Considerations on sliding motion for piecewise smooth systems of Filippov type.

Luca Dieci

School of Mathematics, Georgia Tech, and
Tang Aoqing Visiting Professor, Jilin University

Based on works with Cinzia Elia, Fabio Difonzo, Luciano Lopez.

Dynamics of Evolution Equations
CIRM, Luminy – March 21-25, 2016

[Thanks to CIRM and NSF for support]

The problem.

- Piecewise smooth systems ("switched" systems):

$$x' = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, \dots, m\ ,$$

$t \in [0, T]$, $x(0) = x_0$. Here, $R_i \subseteq \mathbb{R}^n$ are open, disjoint and partition $\mathbb{R}^n$:$\mathbb{R}^n = \overline{\bigcup_i R_i}$.

The problem.

- Piecewise smooth systems ("switched" systems):

$$x' = f(x) \ , \ \ f(x) = f_i(x) \ , \ \ x \in R_i \ , i = 1, \dots, m \ ,$$

$t \in [0, T]$, $x(0) = x_0$. Here, $R_i \subseteq \mathbb{R}^n$ are open, disjoint and partition $\mathbb{R}^n$:$\mathbb{R}^n = \overline{\bigcup_i R_i}$.

- In each region R_i, we have a standard differential equation with smooth vector field f_i. On the boundaries .... ?

The problem.

- Piecewise smooth systems ("switched" systems):

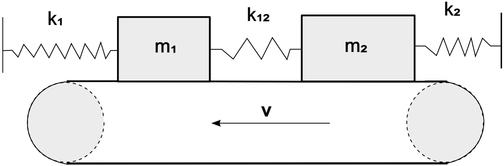
$$x' = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, \dots, m\ ,$$

  $t \in [0, T]$, $x(0) = x_0$. Here, $R_i \subseteq \mathbb{R}^n$ are open, disjoint and partition $\mathbb{R}^n$:$\mathbb{R}^n = \overline{\bigcup_i R_i}$.

- In each region R_i, we have a standard differential equation with smooth vector field f_i. On the boundaries .... ?

- Assume that the R_i's are separated (locally) by surfaces characterized as zero sets of smooth functions.

- A lot of activity and open mathematical problems. Widely used in applications.
  (Filippov, Utkin, Sontag, Cortes, Acary-Brogliato, $\cdots$).
- Systems with delays, models of relays, switches, gates, thermostats and refrigeration processes.
- Bang-bang controls, controllers in fields with obstacles, and generally VSC.
- Also mechanical systems (stick-slip).



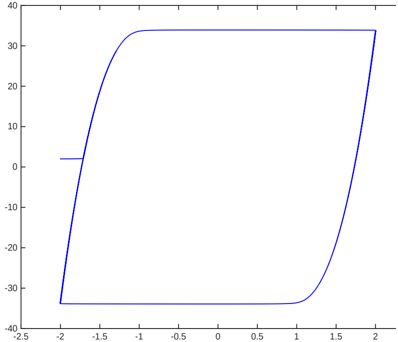
... the model ...

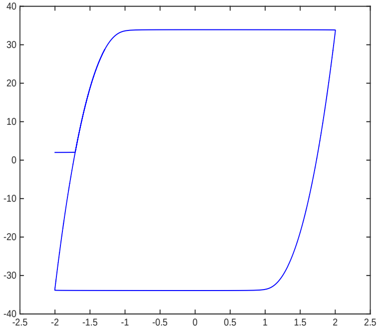
- Natura non facit saltum (Linnaeus, **1751**) ... is model innatural, wrong?

- Natura non facit saltum (Linnaeus, **1751**) ... is model innatural, wrong?
- Probably neither . . . , perhaps not all "natural" processes are smooth, or the PWS model is incomplete, or it is just a convenient simplification of an underlying model, or the **skeleton** of it . . .

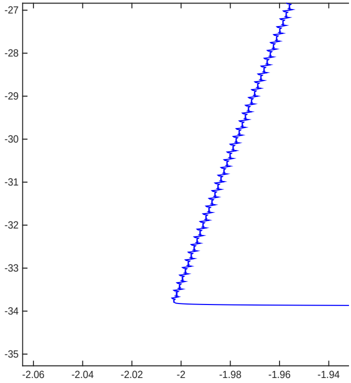
- Natura non facit saltum (Linnaeus, **1751**) ... is model innatural, wrong?
- Probably neither . . . , perhaps not all "natural" processes are smooth, or the PWS model is incomplete, or it is just a convenient simplification of an underlying model, or the **skeleton** of it . . .
- Can we see the difference?

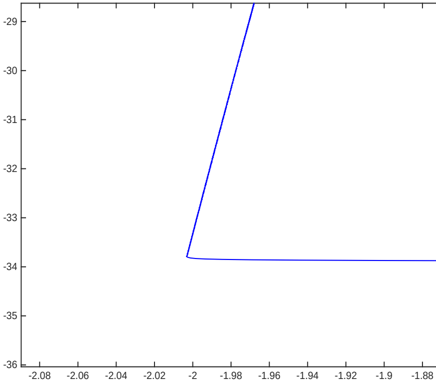
- Natura non facit saltum (Linnaeus, **1751**) ... is model innatural, wrong?
- Probably neither ..., perhaps not all "natural" processes are smooth, or the PWS model is incomplete, or it is just a convenient simplification of an underlying model, or the **skeleton** of it ...
- Can we see the difference?





- Natura non facit saltum (Linnaeus, **1751**) ... is model innatural, wrong?
- Probably neither ..., perhaps not all "natural" processes are smooth, or the PWS model is incomplete, or it is just a convenient simplification of an underlying model, or the **skeleton** of it ...





- Seidman calls the PWS system the **residue of model reduction**: . . . *apparent discontinuity is actually a continuous process having a fine structure on a more rapid time scale*

... the model ...

- Seidman calls the PWS system the **residue of model reduction**: . . . *apparent discontinuity is actually a continuous process having a fine structure on a more rapid time scale*
- Utkin looks at the PWS system as unable to describe *what happens in the boundary layer*

... the model ...

- Seidman calls the PWS system the **residue of model reduction**: ... *apparent discontinuity is actually a continuous process having a fine structure on a more rapid time scale*
- Utkin looks at the PWS system as unable to describe *what happens in the boundary layer*
- Whatever the case, our key point is that: *If a modeling simplification took place, and is meaningful, then the PWS (reduced) model must have retained sufficient features to understand the dynamics (behavior) of the true (unmodeled, possibly unknown) problem (if any)*

- Seidman calls the PWS system the **residue of model reduction**: . . . *apparent discontinuity is actually a continuous process having a fine structure on a more rapid time scale*
- Utkin looks at the PWS system as unable to describe *what happens in the boundary layer*
- Whatever the case, our key point is that: *If a modeling simplification took place, and is meaningful, then the PWS (reduced) model must have retained sufficient features to understand the dynamics (behavior) of the true (unmodeled, possibly unknown) problem (if any)*
- Thus, we look at PWS model with its own intrinsic **mathematical dignity**, try to develop a mathematical framework and tools to understand its dynamics, and indirectly (possibly) those of unreduced (and un-modeled) problem (if any).

