguish two sets:

$$\mathcal{A} = \{t_0 \mid x(t) \text{ is analytic for } t \text{ in some neighborhood of } t_0\},$$

$$\mathcal{N} = \mathbf{R} \setminus \mathcal{A}.$$

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Note that $\mathcal{A} \subseteq \mathbf{R}$ is open and $\mathcal{N} \subseteq \mathbf{R}$ is closed.

$$\dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t)), \quad \eta(t) = t - r(t)$$

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The sets $\mathcal{A}$ and $\mathcal{N}$ are intimately related to the dynamics of the "history map" $\eta : S^1 \to S^1$, namely

$$\eta(\mathcal{A} \setminus \mathcal{M}) \subseteq \mathcal{A}, \quad \eta(\mathcal{N}) \subseteq \mathcal{N},$$

$$\mathcal{M} = \{t_0 \in \mathbf{R} \mid t_0 \text{ is a local max or min of } \eta(t)\}.$$

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The quantity ν will appear as an eigenvalue of the above integral operator.

Standing Assumptions

$$r : \mathbf{R} \to \mathbf{R} \text{ is continuous,}$$

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We shall later also assume $r(t)$ is analytic for all t.

Integral Operator

$$(Lx)(t) = \int_{\eta(t)}^{t} x(s)\, ds, \quad x \in X,$$

$$X = \{x : \mathbf{R} \to \mathbf{R} \mid \text{continuous and } 2\pi \text{ periodic}\}$$

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Krein-Rutman implies there exists $\nu > 0$ and $x \in X \setminus \{0\}$, with $x \geq 0$, such that

$$Lx = \nu x$$

if and only if the spectral radius equals $\text{rad}(L) > 0$. And if so, one can take $\nu = \text{rad}(L)$.

Theorem. The spectral radius is positive, $\mathrm{rad}(L) > 0$, if and only if

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Remark. If $\eta(t) < t$ (that is, $r(t) > 0$) for every t, then $(*)$ holds and $\mathrm{rad}(L) > 0$. In this case the eigenfunction is unique.

Sketch of Proof. Suppose $(*)$ holds for every t. Using $(*)$ we obtain points

$$t_0 < t_1 < t_2 < \cdots < t_m \equiv t_0 (\mathrm{mod}\ 2\pi)$$

such that

$$t_k \in (\eta(t_{k+1}), t_{k+1}).$$

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This implies (upon iterating) that $\|L^n\| \geq c^n$, and thus $\text{rad}(L) \geq c > 0$.

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Then for any $\tau \geq t$ we have $t \leq \eta(\tau) \leq \tau$, and so

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Gronwall implies $x(t) \equiv 0$ identically, a contradiction.///

We now come to a main result on the analyticity set $\mathcal{A}$.

Theorem. In addition to the standing assumptions (periodicity and nonnegativity) on $r(t)$, assume that

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Then the set of analyticity $\mathcal{A}$ is a nonempty open set with infinitely many connected components (mod 2π). The set of nonanalyticity $\mathcal{N}$ is uncountable. Further, under a "stretching" condition on η the set $\mathcal{N}$ has empty interior and no isolated points, and is thus a generalized Cantor set.

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If $\rho = n\pi$ for an integer n, the sets $\mathcal{A}$ and $\mathcal{N}$ can be described precisely and $\mathcal{N}$ is a Cantor set.

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Iterate the points in $\overline{S(a)}$ backwards to get them in a neighborhood of a (mod 2π), and of b.

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Again suppose $I = [a, b]$ is the only maximal interval.

There exists some point $c \in \mathcal{A}$ with $c \in (b - 2\pi, a)$.

Iterate c backward to get arbitrarily close to a. Then a is a limit point (to the left) of points in $\mathcal{A}$, and of points in $\mathcal{N}$.

Thus $\mathcal{A}$ has infinitely many components near a (and near b).

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Nonanalyticity at a Point

$$\dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t)), \quad \eta(t) = t - r(t)$$

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Assume that

$$\eta(t_0) = t_0, \qquad |\dot{\eta}(t_0)| > 1.$$

An analytic Hartman-Grobman transformation gives

$$\dot{y}(t) = \widetilde{\alpha}(t)y(t) + \widetilde{\beta}(t)y(\mu t), \quad |\mu| > 1$$

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Expand (formal) Taylor series to get

$$y(t) = \sum_{k=0}^{\infty} y_k t^k, \quad y_k = \left(\frac{\mu^{k^2}\sigma^k}{k!}\right) w_k,$$

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Theorem. $w_\infty = 0$ if and only if there exists an analytic solution in a neighborhood of t_0.

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Answer unknown, but if so it would be very interesting: An interval where the solution is everywhere C^∞ but nowhere analytic.

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Then there is a maximal interval $I = [0, \tau]$ for some $\tau \in (0, \frac{\pi}{2})$

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But the endpoints of I' are not points of analyticity. Thus

$$(0, \tau), (\pi - \tau, \pi) \subseteq \mathcal{A}, \qquad 0, \tau, \pi - \tau, \pi \in \mathcal{N}$$

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But (**) is impossible due to the stretching condition:

There exist $k_1 < k_2 < k_3 < \ldots$ such that

$$\mathrm{len}(\eta^{k_{i+1}}(J)) > 2\mathrm{len}(\eta^{k_i}(J))$$