Structural Stability of Uniform Attractors: Topological and Geometrical

Alexandre N. Carvalho
Instituto de **C**iências **M**atemáticas e de **C**omputação
University of **S**ão **P**aulo
São Carlos SP, Brazil

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Seting of the problem

We are interested in the asymptotic behavior of solutions of initial value problems of the form

$$\begin{cases} \dot{u} = f(t,u), \ \ t > \tau \\ u(\tau) = u_0 \in X, \end{cases} \tag{1}$$

where X is a Banach space, $f : J \times D \subset \mathbb{R} \times X \to X$ ($J = \mathbb{R}$ or $\mathbb{R}^+$, $\bar{D} = X$) is a map belonging to some metric space $\mathcal{C}$. Assume that, for each $u_0 \in X$ and $\tau \in J$ the solution of (1) is defined for all $t \geq \tau$; that is,

- for each $u_0 \in X$, there is a unique continuous function $[\tau, \infty) \ni t \mapsto u(t,\tau,f,u_0) \in X$ 'satisfying' (1).

If f is **time independent**, $u(t, \tau, f, u_0) = u(t - \tau, 0, f, u_0)$ and the asymptotic behavior of solutions can be seen

- Making the $t \to \infty$ (seeing what happens to the state at the final time t when t is driven further to the future)
  or
- Making $\tau \to -\infty$ (seeing what happens to the state at fixed time t when the initial time τ is driven further to the past).

On the other hand, if f is **time dependent** these two asymptotics give rise to completely different scenarios.

- The asymptotics w.r.t. the elapsed time $t-\tau$ (when $t-\tau \to \infty$) is called **forwards dynamics** whereas the asymptotics w.r.t. τ (when $\tau \to -\infty$ and t is arbitrary but fixed) is called **pullback dynamics** and they are in general unrelated.

It is natural that they be unrelated (in the non-autonomous case) for the set of vector fields driving the solution may be completely different.

Based on this, two main approaches have been developed in order to study attractors for (1):

- The pullback attractor ([13, 26]): An invariant family of sets for the evolution process which is pullback (in general, not forwards) attracting and
- The uniform attractor ([16]): a non-invariant minimal compact set attracting bounded subsets of X forwards in time (uniformly w.r.t. the initial time τ).

- The **pullback attractors** main feature is the invariance and that gives it an intrinsic relation with the associated evolution process.
- The **uniform attractors** main feature is the forwards attraction which the pullback attractors will not, in general, possess.

- Exploiting the invariance properties of the pullback attractors one can prove their continuity (upper and lower semicontinuity) and characterization in terms of its internal structures ([2, 5, 9, 10, 13])

- The same kind of study of inner structures had no parallel for uniform attractors due to its dissociation from the original dynamical system.

Main Objective

- Our aim is to 'relate' pullback and uniform attractors, to be able to give a description of the uniform attractor and to understand some of its dynamical structures.
- That will allow us to talk about continuity and topological structural stability of uniform attractors, for a non-autonomous perturbation of a semigroup.
- The geometric structural stability of uniform attractors can also be accomplished (under suitable assumptions - ongoing research).

Non-autonomous semiflows and attractors

For f in $\mathcal{C}$ (for $J = \mathbb{R}^+$),

$$\Sigma = \text{closure of } \{f(s+\cdot,\cdot) : \ s \in \mathbb{R}^+\} \ \text{ w.r.t. the metric } \rho \text{ of } \mathcal{C}.$$

For each $t \geq 0$, define the shift operator $\theta_t : \Sigma \to \Sigma$ by

$$\theta_t f(\cdot,\cdot) = f(t+\cdot,\cdot).$$

The semigroup $\{\theta_t : t \geq 0\}$ is called the **driving semigroup** and we assume that it has a global attractor $\mathscr{S}$ in Σ.