- Which features must be looked at? What has been retained? Is the task well posed?

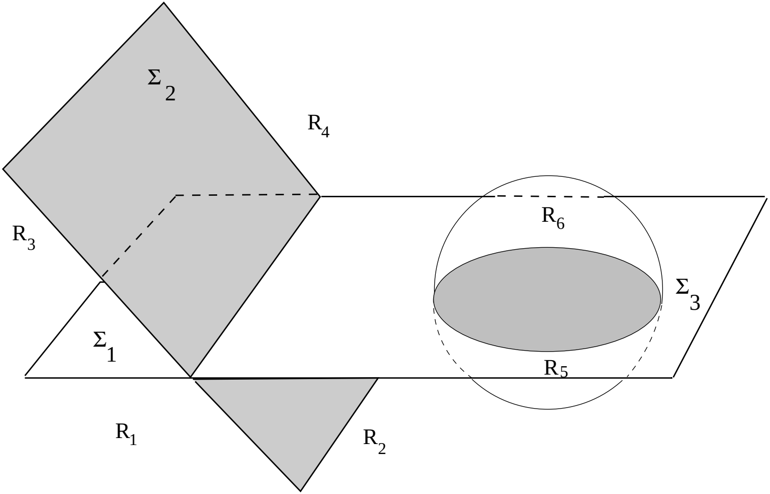
- Which features must be looked at? What has been retained? Is the task well posed?
- First, robustness of problem configuration:

... the model ...

- Which features must be looked at? What has been retained? Is the task well posed?
- First, robustness of problem configuration:
- If the separating surface, call it Σ, has codimension d, then (locally, in a neighborhood of Σ) there are 2^d regions R_i's and therefore 2^d vector fields f_i's:

$$\Sigma = \{x \in \mathbb{R}^n \; : \; h(x) = 0 \, , \quad h : \; \mathbb{R}^n \to \mathbb{R}^d\} \, ,$$

where $h(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_d(x) \end{bmatrix}$, $\nabla h_j(x) \neq 0$, $j = 1, \dots, d$, and the vectors $\{\nabla h_1(x), \dots, \nabla h_d(x)\}$ are linearly independent, and smooth ($\mathcal{C}^1$) for all $x \in U_\Sigma$.



Σ 2
R4
R3
R6
Σ 3
Σ 1
R5
R1
R2

- Case of interest: (a portion of) the surface Σ attracts nearby trajectories, and it is reached in finite time. [Of course, time arrow does matter.]

- Case of interest: (a portion of) the surface Σ attracts nearby trajectories, and it is reached in finite time. [Of course, time arrow does matter.]
- As a consequence, solution trajectories with initial data on (part of) Σ will be forced to remain on it. Motion (if defined) must continue on Σ, sliding motion. (Irreversible).

- Case of interest: (a portion of) the surface Σ attracts nearby trajectories, and it is reached in finite time. [Of course, time arrow does matter.]
- As a consequence, solution trajectories with initial data on (part of) Σ will be forced to remain on it. Motion (if defined) must continue on Σ, sliding motion. (Irreversible).
- Important to characterize finite time attractivity of Σ, and ideally to decide what to do if a trajectory reaches Σ, and when/if/how it should leave it. [Here, we'll assume that there are finitely many *events*: changes of regime].

... the model ...

- Case of interest: (a portion of) the surface Σ attracts nearby trajectories, and it is reached in finite time. [Of course, time arrow does matter.]
- As a consequence, solution trajectories with initial data on (part of) Σ will be forced to remain on it. Motion (if defined) must continue on Σ, sliding motion. (Irreversible).
- Important to characterize finite time attractivity of Σ, and ideally to decide what to do if a trajectory reaches Σ, and when/if/how it should leave it. [Here, we'll assume that there are finitely many *events*: changes of regime].
- First of all, a solution concept is needed.

Filippov convexification.

1. Consider the set valued function

$$F(x) \; = \; \text{co}\{\lim_{k\to\infty} f(x_k), \; x_k \to x, \; x_k \in R_i\} \, .$$

In other words, $F(x)$ is the convex hull of the values of $f(x)$ obtained approaching x through (smooth) regions R_i.

Filippov convexification.

1. Consider the set valued function

$$F(x) \;=\; \text{co}\{\lim_{k\to\infty} f(x_k),\ x_k \to x,\ x_k \in R_i\}\,.$$

   In other words, $F(x)$ is the convex hull of the values of $f(x)$ obtained approaching x through (smooth) regions R_i.

2. Consider the differential inclusion obtained by replacing f with F: $x' \ \in F(x)$, and a *Filippov solution* is a classical solution of this differential inclusion.

Filippov convexification.

1. Consider the set valued function

$$F(x) \;=\; \text{co}\{\lim_{k\to\infty} f(x_k),\; x_k \to x,\; x_k \in R_i\}\,.$$

   In other words, $F(x)$ is the convex hull of the values of $f(x)$ obtained approaching x through (smooth) regions R_i.

2. Consider the differential inclusion obtained by replacing f with F: $x' \;\in F(x)$, and a *Filippov solution* is a classical solution of this differential inclusion.

- Existence is a classical result [Filippov]. Uniqueness is more complicated since it is necessary to characterize what happens on the boundaries of the regions R_i's.

- So, Filippov idea is to consider:

$$x' \in F(x) = \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \ , \ \ \lambda_i(x) \geq 0 \ , \ \text{and} \ \ \sum_{i=1}^{2^d} \lambda_i(x) = 1 \, .$$

- So, Filippov idea is to consider:

$$x' \ \in \ F(x) = \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \ , \ \ \lambda_i(x) \geq 0 \ , \ \text{and} \ \ \sum_{i=1}^{2^d} \lambda_i(x) = 1 \, .$$

- Sliding motion on Σ must occur with a vector field that lies in the tangent plane at $x \in \Sigma$:

- So, Filippov idea is to consider:

$$x' \in F(x) = \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \ , \ \ \lambda_i(x) \geq 0 \ , \ \text{and} \ \ \sum_{i=1}^{2^d} \lambda_i(x) = 1 \, .$$

