These are the notes of my mini-course "Groups with the Haagerup property", taught at CIRM from June 30 to July 3, 2015.

As the scribblings indicate, these notes were initially not intended for distribution.

By a sad twist of fate, Prof. Uffe Haagerup died untimely on July 5, 2015. I dedicate these notes to his memory.

GROUPS WITH THE HAAGERUP PROPERTY
A.V, Luminy 30 june - 3 july 2015

1. ~~Motivation~~ Introduction

Let G be ~~a~~ a locally compact σ-compact group acting by isometries on a metric space (X,d). We say that the action is (metrically) proper if $\lim_{g\to\infty} d(x,gx)=+\infty$ $\forall x\in X$. (this means that all orbits go to infinity).

Examples: $\mathbb{Z}^n$, $\mathbb{R}^n$ act properly by translations on n-dimensional Euclidean space $\mathbb{E}^n$.

Thm (BIEBERBACH) Let G be a discrete group acting isometrically properly on $\mathbb{E}^n$. Then G is crystallographic, i.e G fits in a short exact sequence
$$0\to\mathbb{Z}^k\to G\to F\to 1$$
where $k\leq n$, F is finite, and $\mathbb{Z}^k$ acts by translations. (in case $k=n$).

Definition (Gromov 1990): G is a-(T)-menable if G admits a (metrically) proper action on a Hilbert space.

Gromovian pun: close to amenable, far from property (T).
Indeed:

Thm (Delorme - Guichardet 1973): G has property (T) $\Leftrightarrow$ every isometric action on a Hilbert space has a globally fixed point

(clearly: $\{$a-(T)-menable$\}\cap\{$prop (T)$\}=\{$compact$\}$.)

Definition: G is amenable if, for every $\varepsilon>0$ and every compact set $K \subset G$, there exists a relatively compact set $U \subset G$ s.t. $\frac{m(KU \Delta U)}{m(U)} < \varepsilon$. ($U$ is a ε-Følner set for K).

2. ~~Aff~~ Isometric actions on Hilbert spaces

[Banach-Mazur thm: Every (surjective) isometry of a real Banach space is affine (i.e. composition of a linear isometry and translation)

Pf. for Hilbert spaces: Metric characterization of segments:

$[x\,y] = \{z \in \mathcal{H} : d(x,z) + d(z,y) = d(x,y)\}$.

$\Rightarrow$ every isometry preserves segments.

Now (Darboux): every map preserving segments is affine. $\boxtimes$

So: every isometry α of $\mathcal{H}$ is of the form $\alpha(v) = Uv + b$, U a linear isometry. So, if $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ is a homomorphism we have $\alpha(g)v = \pi(g)v + b(g)$ where π is an isometric representation and $b : G \to \mathcal{H}$ is a 1-cocycle with respect to π: $b(gh) = \pi(g)b(h) + b(g)$.

$Z^1(G,\pi) = \{$1-cocycles w.r.t $\pi\}$.

[Consequence: G is a-(T)-menable if $\exists\,(\pi, b)$ s.t. $\lim_{g\to\infty} \|b(g)\| = +\infty$.

[Prop (Bekka-Cherix-V 1991): Amenable groups are a-(T)-menable

Pf: Write $G = \bigcup_n K_n$, K_n increasing sequence of compact sets. Let $U_n \subset G$ be such that $\frac{m(K_n U_n \Delta U_n)}{m(U_n)} < 2^{-n}$.

Let λ be the left regular representation of G on $L^2(G)$

$(\lambda(g)\xi)(h) = \xi(g^{-1}h)$.

Set $b_n(g) = n(\lambda(g)\xi_n - \xi_n)$ where $\xi_n(h) = \begin{cases} \frac{1}{m(U_n)^{1/2}} & \text{if } h \in U_n \\ 0 \end{cases}$ 3

b_n is a 1-cocycle w.r.t. λ.

Set then $\pi = \bigoplus_n \lambda = \infty\lambda$, $b = \bigoplus_n b_n$.

For $g \in K_{n_0}$, we have for $n \geq n_0$:

$$\|b_n(g)\|^2 = n^2 \frac{m(gU_n \Delta U_n)}{\mu(U_n)} \leq n^2 2^{-n}$$

$\Rightarrow b(g) \in \ell^2(\mathbb{N}) \otimes L^2(G)$.