For each $\sigma \in \Sigma$, consider the semiflow

$$\mathbb{R}^+ \times X \ni (t, u_0) \mapsto \varphi(t,\sigma)u_0 \in X$$

where, for each $u_0 \in X$, $\mathbb{R}^+ \ni t \mapsto \varphi(t,\sigma)u_0 \in X$ is the solution of the initial value problem

$$\begin{aligned} \dot{u} &= \sigma(t,u), t > 0, \\ u(0) &= u_0 \in X. \end{aligned} \tag{2}$$

The family of maps $(t,\sigma) \in \mathbb{R}^+ \times \Sigma \mapsto \varphi(t,\sigma) \in \mathcal{C}(X)$, 'satisfy'

- $\varphi(0,\sigma) = \mathrm{Id}_X$ for all $\sigma \in \Sigma$,
- $\mathbb{R}^+ \times \Sigma \ni (t,\sigma) \mapsto \varphi(t,\sigma)u \in X$ is continuous, and
- for all $t, s \geq 0$ and $\sigma \in \Sigma$, $\varphi(t+s,\sigma) = \varphi(t,\theta_s\sigma)\varphi(s,\sigma)$, the 'cocycle property'.

The association with the differential equation is no longer required.

Hereafter, we use the pair $(\varphi, \theta)_{(X,\Sigma)}$, called a **non-autonomous dynamical system** (NDS), to study the asymptotics of (1).

Now we define the associated **skew-product semigroup** ([31, 32]) on $\mathbb{X} = X \times \Sigma$ (with the product metric) by setting

$$\Pi(t)(u, \sigma) = (\varphi(t, \sigma)u, \theta_t\sigma), \ \ t \geq 0.$$

The semigroup property of θ_t and the cocycle property of φ ensure that $\Pi(\cdot)$ satisfies the semigroup property.

Thus, to a non-autonomous differential equation, we associate the dynamical systems:

(a) The *driving semigroup* $\{\theta_t : t \geq 0\}$ on Σ,
(c) The *skew-product semigroup* $\{\Pi(t) : t \geq 0\}$ on $X \times \Sigma$,
(b) The NDS $(\varphi, \theta)_{(X,\Sigma)}$ and
(d) The evolution process $S(t,\tau)u_0 = \varphi(t-\tau, \theta_\tau f)u_0$, when $J = \mathbb{R}$.

Each of these dynamical systems may have an associated attractor:

(i) The global attractor $\mathscr{S}$ for the driving semigroup θ_t,

(ii) The global attractor $\mathbb{A}$ for the skew-product semiflow $\Pi(t)$,

(iv) The pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$ for $S(t,\tau)$,

(v) The uniform attractor $\mathcal{A}$ for the NDS $(\varphi,\theta)_{(X,\Sigma)}$,

Relationship between attractors

Definition ([15, 16, 35])

The NDS $(\varphi, \theta)_{(X,\Sigma)}$ is **uniformly asymptotically compact** (UAC) if there exists a compact set $K \subset X$ such that

$$\lim_{t\to\infty} \sup_{\sigma\in\Upsilon} \mathrm{dist}_H(\varphi(t,\sigma)B, K) = 0, \tag{3}$$

for every bounded subset B of X and bounded set Υ of Σ.

Theorem

The NDS $(\varphi, \theta)_{(X,\Sigma)}$ is UAC and θ_t has a global attractor $\mathscr{S}$ if and only if $\{\Pi(t) : t \geq 0\}$ has a global attractor $\mathbb{A}$. In this case the NDS $(\varphi, \theta)_{(X,\Sigma)}$ has a uniform attractor $\mathcal{A}$ and $\mathcal{A} = \Pi_X \mathbb{A}$.

Theorem (Theorem 2.7 in [5])

Assume that $\Pi(t)$ *has a global attractor* $\mathbb{A}$ *(hence* θ_t *has a global attractor* $\mathscr{S}$*). If* $\eta : \mathbb{R} \to \mathscr{S}$ *is a global solution for* θ_t*, the process*

$$T_\eta(t,s)u = \varphi(t-s,\eta(s))u, \ u \in X, \ t \geq s,$$

has a pullback attractor $\{A_\eta(t) : t \in \mathbb{R}\}$ *and*

$$\mathbb{A} = \bigcup_{\eta} \bigcup_{t\in\mathbb{R}} A_\eta(t) \times \{\eta(t)\},$$

where the first union is taken over all global solutions $\eta : \mathbb{R} \to \mathscr{S}$ *of* θ_t*. Moreover, the NDS* $(\varphi,\theta)_{(X,\Sigma)}$ *has a uniform attractor* $\mathcal{A}$ *with*

$$\mathcal{A} = \Pi_X \mathbb{A} = \bigcup_{\eta} \bigcup_{t\in\mathbb{R}} \mathcal{A}_\eta(t).$$