- Sliding motion on Σ must occur with a vector field that lies in the tangent plane at $x \in \Sigma$:
  - We call *Filippov (sliding) vector field* a smooth vector field of the form

$$\text{(a)} \quad f_{\mathsf{F}} := \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \ , \ \text{with} \ \ \lambda_i(x) \geq 0 \ , \ \ \sum_{i=1}^{2^d} \lambda_i(x) = 1 \, ,$$

$$\text{(b)} \quad (\nabla h_j(x))^T f_{\mathsf{F}}(x) = 0 \ , \ \ \text{for all} \ \ j = 1, \dots, d \, .$$

- So, Filippov idea is to consider:

$$x' \;\in\; F(x) = \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \;,\;\; \lambda_i(x) \geq 0 \;,\; \text{and} \;\; \sum_{i=1}^{2^d} \lambda_i(x) = 1\,.$$

- Sliding motion on Σ must occur with a vector field that lies in the tangent plane at $x \in \Sigma$:
  - We call *Filippov (sliding) vector field* a smooth vector field of the form

$$\begin{aligned}
&\text{(a)} \quad f_{\mathsf{F}} := \sum_{i=1}^{2^d} \lambda_i(x) f_i(x) \;,\; \text{with} \;\; \lambda_i(x) \geq 0 \;,\;\; \sum_{i=1}^{2^d} \lambda_i(x) = 1\,, \\
&\text{(b)} \quad (\nabla h_j(x))^T f_{\mathsf{F}}(x) = 0 \;,\;\; \text{for all} \;\; j = 1, \ldots, d\,.
\end{aligned}$$

- Well understood process in case Σ has co-dimension 1, with a lot of work here still being done (also in the planar case), including periodic orbits, bifurcation studies, numerical methods.

Label R_1: $h(x) < 0$, R_2: $h(x) > 0$. Define

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \nabla h(x)^T f_1(x) \\ \nabla h(x)^T f_2(x) \end{bmatrix} \ , \quad x \in \Sigma \, ,$$

$\Rightarrow$ attractivity in finite time (trajectories enter Σ transversally):

$$w_1(x) \geq a > 0 \qquad \text{and} \qquad w_2(x) \leq -b < 0 \ .$$

- Have a unique Filippov (sliding) vector field

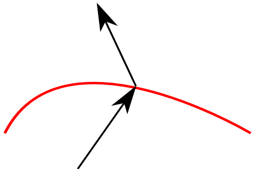
$$f_{\mathsf{F}} = (1-\alpha) f_1 + \alpha f_2 \ , \quad \alpha \ : \ \alpha = w_1/(w_1 - w_2) \, .$$

- If $\alpha = 0$ (resp. $\alpha = 1$), f_1 (resp. f_2), is tangent to Σ. Expect trajectory to exit Σ and enter in R_1 (resp. R_2). These are tangential (and smooth) exits: predicted by first order Filippov theory.
- Well defined Filippov sliding vector field also for repulsive Σ:

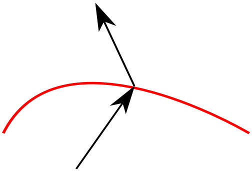
$$w_1(x) \leq -c < 0 \qquad \text{and} \qquad w_2(x) \geq c > 0 \, , \ x \in \Sigma \, .$$

But ... sliding motion unstable, no uniqueness, can leave at any time with f_1 or f_2: non-tangential (and non-smooth) exits.

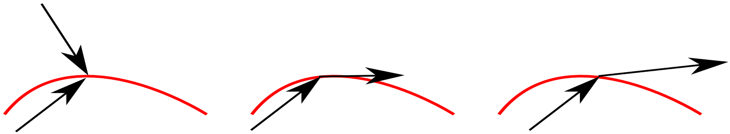
- Transversal crossing



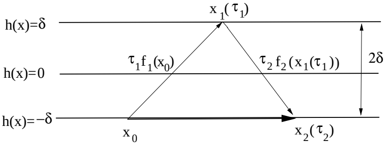
- Transversal crossing



- Sliding and tangential (smooth) exit



- Among several validations of Filippov sliding vector field:
  - limiting behavior of Euler iterates (and other 1-step methods)



  - limiting behavior of Sotomayor-Teixeira regularization

$$\dot{x} = (1 - \alpha_\epsilon(h(x)))\, f_1(x) + \alpha_\epsilon(h(x)) f_2(x)\,,$$

with (for example)

$$\alpha_\epsilon(z) = \begin{cases} 1 & z > \epsilon \\ \frac{1}{2} + \frac{z}{4\epsilon}(3 - (\frac{z}{\epsilon})^2) & z \in [-\epsilon, \epsilon] \\ 0 & z < -\epsilon \end{cases}\,.$$

Σ of codimension 2

- Still object of investigation, and present emphasis. Now, $\Sigma_1 = \{\ x: \ h_1(x) = 0\}$, $\Sigma_2 = \{\ x: \ h_2(x) = 0\}$, and we have $\Sigma = \Sigma_1 \cap \Sigma_2$.

Σ of codimension 2

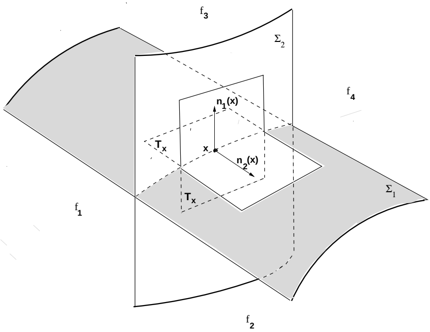
- Still object of investigation, and present emphasis. Now, $\Sigma_1 = \{\ x :\ h_1(x) = 0\}$, $\Sigma_2 = \{\ x :\ h_2(x) = 0\}$, and we have $\Sigma = \Sigma_1 \cap \Sigma_2$.
- There are four different regions R_1, R_2, R_3 and R_4:

$$\begin{array}{llllll} R_1: & \text{when} & h_1 < 0\ ,\ h_2 < 0\ , & R_2: & \text{when} & h_1 < 0\ ,\ h_2 > 0\ , \\ R_3: & \text{when} & h_1 > 0\ ,\ h_2 < 0\ , & R_4: & \text{when} & h_1 > 0\ ,\ h_2 > 0\ . \end{array}$$

Σ of codimension 2

- Still object of investigation, and present emphasis. Now, $\Sigma_1 = \{ x : \ h_1(x) = 0\}$, $\Sigma_2 = \{ x : \ h_2(x) = 0\}$, and we have $\Sigma = \Sigma_1 \cap \Sigma_2$.
- There are four different regions R_1, R_2, R_3 and R_4:

$$R_1 : \quad \text{when} \quad h_1 < 0 \ , h_2 < 0 \ , \qquad R_2 : \quad \text{when} \quad h_1 < 0 \ , h_2 > 0 \ ,$$
$$R_3 : \quad \text{when} \quad h_1 > 0 \ , h_2 < 0 \ , \qquad R_4 : \quad \text{when} \quad h_1 > 0 \ , h_2 > 0 \ .$$



- Let

$$w_i(x) = \begin{bmatrix} w_i^1(x) \\ w_i^2(x) \end{bmatrix} = \begin{bmatrix} \nabla h_1^T f_i \\ \nabla h_2^T f_i \end{bmatrix} , \qquad i = 1, 2, 3, 4 \ .$$

To form

$$f_{\mathsf{F}} = \sum_{i=1}^{4} \lambda_i(x) f_i(x)$$

we need to solve

$$\begin{bmatrix} W \\ \mathbf{1}^T \end{bmatrix} \lambda = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

obviously underdetermined (in general).

