Integral Identity and Measure Estimates for Stationary Fokker-Planck Equation

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1. Itô Stochastic Differential Equations

- **ODE system:**

$$x' = V(x), \qquad x \in R^n.$$

- **Itô SDE:**

$$\mathrm{d}X = V(X)\mathrm{d}t + G(X)\mathrm{d}W, \qquad X \in R^n$$

W – m-dimensional Wiener process,

G – $n \times m$ matrix-valued function,

X – stochastic process:

$X(t)$ – stochastic variable ($\forall t$), i.e.,

measurable map form probability space $(\Omega, \mathcal{U}, P)$ to R^n.

- **Solutions of SDE:**

$$X(t) = x_0 + \int_0^t V(X(t))dt + \int_0^t G(X(t))dW,$$

where $\int_0^t GdW$ is understood in the sense of Itô integral.

- **Itô's Formula**

Assume $n = 1$, $u : R^1 \to R^1$ smooth, $X(\cdot)$ solves SDE.
$Y(t) := u(X(t))$

$$dY = u'dX = u'(Vdt + GdW) \qquad \text{wrong !}$$

Theorem(Itô's chain rule).

$$dY = \left(u'V + \frac{1}{2}G^2u''\right)dt + u'GdW.$$

Fact:

$$dW \approx (dt)^{1/2} \qquad \text{in some sense}$$

2. Fokker-Planck Equation

• **Probability Properties:** Under some conditions

(i) If $X_{s,x}$ is the unique solutions on $[s, T)$ satisfying $X(s) = x$, then the probability function

$$P(s, x, t, B) = \Pr \{X_{s,x}(t) \in B\}, \qquad B \subset \mathcal{B}$$

is a *transition probability*, where $\mathcal{B}$ is the set of all Borel subsets in R^n.

(ii) If

$$P(s, x, t, B) = \int_B p(s, x, t, y) dy,$$

then p is the kernel of the Fokker-Planck equation

- **Fokker-Planck equation:**

$$\begin{cases} u_t = Lu =: \sum_{i,j=1}^{n} \partial_{ij}^2 (a^{ij} u) - \mathrm{div}(Vu), & x \in R^n,\ t > 0, \\ u(x,t) \geq 0, & \int_{R^n} u(x,t)dx = 1, \end{cases}$$

where $(a^{ij}) = \dfrac{GG^\top}{2}$ – *diffusion term*; V – *drift term.*

<u>Assume</u> $\mathcal{A} := (a^{ij}) > 0$ everywhere.

- **Stationary Fokker-Planck equation:** Under some conditions, $u(x,t) \to u(x)$ as $t \to +\infty$, and $u(x)$ satisfies

$$\begin{cases} Lu =: \sum_{i,j=1}^{n} \partial_{ij}^2 (a^{ij} u) - \mathrm{div}(Vu) = 0, & x \in R^n \\ u(x) \geq 0, & \int_{R^n} u(x)dx = 1. \end{cases} \tag{1}$$

3. Motivation

General dynamics issues:

stochastic stability: Classify "dynamics" which are "robust" under the noise perturbations.

– General dynamics subjects are invariant measures (and invariant sets) of ODE.

• *Stochastic stability of invariant measures is directly related to the steady states of the Fokker-Planck equation.*

• **Fact:** Stationary measure $\mu = udx$, where u is a steady state of the F-P eq., is an *invariant measure* of the diffusion process X generated by SDE.

Consider a *null* family

$$A = \{\mathcal{A}_\alpha\}_{\alpha\in\Lambda} = \{(a_\alpha^{ij})\}_{\alpha\in\Lambda}, \qquad \mathcal{A}_\alpha \to 0.$$

Denote by $u_{\mathcal{A}_\alpha}$ the steady states F-P equation with $\mathcal{A} = \mathcal{A}_\alpha$. The stochastic stability is to characterize *the limits of the set of measures* $\{u_{\mathcal{A}_\alpha} dx\}$, *as* $|\mathcal{A}_\alpha|_{C^0} \to 0$.

Questions:

1) Existence of the steady states?