Characterization of the uniform attractor

Definition (lifted-invariant sets)

If $\eta : \mathbb{R} \to \mathscr{S}$ is a global solution of θ_t and $\xi : \mathbb{R} \to X$ is such that

$$\varphi(t-s, \eta(s))\xi(s) = \xi(t), \ \forall t \geqslant s, \text{ and } \xi(0) = x,$$

we say that ξ is a **global solution through** x **on** η.

A set $\mathcal{M} \subset X$ is **lifted-invariant** if for each $x \in \mathcal{M}$ there is a global solution $\eta : \mathbb{R} \to \mathscr{S}$ of θ_t and a global solution $\xi : \mathbb{R} \to X$ through x on η such that $\xi(\mathbb{R}) \subset \mathcal{M}$.

If there is an $\epsilon > 0$ such that $\mathcal{M}$ is the maximal lifted-invariant set in $\mathcal{O}_\epsilon(\mathcal{M})$, we say that $\mathcal{M}$ is an **isolated lifted-invariant set**.

Proposition (Characterization of Uniform Attractors)

Under the assumptions that NDS $(\varphi, \theta)_{(X,\Sigma)}$ is UAC and θ_t has a global attractor, the uniform attractor $\mathcal{A}$ is the **maximal bounded isolated lifted-invariant set of** X.

Dynamically gradient uniform attractors

Definition

We say that a family $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ of subsets of X is a **disjoint family of isolated lifted-invariant sets** if each Ξ_i is an isolated lifted-invariant set and there exists $\epsilon > 0$ such that $\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing$ if $1 \leqslant i < j \leqslant n$.

Definition (Homoclinic structure)

Let $(\varphi, \theta)_{(X,\Sigma)}$ be a NDS with a uniform attractor $\mathcal{A}$ and a disjoint family of isolated lifted-invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ in $\mathcal{A}$.

A **homoclinic structure** in $\mathbf{\Xi}$ is a subset $\{\Xi_{\ell_1}, \cdots, \Xi_{\ell_k}\}$, together with global solutions $\xi_i : \mathbb{R} \to \mathcal{A}$ through x_i on η_i such that

$$\Xi_{\ell_i} \overset{t \to -\infty}{\longleftarrow} \xi_i(t) \overset{t \to \infty}{\longrightarrow} \Xi_{\ell_{i+1}},$$

where $\Xi_{\ell_{k+1}} := \Xi_{\ell_1}$ and if $k = 1$ there exists $\epsilon > 0$ such that

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\xi_1(t), \mathcal{O}_\epsilon(\Xi_{\ell_1})) > 0.$$

Definition (Dynamically Gradient NDS)

A NDS $(\varphi, \theta)_{(X,\Sigma)}$ with a uniform attractor $\mathcal{A}$ is said to be **dynamically gradient** relatively to a disjoint family of isolated lifted-invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ if

- (GU1) for $x \in \mathcal{A}$ and global solution $\xi : \mathbb{R} \to \mathcal{A}$ through x on η, we have that

$$\Xi_j \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_i,$$

  for some $1 \leqslant i, j \leqslant n$.

- (GU2) There are no homoclinic structures in $\boldsymbol{\Xi}$.

As a direct consequence of (GU1) and (GU2), we have that

Proposition

If a NDS $(\varphi, \theta)_{(X,\Sigma)}$ *with a uniform attractor* $\mathcal{A}$ *is dynamically gradient relatively to a disjoint family of isolated lifted-invariant sets* $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$*, then the associated skew product semiflow* $\{\Pi(t) : t \geqslant 0\}$ *is dynamically gradient relatively to the disjoint family of isolated invariant sets* $\mathfrak{E} = \{\mathbb{E}_1, \cdots, \mathbb{E}_n\}$*, with* $\mathbb{E}_i = \{(x, \sigma) \in \mathbb{A} : x \in \Xi_i\}$.

