A multi-dimensional Birkhoff theorem for non-autonomous Tonelli Hamiltonians

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Background (1)

- M is a d-dimensional closed manifold;
- T^*M is endowed with the symplectic 2-form $\omega = dp \wedge dq = \sum dp_i \wedge dq_i$ where $p \in T_q^*M$;
- the Hamiltonian $H : T^*M \times \mathbb{T} \to \mathbb{R}$ is a C^2-function $(x, t) = (q, p, t) \mapsto H(x, t) = H_t(x)$;
- the Hamiltonian vectorfield X_{H_t} is defined by

$$\omega(X_{H_t}, \delta x) = dH_t \delta x \quad \text{i.e.} \quad X_{H_t} = \begin{pmatrix} \frac{\partial H_t}{\partial p} \\ -\frac{\partial H_t}{\partial q} \end{pmatrix}.$$

Observe that the time t map $\phi_H^{0,t}$ of a Hamiltonian vectorfield preserve the symplectic form ω, i.e. is a symplectic diffeomorphism.

Background (2)

Assume that H is Tonelli i.e.

- complete;
- $\frac{\partial^2 H_t}{\partial p^2}$ is positive definite;
- H_t is superlinear in the fiber direction

$$\forall A, \exists B; \|p\| \geq B \Longrightarrow H_t(q, p, t) \geq A\|p\|.$$

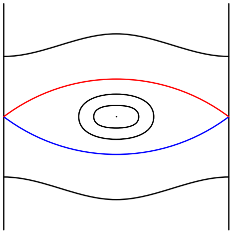
Examples

- close to a completely elliptic fixed point of a symplectic diffeomorphism, the normal form is $(\theta, r) \in \mathbb{T}^n \to \mathbb{R}^n \mapsto (\theta + \alpha + \beta.r, r) + \mathrm{small}$; if the symmetric matrix β is positive definite, we are in this case;
- idem close to an invariant C^∞ invariant Diophantine Lagrangian torus;
- Riemannian metrics.

Motivation: a result of Birkhoff for twist maps

Theorem (Birkhoff)
If a simple loop Γ that is not homotopic to a point is invariant by the time 1 map of a Tonelli Hamitonian of the 2-dimensional annulus $\mathbb{T} \times \mathbb{R}$, then Γ is the graph of Lipschitz map.

Remark It may happen that a simple loop that is homotopic to a point is invariant by a conservative twist map; see the case of the pendulum:



Lagrangian submanifolds (1)

In higher dimension, the notion of Lagrangian submanifold will replace the one of loop.

Definition

A submanifold $N \subset T^*M$ is Lagrangian if $\dim N = d$ and $\omega_{|TN} = 0$.

Examples

- In $T^*\mathbb{T}$, a loop is always Lagrangian;
- a vertical fiber T_q^*M is Lagrangian;
- the zero-section is Lagrangian;
- more generally, a C^1 graph is Lagrangian iff it is the graph of a closed 1-form: $\{(q, dS(q)); q \in M\}$;
- the stable or unstable manifold at a hyperbolic equilibrium is Lagrangian;
- for the so-called completely integrable systems the phase space is foliated by invariant Lagrangian tori;
- some of these tori remain after perturbation (KAM theory).

Lagrangian submanifolds (2)

- Consider a Lagrangian submanifold N. Then, preserving ω along N means just preserving 0. Hence a symplectic dynamics restricted to N can be anything;
- but some restricted dynamics force the submanifold to be Lagrangian:
  - a gradient dynamics for a Morse function on N;
  - M. Herman proved

Proposition *Let $N \subset T^*M$ be a d-dimensional submanifold invariant by a C^1 symplectic diffeomorphism F; if the restricted dynamics $F_{|N}$ is C^1-conjugated to an ergodic rotation of $\mathbb{T}^d$, then N is Lagrangian.*

Lagrangian submanifolds (3)

Question Let F be symplectic diffeomorphism of T^*M that is homotopic to Id and let N be a C^1 submanifold that is:

- invariant by F;
- homotopic to the zero section;
- such that the restricted dynamics $F_{|N}$ is minimal (i.e. all its orbits are dense in N).