2) As $|\mathcal{A}_\alpha|_{C^0} \to 0$, compactness of $\{u_{\mathcal{A}_\alpha} dx\}$?

3) Concentration of $\{u_{\mathcal{A}_\alpha} dx\}$?

4. Steady States

• **Measure-valued solutions**: Meaningful solutions of (1) in the weakest term are actually probability measures.

Definition. A Borel probability measure μ is solution of $L\mu = 0$ if

$$\int_\Omega L^* f d\mu = 0 \qquad \forall f \in C_0^\infty(\Omega)$$

where L^* is the adjoint of L

$$L^* f := \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j} + V \cdot \nabla f \qquad \forall f \in C^2(\Omega).$$

Remark: If $d\mu = u dx$ for some $u \in C^2$, then $Lu = 0$ in classical sense (when coefficients $\in C^2$). i.e., u solves (1) in the classical sense.

Known Results

1) L is unif. elliptic and defined on a compact set

- **Theorem** (Zeeman, 1988): With smooth coefficients, the stationary Fokker-Planck equation defined on a compact manifold admits a unique strong solution.

- When eq. is defined on a bounded $\Omega \subset R^n$, solve

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 1 \end{cases}$$

or

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \Sigma_{ij}\partial_i(a^{ij}u)\nu_j + V \cdot \nu u = 0 & \text{on } \partial\Omega \end{cases}$$

where ν is normal vector.

Remark: *Existence is not true when Ω is unbounded, or L not unif. ellip..*

Example 1. $\Omega = R^n$, $V = 0$, $(a^{ij}) = I$

$$\begin{cases} \triangle u = 0, \\ u(x) \geq 0, \quad \int_{R^n} u(x)dx = 1, \end{cases}$$

Example 2. $\Omega = (0,1)$, $V = 0$, $a(x) = x^2$

$$(x^2 u)'' = 0,\ u \geq 0,\ \int_0^1 u(x)dx = 1.$$

No any measure-valued solution can exist!

Counter examples on uniqueness

Bogachev-Röckner-Stannat (1999, 2002), Shaposhnikov (2008)

$(a^{ij}) = I$, $\exists V \in C^\infty(R^n)$, s.t. (1) has more than one solution.

It actually admits infinitely many linearly independent solutions.

2) The case of $\Omega = R^n$

Khasminskii's Theorem (1960, 1980): Following the Khasminskii's Theorem in SDE, one can conclude that if

a) a^{ij}, $V \in \mathrm{Lip}_{loc}$,

b) $\exists\ U \in C^2(R^n)$ satisfying

$$i)\ \lim_{x\to\infty} U(x) = +\infty; \qquad ii)\ \lim_{x\to\infty} L^*U(x) \leq -\gamma < 0,$$

then (1) admits a unique strong solution.

—**Extensions** to non-Lip. $\mathcal{A}$ and V by

Bensoussan (1988), Skorohod (1989), Veretennikov (1987, 1997, 1999)...Albeverio, Bogachev, Krylov, Röckner, Stannat, Shaposhnikov (1997-2012)

Assume A): $a^{ij} \in W^{1,p}_{loc}$, $V \in L^p_{loc}$, with $p > n$.

Theorem(Albeverio, Bogachev, Krylov, Röckner, Stannat,Shaposhnikov, 1997-2012). Assume A) and b), then (1) admits a unique positive solution in $W^{1,p}_{loc}$.

Under A),

Regularity : B–K–R (1997, 2001).

Uniqueness : A–B–R (1999), B–R–S (1999, 2002) under $L^*U \leq \alpha U$ ($\forall |x| \gg 1$) for some $\alpha > 0$.

Existence : B–R (2000) under

$$i) \lim_{x\to\infty} U(x) = +\infty; \qquad ii)' \quad \lim_{x\to\infty} L^*U(x) = -\infty.$$

B-R-Sha (2012) under b).

2) Equations defined in $\Omega \subset R^n$

Main analysis: measure estimates

Example.