- Let

$$w_i(x) = \begin{bmatrix} w_i^1(x) \\ w_i^2(x) \end{bmatrix} = \begin{bmatrix} \nabla h_1^T f_i \\ \nabla h_2^T f_i \end{bmatrix} \ , \qquad i = 1, 2, 3, 4 \ .$$

  To form

$$f_{\mathsf{F}} = \sum_{i=1}^{4} \lambda_i(x) f_i(x)$$

  we need to solve

$$\begin{bmatrix} W \\ \mathbf{1}^T \end{bmatrix} \lambda = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ ,$$

  obviously underdetermined (in general).
- **Note**: algebraic nature of ambiguity.

Ways in which this ambiguity has been "removed."

a) Restrict to problems where there is no ambiguity (e.g., stick-slip model).

b) Select a specific sliding field on Σ. Two choices studied. Bilinear interpolant (a few people, including us)

$$\dot{x} = (1-\alpha)\left((1-\beta)f_1 + \beta f_2\right) + \alpha\left((1-\beta)f_3 + \beta f_4\right) \ ,$$

$$(\alpha, \beta) \in (0,1)^2 \ : \ W\lambda_{\mathrm{B}} = 0 \quad \text{with } \lambda_{\mathrm{B}} := \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix} \ .$$

Nonlinear system to solve.

Moments method (D-Difonzo). For $x \in \Sigma$, solve

$$M\lambda_{\mathrm{M}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ , \text{ where } M := \begin{bmatrix} W \\ e^T \\ d^T \end{bmatrix} \ , \ d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix} \ ,$$

and $d_i = \|w_i\|_2$, $i = 1, \ldots, 4$.

c) Globally regularize the PWS system. Thus far, only one technique has really been studied (a few people, including us): *bilinear regularization*:

$$\begin{aligned}\dot{x} =&(1-\alpha_{\epsilon_1}(h_1(x)))[(1-\beta_{\epsilon_2}(h_2(x)))f_1+\beta_{\epsilon_2}(h_2(x))f_2(x)]+\\ &\alpha_{\epsilon_1}(h_1(x))[(1-\beta_{\epsilon_2}(h_2(x)))f_3++\beta_{\epsilon_2}(h_2(x))f_4(x)]\,.\end{aligned}$$

where α_{ϵ_1}, β_{ϵ_2}, are smooth step functions as in the co-d 1 case.

c) Globally regularize the PWS system. Thus far, only one technique has really been studied (a few people, including us): *bilinear regularization*:

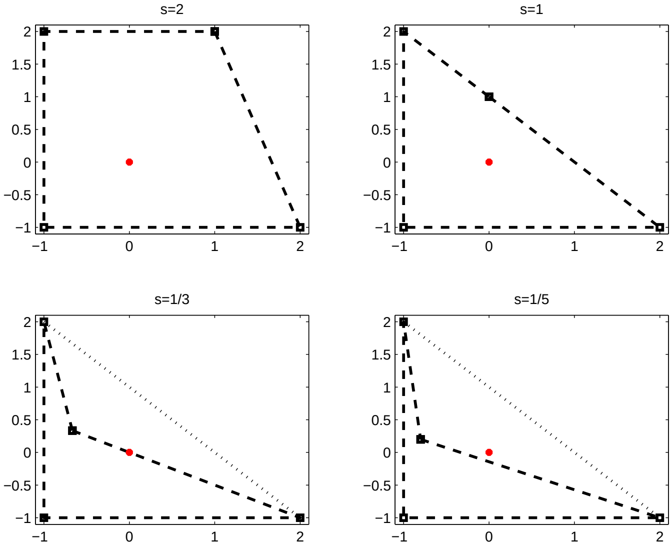
$$\begin{aligned}\dot{x} =&(1-\alpha_{\epsilon_1}(h_1(x)))[(1-\beta_{\epsilon_2}(h_2(x)))f_1+\beta_{\epsilon_2}(h_2(x))f_2(x)]+\\ &\alpha_{\epsilon_1}(h_1(x))[(1-\beta_{\epsilon_2}(h_2(x)))f_3++\beta_{\epsilon_2}(h_2(x))f_4(x)]\,.\end{aligned}$$

where α_{ϵ_1}, β_{ϵ_2}, are smooth step functions as in the co-d 1 case.
$\longrightarrow$ Different regularizations are not equivalent to one another.

d) Other: Euler, SDE, hysteresis (delay), minimum variation. Limited results, in rather restrictive cases.

- Here, not interested in the selection process of a sliding vector field, but rather in what is the dynamical impact of the ambiguity.
- Is the ambiguity in the trajectory selection reflecting into a dynamics concern?
- Or: "can we (at least) say what should happen?"
- Our viewpoint: it is Σ 's properties with respect to the nearby vector fields (namely, attractivity) that give appropriate insight.
- Punchline is that "sliding" is a meaningful idealization as long as Σ is attractive, even if one cannot generally uniquely determine how sliding should take place. At the same time, trajectories can be perturbed off Σ, and should not remain on Σ, if Σ is not attracting.

• .... there are (too many) Filippov vector fields, the convex hull does not know about attractivity of Σ ....



Attractivity

- There are two fundamentally different ways in which Σ can attract nearby trajectories: through sliding, or in a spiral-like manner. In all cases, the vectors w_i, $i = 1, 2, 3, 4$, projections of the vector fields along the normals, are the key.