Conversely, if $\{\Pi(t) : t \geqslant 0\}$ *is dynamically gradient relatively to a disjoint family of isolated invariant sets* $\mathfrak{E} = \{\mathbb{E}_1, \cdots, \mathbb{E}_n\}$ *and*

$$\Xi_i \cap \Xi_j = \varnothing,\ 1 \leqslant i < j \leqslant n, \qquad \Xi_i := \Pi_X \mathbb{E}_i,\ 1 \leq i \leq n,$$

then $(\varphi, \theta)_{(X,\Sigma)}$ *is dynamically gradient relatively to the disjoint family of isolated lifted-invariant sets* $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$.

Topological Structural Stability

Theorem

Let $(\varphi_\nu, \theta)_{(X,\Sigma_\nu)}$ be a NDS and $\{\Pi_\nu(t) : t \geq 0\}$ be the associated skew-product semiflows, $\nu \in [0,1]$, and assume that

(a) *For each $\nu \in [0,1]$, $\{\Pi_\nu(t) : t \geq 0\}$ has a global attractor $\mathbb{A}_\nu$ and $\overline{\cup_{\nu\in[0,1]}\mathbb{A}_\nu}$ is compact. Hence $(\varphi_\nu, \theta)_{(X,\Sigma_\nu)}$ has a uniform attractor $\mathcal{A}_\nu = \Pi_X \mathbb{A}_\nu$ and θ_t has a global attractor $\mathscr{S}_\nu$ in Σ_ν.*

(b) *$\Sigma_0 = \{\sigma_0\}$ and $\varphi_0(t, \sigma_0) \equiv S_0(t)$, where $\{S_0(t) : t \geqslant 0\}$ is a dynamically gradient semigroup relatively to a disjoint family of isolated invariants $\mathcal{E}_0 = \{E_{0,1}, \cdots, E_{0,n}\}$.*

(c) *For each $\nu \in [0,1]$, there is a disjoint family of isolated lifted-invariant sets $\mathcal{E}_\nu = \{E_{\nu,1}, \cdots, E_{\nu,n}\} \subset \mathcal{A}_\nu$ such that*

$$\max_{1\leqslant i\leqslant n} \mathrm{Dist_H}(E_{\nu,i}, E_{0,i}) \to 0, \text{ as } \nu \to 0^+.$$

(d) $\sup_{\sigma_\nu \in \Sigma_\nu} [\mathrm{dist}(\varphi_\nu(t,\sigma_\nu)x, \varphi_0(t,\sigma_0)x) + \rho(\sigma_\nu,\sigma_0)] \stackrel{\nu\to 0^+}{\longrightarrow} 0$, uniformly in compact subsets of $\mathbb{R}^+ \times X$.

(e) There is a $\delta > 0$ such that if $\mathbb{R} \ni t \mapsto \xi_\nu(t) \in \mathcal{A}_\nu$ is a global solution through x on η, $t_0 \in \mathbb{R}$ and $\mathrm{dist}(\xi_\nu(t), E_{\nu,i}) \leqslant \delta$, for all $t \leqslant t_0$ ($t \geqslant t_0$), then $\mathrm{dist}(\xi_\nu(t), E_{\nu,i}) \to 0$ as $t \to -\infty$ ($t \to \infty$).

Then there exists $\nu_0 > 0$ such that the NDS $(\varphi_\nu, \theta)_{(X,\Sigma_\nu)}$ is a dynamically gradient relatively to the disjoint family of isolated lifted-invariant sets $\mathcal{E}_\nu = \{E_{\nu,1}, \cdots, E_{\nu,n}\}$, for all $\nu \in [0, \nu_0]$.