For an exterior domain $\mathcal{N} = \Omega \setminus K$ for some $K \subset\subset \Omega$,

$$\int_{\mathcal{N}} u_{\mathcal{A}} dx \leq C|\mathcal{A}|_{C^0(K)} \int_K u_{\mathcal{A}} dx,$$

This implies existence result, also the compactness of $\{u_{\mathcal{A}_\alpha} dx\}$ *by*

Prokhorov Theorem: a set $\mathcal{M}$ of measures on Ω is relatively sequentially compact if it is *tight* , i.e., for any $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\epsilon) < \epsilon$ for all $\mu \in \mathcal{M}$.

- *More delicate estimates* are obtained, which can imply the existence when *the equation allows degenerate.*
- *Lower bound estimates* are also obtained, which can imply *non-existence* results.

Basic Lemmas

Lemma 1(Integral Identity).

Assume A). Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of (1) in Ω. Then for any generalized Lipschitz domain $\Omega' \subset\subset \Omega$ and any function $F \in C^2(\bar{\Omega}')$ with $F|_{\partial\Omega'}$ =constant,

$$\int_{\Omega'} (L^* F)u \, dx = \int_{\partial\Omega'} (a^{ij}\partial_i F \nu_j)u \, ds,$$

where for a.e. $x \in \partial\Omega'$, $(\nu_j(x))$ denotes the unit outward normal vector of $\partial\Omega'$ at x.

- *In application Ω' is often chosen as sublevel sets of a function U*

$$\Omega_\rho = \{x \in \Omega : U(x) < \rho\}.$$

Definition. A non-negative $U \in C(\Omega)$ is said to be a *compact function* if

1) $\lim_{x\to\partial\Omega} U(x) = \sup_\Omega U := \rho_M$;

2) $U(x) < \rho_M, \quad x \in \Omega$.

Remark: Here $\partial\Omega$ and $x \to \partial\Omega$ are understood under the topology of the extended Euclidean space

$$E^n = R^n \cup \partial R^n, \qquad \partial R^n = \{x_*^\infty : x_* \in S^{n-1}\},$$

where x^∞ is the infinity element of the ray through x, with identifying

$$E^n \longleftrightarrow \bar{B}_1(0)$$

through $p : E^n \to \bar{B}_1(0)$, $\partial R^n \to \partial B_1(0)$.

Example.

Unbounded compact functions in R^n: $\quad \lim_{x\to\infty} U(x) = +\infty$.

Let $u \in C(\Omega)$ and let $U \in C^1(\Omega)$ be a compact function.

Consider the measure function

$$y(\rho) := \int_{\Omega_\rho} u \, dx, \qquad \rho \in (0, \rho_M),$$

and the open set

$$\mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, \; x \in U^{-1}(\rho)\},$$

where $\rho_M = \sup_\Omega U$.

Lemma 2 (Differential Formula). The measure function y is of the class C^1 on $\mathcal{I}$ with derivatives

$$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \, ds, \qquad \rho \in \mathcal{I}.$$

Definition. Let U be a C^2 compact function in Ω.

1. U is called a *(stochastic) Lyapunov function* (resp. *(stochastic) anti-Lyapunov function*), if there is a neighborhood $\mathcal{N}$ of $\partial\Omega$ and a constant $\gamma > 0$, such that

$$L^* U(x) \leq -\gamma, \quad (\text{resp.} \geq \gamma), \qquad x \in \mathcal{N}. \tag{2}$$

2. U is called a *(stochastic) weak Lyapunov function* (resp. *(stochastic) weak anti-Lyapunov function*), if $\gamma = 0$ in (2).

Remark: Recall that a classical Lyapunov function U for an ODE system is such that

$$V \cdot \nabla U(x) \leq -\gamma < 0, \qquad |x| \gg 1,$$

which implies the existence of global attractor.