Is N necessarily Lagrangian?

- The answer is yes for $d = 1, 2$.
- The answer is no if if you ask nothing for the restricted dynamics. Consider $H : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $H(\theta, r) = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2)$ and and

$$N = \{(\theta; \cos 2\pi\theta_3, \sin 2\pi\theta_3, 0); \theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3\}.$$

This last example can be modified in such a way that it is not a graph.

Motivation: a result of Birkhoff for twist maps

Theorem (Birkhoff)

If a simple loop Γ that is not homotopic to a point is invariant by the time 1 map of a Tonelli Hamitonian of the 2-dimensional annulus $\mathbb{T} \times \mathbb{R}$, then Γ is the graph of Lipschitz map.

Existing generalizations (1)

- Replace loop by Lagrangian submanifold;
- introduce some topological condition for the submanifold.

Theorem (Arnaud, 2010)
Let M be a closed manifold. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and let $N \subset T^*M$ be an invariant by the Hamiltonian flow Lagrangian submanifold that is Hamiltonianly isotopic to a Lagrangian graph. Then N is a Lagrangian graph.

Remark (1) This result was enlarged to the case of Lipschitz Lagrangian manifolds by P. Bernard and J. dos Santos.
(2) There were previous results of Bialy and Polterovich for Tonelli Hamiltonians on $T^*\mathbb{T}^d$; they assumed that the manifold is a Lagrangian torus that is homologous to the zero section and such that the restricted dynamics is chain recurrent (for example C^0-conjugated to a rotation).

Existing generalizations (2)

In 1989, Michel Herman proved a similar result for a manifold that is:

- compact and Lagrangian;
- with a Maslov class equal to 0;
- invariant by an exact symplectic twist map of $\mathbb{T}^d \times \mathbb{R}^d$ that is C^1-close enough to a completely integrable symplectic twist map;
- such that the restricted dynamics is chain recurrent.

Remark: Herman did not assume that the manifold is a torus, but had a strong hypothesis on the restricted dynamics.

Main result

Theorem (Arnaud & Venturelli, 2016)
Let M be a closed manifold, let $H : T^*M \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a Tonelli 1-time periodic Hamiltonian, and let $N \subset T^*M$ be a Lagrangian submanifold Hamiltonianly isotopic to a Lagrangian graph. If N is invariant by the time one map associated to H, then N is a Lagrangian graph.

Questions
Can the assumption Hamiltonian isotopic to the zero section be replaced by homotopic to the zero section, isotopic to the zero section?
Is the same result true for twist diffeomorphisms in all dimensions?

The proof: first step

In this part, we extend the Hamiltonian and the Lagrangian submanifold by adding two units to the dimension. $c \in \mathbb{R}$.

1st attempt

$$\mathcal{H}(q, p, \tau, E) = H(q, p, \tau) + E;$$

$$\mathcal{N}_c = \{\phi^t_{\mathcal{H}}(q, p, 0, c - H(q, p, 0)); (q, p) \in N; t \in [0, 1]\}.$$

Problem $\mathcal{N}_c$ has a boundary. And if we identify $\tau = 0$ with $\tau = 1$, we cannot extend the isotopy.

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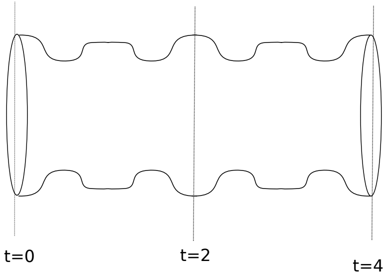
2nd attempt Multiplying H by some function $\eta(t)$, we obtain a 2-periodic Hamiltonian flow X_K whose integral curves are reparametrizations of segments of integral curves of X_H. The orbits slow down and turn back at integer times.



The proof: second step (1)

As the extended submanifold $\mathcal{N}$ is Hamiltonianly isotopic to the zero section of $T^*\mathcal{M}_2 = T^*(M \times \mathbb{R}/2\mathbb{Z})$, it admits a generating function.

Vague idea of generating function If your manifold is not the graph of dS for some $S : \mathcal{M}_2 \to \mathbb{R}$, add some variables in such a way it becomes a graph.