- There are two fundamentally different ways in which Σ can attract nearby trajectories: through sliding, or in a spiral-like manner. In all cases, the vectors w_i, $i = 1, 2, 3, 4$, projections of the vector fields along the normals, are the key.
- Below, we will let $\Sigma_1^{\pm} = \{x : \; h_1(x) = 0 \;, \; h_2(x) \gtrless 0\}$, and similarly for $\Sigma_2^{\pm}$. Also, let $f_{\mathsf{F}}{}_{1,2}^{\pm}$ the sliding vector fields (whenever properly defined) on the sub-surfaces $\Sigma_{1,2}^{\pm}$. These are co-d 1 Filippov sliding vector fields. Say:

$$f_{\mathsf{F}}{}_1^{+} = (1-\alpha^+) f_2 + \alpha^+ f_4 \;, \; \alpha^+ = \left[\frac{\nabla h_1^T f_2}{\nabla h_1^T (f_2 - f_4)} \right]_{x \in \Sigma_1^+} .$$

Definition (D, Elia, Lopez)

(a) For $j = 1, \dots, 4$, and $x \in R_j$, w_j^1 and w_j^2 do not have the same signs as the pair $(h_1(x), h_2(x))$, and $(w_j^1, w_j^2) \neq 0$ on U;

(b) At least one of the following conditions is satisfied on U:

(1^+) $\det \begin{bmatrix} w_2^1 & w_4^1 \\ 1 & 1 \end{bmatrix} > 0$ together with (1_a^+):
$(1-\alpha^+)w_2^2 + \alpha^+ w_4^2 < 0;$

(1^-) $\det \begin{bmatrix} w_3^1 & w_1^1 \\ 1 & 1 \end{bmatrix} < 0$ together with (1_a^-):
$(1-\alpha^-)w_1^2 + \alpha^- w_3^2 > 0;$

(2^+) $\det \begin{bmatrix} w_4^2 & w_3^2 \\ 1 & 1 \end{bmatrix} < 0$ together with (2_a^+):
$(1-\beta^+)w_3^1 + \beta^+ w_4^1 < 0;$

(2^-) $\det \begin{bmatrix} w_1^2 & w_2^2 \\ 1 & 1 \end{bmatrix} > 0$ together with (2_a^-):
$(1-\beta^-)w_1^1 + \beta^- w_2^1 > 0;$

(c) If any of $(1^\pm)$ or $(2^\pm)$ is satisfied, then $(1_a^\pm)$ or $(2_a^\pm)$ must be satisfied as well.

- Condition (a) implies that the vector fields f_j, $j = 1, \ldots, 4$, must point towards at least one of $\Sigma_{1,2}$.

- Condition (a) implies that the vector fields f_j, $j = 1, \dots, 4$, must point towards at least one of $\Sigma_{1,2}$.
- Condition (b) guarantees that there is attractive sliding towards Σ along at least one of the $\Sigma_{1,2}^{\pm}$.

- Condition (a) implies that the vector fields f_j, $j = 1, \dots, 4$, must point towards at least one of $\Sigma_{1,2}$.
- Condition (b) guarantees that there is attractive sliding towards Σ along at least one of the $\Sigma_{1,2}^{\pm}$.
- Condition (c) states that if attractive sliding occurs along $\Sigma_{1,2}^{\pm}$ it must be towards Σ.

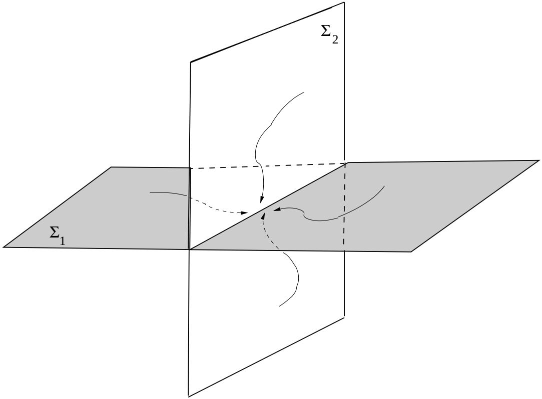
- Condition (a) implies that the vector fields f_j, $j = 1, \dots, 4$, must point towards at least one of $\Sigma_{1,2}$.
- Condition (b) guarantees that there is attractive sliding towards Σ along at least one of the $\Sigma^{\pm}_{1,2}$.
- Condition (c) states that if attractive sliding occurs along $\Sigma^{\pm}_{1,2}$ it must be towards Σ.
- There may be repulsive sliding along $\Sigma^{\pm}_{1,2}$, but –if so– it must be towards Σ.

- Condition (a) implies that the vector fields f_j, $j = 1, \ldots, 4$, must point towards at least one of $\Sigma_{1,2}$.
- Condition (b) guarantees that there is attractive sliding towards Σ along at least one of the $\Sigma^{\pm}_{1,2}$.
- Condition (c) states that if attractive sliding occurs along $\Sigma^{\pm}_{1,2}$ it must be towards Σ.
- There may be repulsive sliding along $\Sigma^{\pm}_{1,2}$, but –if so– it must be towards Σ.

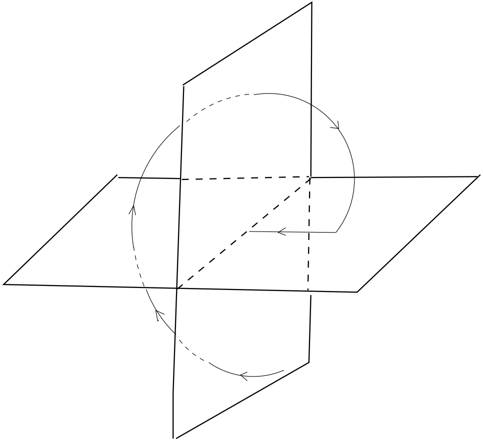
Lemma

If Σ is attractive through sliding then solution trajectories from U_Σ reach Σ in finite time.

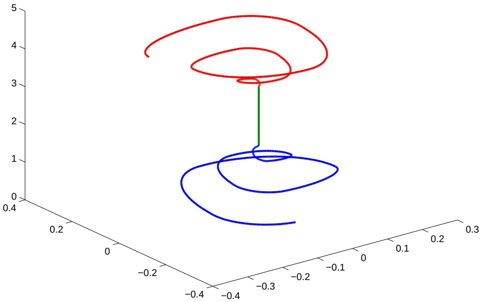
Example: Nodally attractive



Example: partially attractive through sliding



Spiral attractivity of Σ



- Spiral Attractivity of Σ (clockwise), characterized by this number (CNSNS 2015)

$$\mu \;=\; \frac{w_1^2(x)w_3^1(x)w_4^2(x)w_2^1(x)}{w_1^1(x)w_3^2(x)w_4^1(x)w_2^2(x)}\,, \quad x\in\Sigma\ .$$

- Spiral Attractivity of Σ (clockwise), characterized by this number (CNSNS 2015)

$$\mu \;=\; \frac{w_1^2(x)w_3^1(x)w_4^2(x)w_2^1(x)}{w_1^1(x)w_3^2(x)w_4^1(x)w_2^2(x)}\,,\;\; x \in \Sigma\;.$$

- Under appropriate assumptions, have $\mu > 0$, and if $\mu \le \mu_0 < 1$, then Σ (and sliding regime on Σ) is **spirally attractive**, and Σ is reached in finite time. [μ has a similar flavor to a Floquet multiplier.]