Measure Estimates

For $U \in C^1$, let h, H be two non-negative, locally bounded functions on $[0, \rho_M)$ such that $\forall \rho \in [0, \rho_M)$

$$h(\rho) \leq \sum a^{ij}(x)\partial_i U(x)\partial_j U(x) \leq H(\rho), \qquad x \in U^{-1}(\rho).$$

Theorem 1. Assume that (1) has a Lya. funct. U with Lya. const. γ. Then $\exists \rho_m < \rho_M$ s.t. for any measure solution μ of (1) in Ω,

$$\mu(\Omega \setminus \Omega_\rho) \leq \gamma^{-1} C_{\rho_m,\rho}(\sup_{(\rho_m,\rho)} H)\mu(\Omega_\rho), \quad \rho \in [\rho_m,\ \rho_M),$$

where the constant $C_{\rho_m,\rho} \sim \rho_m, \rho$.

Remark. This implies

1. existence if there exists a Lyapunov function;

2. compactness of the set $\{u_{\mathcal{A}_\alpha} dx\}$ if there exists a uniform Lyapunov function w.r.t. the family $A = \{\mathcal{A}_\alpha\}$ as $\mathcal{A}_\alpha \to 0$.

Theorem 2. Let U be a compact function such that for a.e. ρ close to ρ_M,

$$\nabla U(x) \neq 0, \qquad \forall x \in U^{-1}(\rho).$$

I) If U is Lyapunov, then for any measure solution μ:

$$\mu(\Omega \setminus \Omega_\rho) \leq e^{-\gamma \int_{\rho_m}^{\rho} \frac{1}{H(t)} dt}, \qquad \rho \in [\rho_m, \rho_M).$$

II) If U is anti-Lyapunov with γ being an anti-Lyapunov constant, then for any measure solution μ:

$$\mu(\Omega_\rho \setminus \Omega^*_{\rho_m}) \geq \mu(\Omega_{\rho_0} \setminus \Omega^*_{\rho_m}) e^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)} dt}, \qquad \rho \in (\rho_0, \rho_M),$$

where $\Omega^*_{\rho_m} = \{x \in \Omega : U(x) \leq \rho_m\}$, $\rho_0 > \rho_m$.

Set

$$\mathcal{B}^*(\mathcal{A}) = \left\{ \text{compact } U : \int_{\rho_m}^{\rho_M} \frac{1}{H(t)} dt = \infty \right\}$$

Remark. This implies

1)existence of stationary measures in the degenerate case $\mathcal{A} \geq 0$ if $V \in C^0$ and there exists a Lyapunov function in $\mathcal{B}^*(\mathcal{A})$;

2)non-existence of stationary measures if there exists an anti-Lyapunov function in $\mathcal{B}^*(\mathcal{A})$.

One can formulate a necessary and sufficient condition for the existence of solutions of $(1)_\epsilon$ *with* $\mathcal{A}_\epsilon = \epsilon\mathcal{A}$.

• **Corollary** (Cases with small noise):

Assume A) and $0 < \lambda I \leq \mathcal{A} \leq \Lambda I$ in R^n.

If $\exists U \in C^2$ with $C^{-1}I \leq (D^2U) \leq CI \quad \forall |x| \gg 1$, s.t.

$$\lim_{x\to\infty} V \cdot \nabla U = \mu \quad (\pm\infty \text{ is allowed.})$$

Then, $\mu < 0$ iff $(1)_\epsilon$ admits a unique solution in $W^{1,p}_{loc}$, for $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 \ \sim \ n, \mu, \lambda, \Lambda$, and C.

Application. Take $U(x) = |x|^2$, $x \in R^n$.

$$\lim_{x \to \infty} V \cdot x < 0 \quad \text{iff } (1)_\epsilon \text{ admits unique solution .}$$

Theorem 3. Assume $h > 0$ for ρ close to ρ_M.

I) If U is a weak Lyapunov function, then for any $\rho_0 \in (\rho_m, \rho_M)$,

$$\mu(\Omega \setminus \Omega_{\rho_m}) \leq \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) e^{\int_{\rho_0}^{\rho_M} (\tilde{H}(\rho))^{-1} d\rho},$$

where $\tilde{H}(\rho) = h(\rho) \int_{\rho_m}^{\rho} \frac{1}{H(s)} ds$, $\rho \in [\rho_m, \rho_M)$.