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Definition

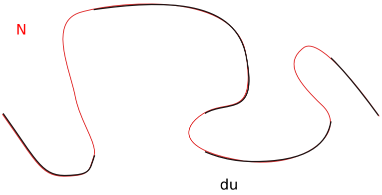
The C^2 function $S : (z, \xi) \in \mathcal{M}_2 \times \mathbb{R}^k \to S(z, \xi) \in \mathbb{R}$ is a generating function for $\mathcal{N}$ if

- 0 is a regular value for $\frac{\partial S}{\partial \xi}$;
- $i_S : \Sigma_S = \{\frac{\partial S}{\partial \xi} = 0\} \to T^*\mathcal{M}_2$ defined by $i_S(z, \xi) = (z, \frac{\partial S}{\partial q}(z, \xi))$ is an embedding with image $\mathcal{N}$;
- outside a compact set, $S(z, \xi) = Q(\xi)$ where Q is a non-degenerate quadratic form.

The proof: second step (2)

We construct a so-called graph selector for S.

We select in every fiber $\{z\} \times \mathbb{R}^k$ a critical value $u(z)$ of $S(z, .)$. Then $u : \mathcal{M}_2 \to \mathbb{R}$ is Lipschitz, C^1 on an open and dense open subset $U_0 \subset \mathcal{M}_2$ with full Lebesgue measure and such that for every $z \in U_0$



$$(z, du(z)) \in \mathcal{N} \quad \text{and} \quad u(z) = S \circ i_S^{-1}(z, du(z)).$$

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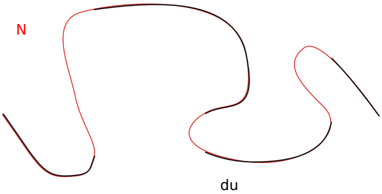

$$(z, du(z)) \in \mathcal{N} \quad \text{and} \quad u(z) = S \circ i_S^{-1}(z, du(z)).$$

Proposition *There exist a real constant c such that*

- $\forall q \in M, u(q,1) = u(q,0) - c;$
- $\forall (q,p,0,E) \in \mathcal{N}, S \circ i_S^{-1}(q,p,1,E) = S \circ i_S^{-1}(q,p,0,E) - c.$

The proof: third step

Replacing $u(q,t)$ by $v(q,t) = u(q,t) + ct$, changing the paramerization in time and extending this function in a time 1-periodic way, we obtain a Lipschitz function $u_0 : \mathcal{M}_1 = M \times \mathbb{T} \to \mathbb{R}$ such that

- u_0 is C^1 above a dense open subset $\mathcal{U}_0 \subset \mathcal{M}_1$ with full Lebesgue measure;
- there exists a primitive s_0 of the 1-form $p.dq + Ed\tau$ along $\mathcal{N}_c$ such that for every $z \in \mathcal{U}_0$

$$(*) \quad (z, du_0(z)) \in \mathcal{N}_c \quad \text{and} \quad u_0(z) = s_0(z, du_0(z));$$

- $(*)$ is true at every point where u_0 is differentiable and $\mathcal{H}(z, du_0(z)) = c$.

The proof: fourth step (1)

Let $L : TM \times \mathbb{T} \to \mathbb{R}$ be the Lagrangian that is associated to H via Fenchel duality

$$L(q, v, t) = \sup_{p \in T_q^* M} (p.v - H(q, p, t)).$$

Then u_0 is dominated by $L + c$

$$u_0(\gamma(b), b) - u_0(\gamma(a), a) \leq \int_a^b (L(\gamma(t), \dot{\gamma}(t), t) + c) dt$$

for every continuous and piecewise C^1 arc $\gamma : [a, b] \to M$.