- Spiral Attractivity of Σ (clockwise), characterized by this number (CNSNS 2015)

$$\mu \;=\; \frac{w_1^2(x)w_3^1(x)w_4^2(x)w_2^1(x)}{w_1^1(x)w_3^2(x)w_4^1(x)w_2^2(x)}\,, \;\; x \in \Sigma \;.$$

- Under appropriate assumptions, have $\mu > 0$, and if $\mu \leq \mu_0 < 1$, then Σ (and sliding regime on Σ) is **spirally attractive**, and Σ is reached in finite time. [μ has a similar flavor to a Floquet multiplier.]
- Of course, there is a counterpart for the counterclockwise case.

Exit Points

We differentiate between three types, always within the framework of a generic first derivative theory.

- *Tangential Exit Points*. Those values $x \in \Sigma$ where one (and just one) of the $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is itself tangent to Σ. The corresponding $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is an *exit vector field*.

Exit Points

We differentiate between three types, always within the framework of a generic first derivative theory.

- *Tangential Exit Points.* Those values $x \in \Sigma$ where one (and just one) of the $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is itself tangent to Σ. The corresponding $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is an *exit vector field*.
- *Non-tangential Exit Points.* Those values $x \in \Sigma$ where one (and just one) of the f_i's is tangent to either Σ_1 or Σ_2. The f_i is an exit vector field.

Exit Points

We differentiate between three types, always within the framework of a generic first derivative theory.

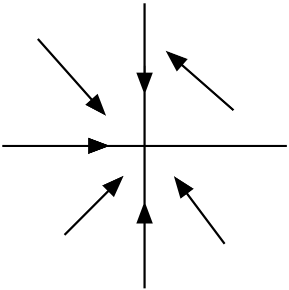
- *Tangential Exit Points*. Those values $x \in \Sigma$ where one (and just one) of the $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is itself tangent to Σ. The corresponding $f_{\mathsf{F}}{}^{\pm}_{1,2}$ is an *exit vector field*.
- *Non-tangential Exit Points*. Those values $x \in \Sigma$ where one (and just one) of the f_i's is tangent to either Σ_1 or Σ_2. The f_i is an exit vector field.
- *Spiral Exit Point*. One has $\mu = 1$. Now, there is no preferred direction along which one should leave Σ. Any of f_1, f_2, f_3, f_4, is a possible choice. [Similar to repulsive sliding on a co-dimension 1 surface].

Exit Points

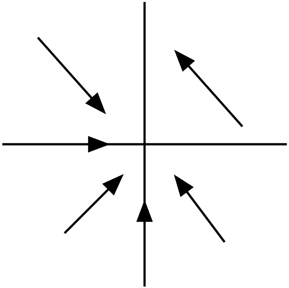
We differentiate between three types, always within the framework of a generic first derivative theory.

- *Tangential Exit Points*. Those values $x \in \Sigma$ where one (and just one) of the $f_{\mathsf{F}_{1,2}^{\pm}}$ is itself tangent to Σ. The corresponding $f_{\mathsf{F}_{1,2}^{\pm}}$ is an *exit vector field*.
- *Non-tangential Exit Points*. Those values $x \in \Sigma$ where one (and just one) of the f_i's is tangent to either Σ_1 or Σ_2. The f_i is an exit vector field.
- *Spiral Exit Point*. One has $\mu = 1$. Now, there is no preferred direction along which one should leave Σ. Any of f_1, f_2, f_3, f_4, is a possible choice. [Similar to repulsive sliding on a co-dimension 1 surface].
- **NB**: All of these exit points can be detected by looking at the entries of W.

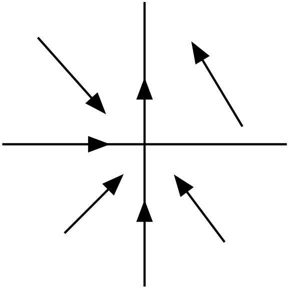
Tangential exit point.



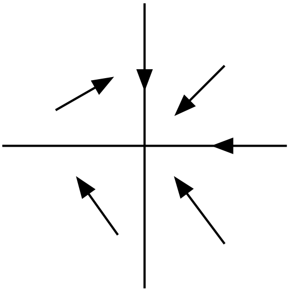
Tangential exit point.



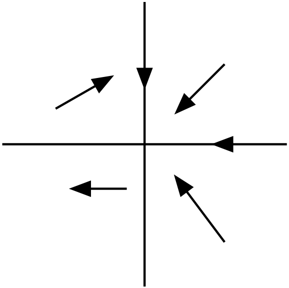
Tangential exit point.



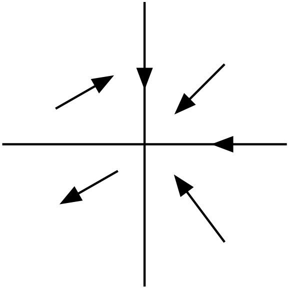
Non-tangential exit point.



Non-tangential exit point.



Non-tangential exit point.



Sliding vector field

How do the bilinear interpolant and moments vector fields relate to attractivity of Σ?

(D-Elia-Lopez, JDE, 2013)

The bilinear vector field is well defined, smoothly varying, both when Σ is attractive through sliding, and spirally. **But**, it does not (in general) align smoothly with an exit vector field at tangential exit points.

(D-Difonzo, JDDE-2014)

The moments vector field is well defined, smoothly varying, both when Σ is attractive through sliding, and spirally. **And**, it aligns smoothly with the exit vector field at tangential exit points.

- Neither of them can smoothly align with a non-tangential exit vector field.
- Extension of moments method gives unique sliding vector field also for (nodally) attractive manifolds of higher codimension (multilinear interpolant does not).

- Impact of ambiguity? Look at problems in $\mathbb{R}^3$.

- Impact of ambiguity? Look at problems in $\mathbb{R}^3$.
- Consider the case in which $\Sigma = \Sigma_1 \cap \Sigma_2$ in $\mathbb{R}^3$, hence, Σ is a (piece of a) curve and –by tangency– all Filippov vector fields on Σ are parallel, though have different norms.

Sliding in $\mathbb{R}^3$

- Impact of ambiguity? Look at problems in $\mathbb{R}^3$.
- Consider the case in which $\Sigma = \Sigma_1 \cap \Sigma_2$ in $\mathbb{R}^3$, hence, Σ is a (piece of a) curve and –by tangency– all Filippov vector fields on Σ are parallel, though have different norms.
- Assume that exit points are tangential and that if an exit point is reached, then all trajectories exit from Σ. Note that not all trajectories necessarily exit smoothly (this depends on the particular sliding vector field).

- Impact of ambiguity? Look at problems in $\mathbb{R}^3$.
- Consider the case in which $\Sigma = \Sigma_1 \cap \Sigma_2$ in $\mathbb{R}^3$, hence, Σ is a (piece of a) curve and –by tangency– all Filippov vector fields on Σ are parallel, though have different norms.
- Assume that exit points are tangential and that if an exit point is reached, then all trajectories exit from Σ. Note that not all trajectories necessarily exit smoothly (this depends on the particular sliding vector field).
- Further, assume that no Filippov vector field f_{F} (in the convex hull of the f_i's) has an equilibrium on Σ.

