Shuzo Izumi: Zero-estimates from geometric point of view.

Take a finite set of holomorphic functions $\mathbf{\Phi} := \{\Phi_1, ..., \Phi_m\}$ on an open subset $\Omega \subset \mathbb{C}^n$. We are interested in the vanishing order $\mathrm{ord}_b F \circ \mathbf{\Phi}$ of the composition $F \circ \mathbf{\Phi} = F(\Phi_1, ..., \Phi_m)$ at $b \in \Omega$ where F varies across polynomials in m variables. A large value of $\mathrm{ord}_b\, F \circ \mathbf{\Phi}$ implies that F is an *approximate algebraic relation* among Φ_i (dual to approximate solution) at b. Let us put

$$\theta_{\mathbf{\Phi},b}(k) := \sup\{\mathrm{ord}_b F \circ \mathbf{\Phi} : \deg F \le k,\ F \circ \mathbf{\Phi} \ne 0\},$$
$$\alpha_b(\mathbf{\Phi}) := \limsup \log_k \theta(k).$$

We see that always $\theta_{\mathbf{\Phi},b}(k) \ge k$ and $\alpha_a(\mathbf{\Phi}) \ge 1$ holds.

Consider the germ at $\mathbf{0} = (0,0,0)$ of the image the map defined by

$$x_1 = \Phi_1 = t_1,\ x_2 = \Phi_2 = t_1 t_2,\ x_3 = \Phi_3 = \exp t_1 - 1.$$

The algebraic Zariski closure $\overline{X}_{\mathbf{0}}$ of the image of Φ is $\mathbb{C}^3$ but there is a transcendental analytic relation $x_3 = \exp x_1 - 1$, which defines the analytic Zariski closure $X_{\mathbf{0}}$. Existence of such an additional transcendental relation (the gap of two kinds of Zariski closures) is judged by α:

$$\alpha_b(\Phi) = 1 \Leftrightarrow \dim \overline{X}_a = \dim X_a.$$

In this talk, we show a few topics on zero-estimates. In particular, if we ignore points of a small subset of Ω, we obtain an upper estimates of them even in the case of quite general holomorphic functions.