The proof: fourth step (2)

A C^1 curve $\gamma : I \to M$ is (u_0, L, c)-calibrated if for all $a, b \in I$ such that $a < b$

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Proposition *If $\gamma : I \to M$ is (u_0, L, c)-calibrated, then*

- *u_0 is differentiable at every $(\gamma(t), t)$ with t in the interior of I;*
- *$t \mapsto (\gamma(t), t)$ is the projection of a piece of orbit for $\mathcal{H}$ that is*

$$t \mapsto (\gamma(t), t, du_0(\gamma(t), t));$$

- *this piece of orbit is contained in the level $\{\mathcal{H} = c\}$.*

Last step (1)

Defect of a curve

If $\gamma : [a, b] \to M$ is C^1, then
$\delta(\gamma) = \int_a^b (L(\gamma(t), \dot\gamma(t), t) + c) - (u_0(\gamma(b), b) - u_0(\gamma(a), a))$.

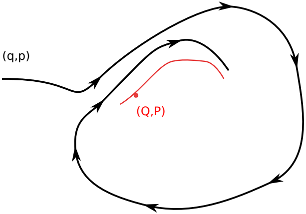
- we have $\delta(\gamma) \geq 0$ and $\delta(\gamma) = 0$ iff γ is (u_0, L, c)-calibrated;
- if (γ_n) C^1-converges to γ, then $\lim_{n\to\infty} \delta(\gamma_n) = \delta(\gamma)$;
- if $I \subset J$, then $\delta(\gamma_{|I}) \leq \delta(\gamma_{|J})$.

Lemma *If $(q, p) \in N$ has for extended orbit $(q(t), t, p(t), E(t))$, then $\int_a^b (L(q(t), \dot q(t), t) + c)dt$*

$$= s_0(q(b), b, p(b), E(b)) - s_0(q(a), a, p(a), E(a))).$$

Last step (2)

Let $\Gamma(0) = (q, 0, p, E) \in \mathcal{N}_c$. Then for every $\Gamma_\pm(0) = (q_\pm, 0, p_\pm, E_\pm)$ in the α or ω-limit set of $(q, 0, p, E)$, the projection of the orbit of $(q_\pm, 0, p_\pm, E_\pm)$ is (u_0, L, c)-calibrated.



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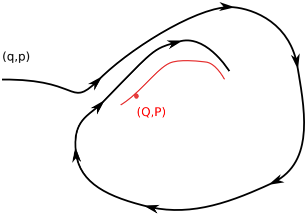


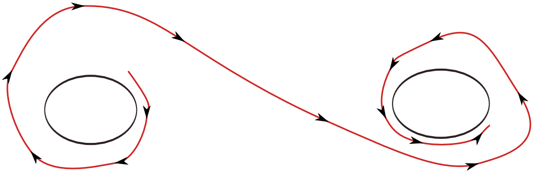
Argument: fix $a < b$. There exists a sequence (k_n) of integer times so that $\lim_{n\to\infty} \phi_{\mathcal{H}}^{k_n}(q, 0, p, E) = (q_+, 0, p_+, E_+)$ and $k_{n+1} - k_n \geq b - a$.

Then $\delta(q_{|[k_n+a, k_{n+1}+a]}) = s_0(\Gamma(k_{n+1} + a)) - s_0(\Gamma(k_n + a)) - (u_0(q(k_{n+1} + a), k_{n+1} + a) - u_0(q(k_n + a), k_n + a))$ tends to $s_0(\Gamma_+(a)) - s_0(\Gamma_+(a)) - (u_0(q_+(a), a) - u_0(q_+(a), a)) = 0$.

And $\delta(q_{+|[a,b]}) = \lim_{n\to\infty} \delta(q_{|[k_n+a, k_n+b]}) = 0$.

Last step (3)

We have proved that if $\gamma : I \to M$ is (u_0, L, c)-calibrated, then u_0 is differentiable at every $z = (t, \gamma(t))$ with t in the interior of I, that $(z, du_0(z)) \in \mathcal{N}_c$ and that $u_0(z) = s_0(z, du_0(z))$.



If $(q, p) \in N$, denoting by (q_-, p_-) a point in $\alpha(q, p)$ and by (q_+, p_+) a point in $\omega(q, p)$, we deduce:
$\delta(q_{|[a,b]}) \leq \lim_{n\to+\infty} \delta(q_{|[-k_n,k_n]})$, this last limit being

$$(s_0(q_+, 0, p_+, E_+) - s_0(q_-, 0, p_-, E_-)) - (u_0(q_+, 0) - u_0(q_-, 0)) = 0.$$