Theorem (D-Elia-Lopez, JNLS 2015)

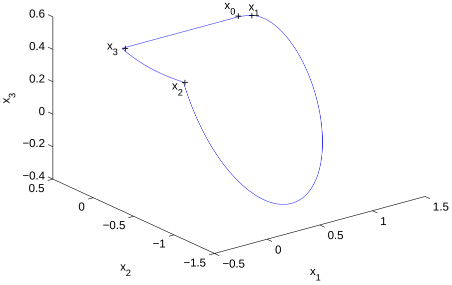
With previous assumptions, the systems $\dot{x} = f_F(x)$, with $f_F(x)$ any smooth Filippov sliding vector field, are all orbitally equivalent.

- A reparametrization of time has taken place, solutions associated to different sliding vector fields are tracing the same orbit, but at different speeds.

- Impact of previous equivalence in the case of periodic orbits.

- Impact of previous equivalence in the case of periodic orbits.
- Suppose for a given choice of vector field f_{F} on Σ, the associated trajectory slides on a portion of it, leaves it at a tangential exit point, then eventually returns to it, after sliding on parts of $\Sigma^{\pm}_{1,2}$, so that altogether it traces a periodic orbit γ, a portion of which is on Σ.

- Impact of previous equivalence in the case of periodic orbits.
- Suppose for a given choice of vector field f_{F} on Σ, the associated trajectory slides on a portion of it, leaves it at a tangential exit point, then eventually returns to it, after sliding on parts of $\Sigma_{1,2}^{\pm}$, so that altogether it traces a periodic orbit γ, a portion of which is on Σ.



- Assume that there are *no equilibria* for any possible Filippov sliding vector field on Σ, and that these sliding vector fields *all exit* at the same tangential exit point. Does it matter which sliding vector field one selects?

- Assume that there are *no equilibria* for any possible Filippov sliding vector field on Σ, and that these sliding vector fields *all exit* at the same tangential exit point. Does it matter which sliding vector field one selects?
- The answer to this question is not immediate, since not all choices of sliding on Σ guarantee smooth exits.
  However, the stability properties of the periodic orbit γ are independent of how one selects a (smooth) sliding vector field on Σ.

- Assume that there are *no equilibria* for any possible Filippov sliding vector field on Σ, and that these sliding vector fields *all exit* at the same tangential exit point. Does it matter which sliding vector field one selects?
- The answer to this question is not immediate, since not all choices of sliding on Σ guarantee smooth exits.
  However, the stability properties of the periodic orbit γ are independent of how one selects a (smooth) sliding vector field on Σ.

Theorem

Under the previous assumptions, the Floquet multipliers of the linearized trajectories are the same, regardless of how we slide on Σ. In the previous scenario, two of them will be equal to 0 and one equal to 1. [Super-stable].

Final considerations: what should happen?

Final questions (D, Elia, 2015) ... .

- Given the ambiguity in selecting a specific sliding vector field, can one at least say what **should happen** to sliding trajectories, particularly insofar as their behavior when they reach "exit points?"

Final questions (D, Elia, 2015) ... .

- Given the ambiguity in selecting a specific sliding vector field, can one at least say what **should happen** to sliding trajectories, particularly insofar as their behavior when they reach "exit points?"
- What happens to trajectories of the smooth globally regularized problem?

$$\begin{aligned}\dot{x} =&(1-\alpha_{\epsilon_1}(h_1(x)))[(1-\beta_{\epsilon_2}(h_2(x)))f_1+\beta_{\epsilon_2}(h_2(x))f_2(x)]+\\ &\alpha_{\epsilon_1}(h_1(x))[(1-\beta_{\epsilon_2}(h_2(x)))f_3++\beta_{\epsilon_2}(h_2(x))f_4(x)]\,.\end{aligned}$$

Final questions (D, Elia, 2015) ... .

- Given the ambiguity in selecting a specific sliding vector field, can one at least say what **should happen** to sliding trajectories, particularly insofar as their behavior when they reach "exit points?"
- What happens to trajectories of the smooth globally regularized problem?

$$\begin{aligned}\dot{x} =& (1-\alpha_{\epsilon_1}(h_1(x)))[(1-\beta_{\epsilon_2}(h_2(x)))f_1 + \beta_{\epsilon_2}(h_2(x))f_2(x)] + \\ & \alpha_{\epsilon_1}(h_1(x))[(1-\beta_{\epsilon_2}(h_2(x)))f_3 + +\beta_{\epsilon_2}(h_2(x))f_4(x)]\,.\end{aligned}$$

- Do regularized solutions converge to the "bilinear" interpolant sliding solution when the parameter(s) go to 0?

- In $\mathbb{R}^3$, with $h_1(x) = x_1$, $h_2(x) = x_2$, through singular perturbation analysis of slow/fast systems ...

- In $\mathbb{R}^3$, with $h_1(x) = x_1$, $h_2(x) = x_2$, through singular perturbation analysis of slow/fast systems . . .

If $(\alpha^*(x_3), \beta^*(x_3))$ (solution of nonlinear system for bilinear method) is an asymptotically stable equilibrium of the fast system, and initial condition is in the basin of attraction of this equilibrium, then the answer is yes.
Further, as long as the solution of the fast system remains asymptotically stable, then the solution of the regularized system remains in a neighborhood of Σ.
Finally, if Σ is attractive along two sub-surfaces, then $(\alpha^*(x_3), \beta^*(x_3))$ is as-stable.

- In $\mathbb{R}^3$, with $h_1(x) = x_1$, $h_2(x) = x_2$, through singular perturbation analysis of slow/fast systems ...

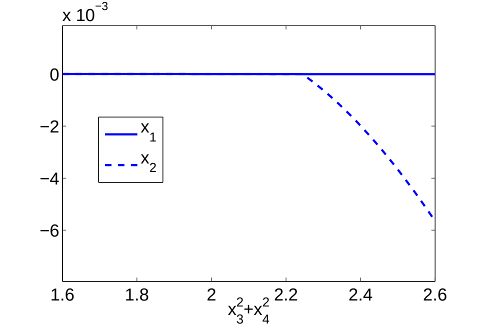
If $(\alpha^*(x_3), \beta^*(x_3))$ (solution of nonlinear system for bilinear method) is an asymptotically stable equilibrium of the fast system, and initial condition is in the basin of attraction of this equilibrium, then the answer is yes.
Further, as long as the solution of the fast system remains asymptotically stable, then the solution of the regularized system remains in a neighborhood of Σ.
Finally, if Σ is attractive along two sub-surfaces, then $(\alpha^*(x_3), \beta^*(x_3))$ is as-stable.