II) If U is a weak anti-Lyapunov function, then for any $\rho_0 \in (\rho_m, \rho_M)$,

$$\mu(\Omega_\rho \setminus \Omega_{\rho_m}) \geq \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) e^{\int_{\rho_0}^{\rho} (\tilde{H}(t))^{-1} dt}, \qquad \rho \in [\rho_0, \rho_M),$$

where $\tilde{H}(\rho) = H(\rho) \int_{\rho_m}^{\rho} \frac{1}{h(s)} ds$, $\rho \in [\rho_m, \rho_M)$.

Remark. This implies

1) existence of steady states if there exists a weak Lyapunov function in

$$\mathcal{B}_*(\mathcal{A}) = \left\{ \text{compact } U : \int_{\rho_m}^{\rho_M} \frac{1}{h(t)} dt < \infty \right\};$$

2) non-existence if there exists a weak anti-Lyapunov function in

$$\mathcal{B}(\mathcal{A}) = \left\{ \text{compact } U : \int_{\rho_m}^{\rho_M} (H(t) \int_{\rho_m}^{t} \frac{1}{h(s)} ds)^{-1} dt = \infty \right\}$$

5. Stochastic Stability of Invariant Sets

Assume that the ODE system generates a local flow φ^t. A *limit measure* is a weak*-limit point of ϵG-stationary measures as $\epsilon \to 0$.

• **Theorem** (Global concentration). If φ^t is dissipative in $\mathcal{U}$, then all limit measures are supported in the global attractor $\mathcal{J}$, i.e., $\mathcal{J}$ is G-stable w.r.t. any G.

• **Theorem** (Local concentration).

1) (Stabilization) If $\mathcal{J}_0$ is a strong local attractor of φ^t, then $\exists\ G$ s.t. all G-limit measures are supported in $\mathcal{J}_0$, i.e., $\mathcal{J}_0$ is G-stable.

2) (Di-stabilization) If $\mathcal{R}_0$ is a strong local repeller of φ^t, then $\exists\ G$ s.t. all G-limit measures are supported away from $\mathcal{R}_0$, i.e., $\mathcal{R}_0$ is G-unstable.

3) (Instability of equilibrium) If $\mathcal{R}_0$ is an equilibrium, then 2) holds for any bounded G.

6. Stochastic Bifurcations

We can also define stochastic global (structural) stability using stationary measures. If the global stability is broken as parameters very, then stochastic bifurcation will occur.

- **Example** (Stochastic Hopf bifurcation): Consider

$$\begin{cases} \mathrm{dx} = (\mathrm{bx} - \mathrm{y} - \mathrm{x}(\mathrm{x}^2 + \mathrm{y}^2))\mathrm{dt} + \sqrt{\epsilon}\mathrm{g}^{11}(\mathrm{x}, \mathrm{y})\mathrm{dW}_1 + \sqrt{\epsilon}\mathrm{g}^{12}(\mathrm{x}, \mathrm{y})\mathrm{dW}_2, \\ \mathrm{dy} = (\mathrm{x} + \mathrm{by} - \mathrm{y}(\mathrm{x}^2 + \mathrm{y}^2))\mathrm{dt} + \sqrt{\epsilon}\mathrm{g}^{21}(\mathrm{x}, \mathrm{y})\mathrm{dW}_1 + \sqrt{\epsilon}\mathrm{g}^{22}(\mathrm{x}, \mathrm{y})\mathrm{dW}_2, \end{cases}$$

where $G(x, y) = (g^{ij}(x, y)) \in W^{1,2\bar{p}}_{loc}$ is non-singular and bounded.

$b \le 0$: $\mathcal{J}_b = \{0\}$. Hence $\mu_{b,\epsilon} \to \delta_0$ as $\epsilon \to 0$. $\Longrightarrow$ $\{0\}$ and δ_0 are G-stable.

$b > 0$: $\mathcal{J}_b = \bar{\Omega}_b$ - the closed disk of radius $\sqrt{b}$. But each limit measure of $\{\mu_{b,\epsilon}\}$ is supported on C_b - the circle with radius $\sqrt{b}$. $\Longrightarrow$ C_b and μ_b (Haar measure on C_b) become G-stable in this case.

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