Aplications - Domains with a handle

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $P, Q \in \bar{\Omega}$. Consider the problem

$$\begin{aligned}
&u_t = \Delta u + f(u),\ t > 0,\ x \in \Omega,\\
&\frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega,\\
&v_t = v_{ss} + g(v),\ t > 0,\ s \in (0,1),\\
&v(t,0) = u(t,P),\ v(t,1) = u(t,Q),\ t > 0,\\
&u(0,x) = u_0(x),\ x \in \Omega, \quad v(0,s) = v_0(s),\ s \in (0,1)
\end{aligned} \tag{4}$$

where $u_0 \in W^{1,p}(\Omega)$, $v_0 \in W^{1,p}(0,1)$, $p > N$ and $f, g : \mathbb{R} \to \mathbb{R}$ are C^1 functions such that $\limsup\limits_{u\to\infty} \frac{f(u)}{u} < 0$ and $\limsup\limits_{v\to\infty} \frac{g(v)}{v} < 0$. Assume that all equilibria of (4) are hyperbolic.

Aplications - Cascade System

Consider the problem

$$
\begin{aligned}
& u_t = u_{xx} + f(u),\ \ t>0,\ \ x\in(0,1),\\
& u_x(t,0) = u_x(t,1) = 0,\ \ t>0,\\
& v_{tt} + \beta v_t = v_{xx} + g(u,v),\ \ t>0,\ \ x\in(0,1),\\
& v_x(t,0) = v_x(t,1) = 0,\ \ t>0,\\
& u(0,x) = u_0(x),\ \ v(0,x) = v_0(x),\ \ v_t(0,x) = v_1(x),\ \ x\in(0,1),
\end{aligned}
\tag{5}
$$

where $u_0 \in H^1(0,1)$, $(v_0,v_1)\in H^1(0,1)\times L^2(0,1)$ and f,g are C^1 functions satisfying suitable growth conditions and dissipation conditions $\limsup_{u\to\infty}\frac{f(u)}{u} < 0$ and $\sup_{u\in\mathbb{R}}\limsup_{v\to\infty}\frac{g(u,v)}{v} < 0$. Assume that all equilibria of (5) are hyperbolic.

Aplication - ODE with Periodic Orbits

Consider the following autonomous equation in polar coordinates

$$\begin{cases} \dot{r} = -r(r-1)(r-2) \\ \dot{\theta} = 1 \end{cases} \tag{6}$$

which has, in the $(x, y)-$plane, $P_0 := (0, 0)$ as a stable fixed point and periodic orbits P_1 (unstable) and P_2 (stable).

The problem (6) generates a semigroup with a global attractor $\mathcal{A}_0 = \{(x, y) : \|(x, y)\| \leqslant 2\}$ which is dynamically gradient relatively to the family of isolated invariant sets $\mathbf{P} = \{P_0, P_1, P_2\}$.

For each $\eta \in [0,1]$, consider the perturbed problem

$$\begin{cases} \dot{r} = -r(r-1)(r-2) + f_\eta(t), \quad t \in \mathbb{R}^+, \\ \dot{\theta} = 1 \end{cases} \tag{7}$$

where $f_0 \equiv 0$, $f_\eta(t) \geq 0$, $\forall \eta,\ t$, $\sup\limits_{t,\eta} f'_\eta(t) < \infty$ and $\sup\limits_{t} f_\eta(t) \overset{\eta \to 0^+}{\longrightarrow} 0$.

It is easy to see that the skew-product semigroup associated to (7) has global attractor and that $\cup_{\nu\in[0,1]}\mathbb{A}_\nu$ is relatively compact.

Note that the region $A_{\alpha_1,\alpha_2} = \{(x, y) \in \mathbb{R}^2 : \alpha_1 \leqslant \|(x, y)\| \leqslant \alpha_2\}$, $1 < \alpha_1 < 2 < \alpha_2$, is positively invariant for any $\sigma_\eta \in \mathscr{S}_\eta$, η is small. Hence, if the restriction of the NDS to A_{α_1,α_2} has a uniform attractor $\mathbf{M}_{2,\eta}$.

Analogous reasoning can be used to prove the existence of isolated lifted-invariant sets $\mathbf{M}_{1,\eta}$ and $\mathbf{M}_{0,\eta}$ for the NDS associated to (7).

Theorem 3 implies that the NDS is dynamically gradient relatively to the disjoint family of lifted-invariant sets $\{\mathbf{M}_{0,\eta}, \mathbf{M}_{1,\eta}, \mathbf{M}_{2,\eta}\}$.

Thank you very much for your attention.

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