- However, the equilibrium of the fast system may be as-stable even if Σ is not attractive. Thus, regularized solution may converge to a sliding solution even if Σ is not attractive!

Example (Tangential Exits)

In $\mathbb{R}^4$, Σ is the (x_3, x_4)-plane. The circle $x_3^2 + x_4^2 = 2$ is made up by tangential exit points.

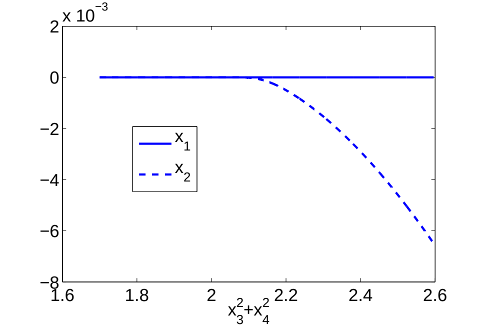
Regularized Integration: $\epsilon_1 = \epsilon_2 = 10^{-5}$



Example (Tangential Exits)

In $\mathbb{R}^4$, Σ is the (x_3, x_4)-plane. The circle $x_3^2 + x_4^2 = 2$ is made up by tangential exit points.

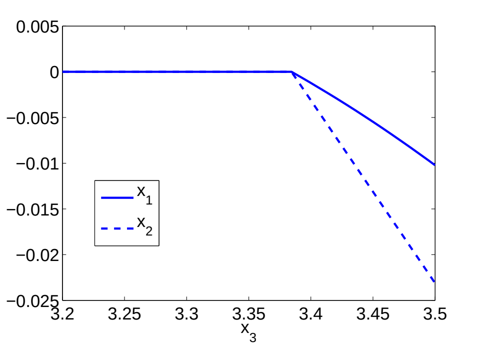
Random Euler: steps centered at $\tau = 10^{-6}$



Example (Non Tangential Exit)

Σ is the x_3 axis, and $x_3 = 3$ is a non-tangential exit point.

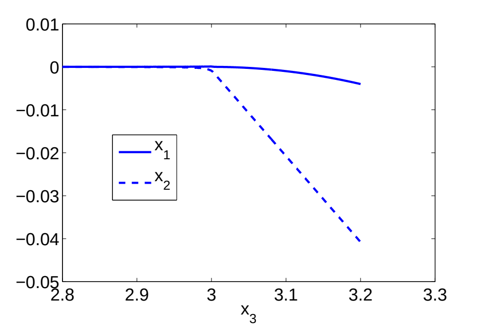
Regularized Integration: $\epsilon_1 = \epsilon_2 = 10^{-6}$



Example (Non Tangential Exit)

Σ is the x_3 axis, and $x_3 = 3$ is a non-tangential exit point.

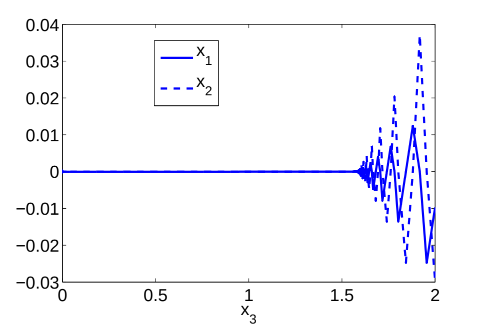
Random Euler: steps centered at $\tau = 10^{-5}$



Example (Spiral Exit)

Σ is the x_3 axis, and $x_3 = 1$ is a spiral exit point.

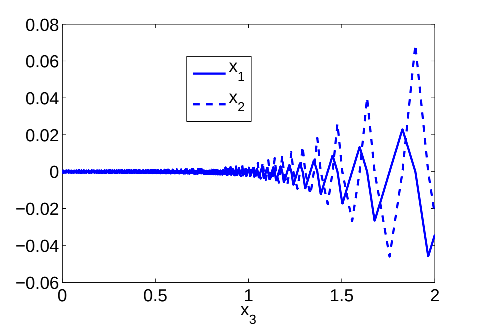
Regularized Integration: $\epsilon_1 = \epsilon_2 = 10^{-5}$



Example (Spiral Exit)

Σ is the x_3 axis, and $x_3 = 1$ is a spiral exit point.

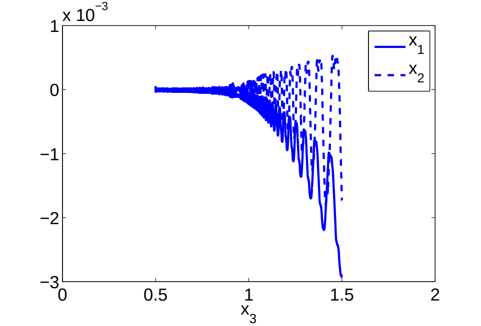
Regularized Integration: $\epsilon_1 = 10^{-4}$, $\epsilon_2 = 10^{-3}$



Example (Spiral Exit)

Σ is the x_3 axis, and $x_3 = 1$ is a spiral exit point.

Random Euler: steps centered at $\tau = 10^{-5}$



Summary

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.

Summary

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.
- Regularization methods have pitfalls deriving from having **superimposed** their own **dynamics** to that of the PWS system. Further, even slight changes in the regularization parameters render different outcomes.

Summary

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.
- Regularization methods have pitfalls deriving from having **superimposed** their own **dynamics** to that of the PWS system. Further, even slight changes in the regularization parameters render different outcomes.
- "Safe" conclusions:

Summary

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.
- Regularization methods have pitfalls deriving from having **superimposed** their own **dynamics** to that of the PWS system. Further, even slight changes in the regularization parameters render different outcomes.
- "Safe" conclusions:
  (a) be leery of techniques whose dynamics are not compatible with PWS system;

Summary

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.
- Regularization methods have pitfalls deriving from having **superimposed** their own **dynamics** to that of the PWS system. Further, even slight changes in the regularization parameters render different outcomes.
- "Safe" conclusions:
  - (a) be leery of techniques whose dynamics are not compatible with PWS system;
  - (b) strive to make statements on dynamics which do not depend on the specific choice of selection. Typically, these are going to be coarse results.

- Random Euler behavior supports our insight into the key interplay between attractivity and loss of it.
- Regularization methods have pitfalls deriving from having **superimposed** their own **dynamics** to that of the PWS system. Further, even slight changes in the regularization parameters render different outcomes.
- "Safe" conclusions:
  - (a) be leery of techniques whose dynamics are not compatible with PWS system;
  - (b) strive to make statements on dynamics which do not depend on the specific choice of selection. Typically, these are going to be coarse results.
- More personal ... if need to simulate problem, use sliding vector fields able to detect smooth tangential exits while monitoring other first order exits.