Although each b_n is bounded, b is proper. Observe that, if $hU_n \cap U_n = \emptyset$, then $\|\lambda(g)\xi_n - \xi_n\|^2 = 2$.

Fix $R > 0$. If $\|b(g)\|^2 \leq R^2$, then $\forall n \geq 0$:

$$\|b_n(g)\|^2 \leq R^2 \Leftrightarrow \|\lambda(g)\xi_n - \xi_n\|^2 \leq \frac{R^2}{n^2}.$$

Take $n \gg 0$ so that $\frac{R^2}{n^2} < 2$. Then $gU_n \cap U_n \neq \emptyset$. But $\{g \in G : gU_n \cap U_n \neq \emptyset\}$ is relatively compact, as <u>G acts</u> properly on itself. $\boxtimes$

<u>How to construct proper isometric actions?</u>

Suppose we are given a G-space X, a Hilbert space $\mathcal{H}$ carrying a unitary rep. π of G, and a continuous map $c : X \times X \to \mathcal{H}$ such that:

- $c(x,y) + c(y,z) = c(x,z) \quad \forall x,y,z \in X$
- $c(gx,gy) = \pi(g)c(x,y) \quad \forall x,y \in X, g \in G$
- $\lim_{g\to\infty} \|c(gx,x)\| = +\infty$ for $x \in X$.

Then G is a-(T)-menable. Indeed, set $b(g) = c(gx_0, x_0)$. Then $b(gh) = c(ghx_0, x_0) = c(ghx_0, gx_0) + c(gx_0, x_0)$
$= \pi(g)b(h) + b(g)$, and b is proper by 3rd assumption.

<u>Examples:</u>

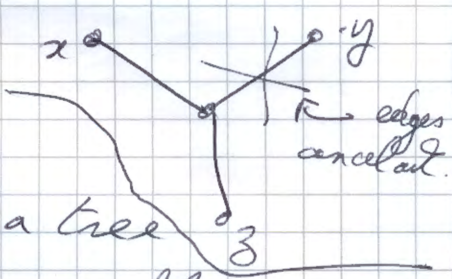
1) Groups acting on trees. ~~T~~ Let $T=(V,E)$ be a tree (= connected graph without circuit). Let $\mathbb{E}$ be the set of oriented edges. Let π be the canonical rep. of G on $\ell^2(\mathbb{E})$. ~~St~~ Set

$$c: V\times V \longrightarrow \ell^2(\mathbb{E}) : (x,y) \longmapsto c(x,y)$$

where $c(x,y)(e) = \begin{cases} 0 \text{ if } e \notin [x,y] \\ \cancel{+1 \text{ if}} \; +1 \text{ if } x \rightsquigarrow \xrightarrow{\;} \rightsquigarrow y \,;\; -1 \text{ if } x \rightsquigarrow e \rightsquigarrow y \end{cases}$ $(e \in \mathbb{E})$

Then $c(gx, gy) = \pi(g)\, c(x,y) \quad \forall g \in \mathrm{Aut}\, T$

$c(x,y) + c(y,z) = c(x,z)$ because



Since $\|c(x,y)\|^2 = 2\, d(x,y)$

we see: every group acting properly on a tree (i.e. $\lim_{g\to\infty} d(gx,x) = +\infty$) is a-(T)-menable.

(example: $\mathbb{F}_n$, $SL_2(\mathbb{Q}_p)$)

$\longrightarrow$ ~~4~~ 4th lecture

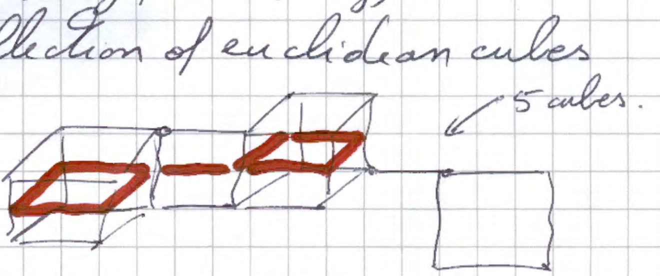
2) Def: (Haglund - Paulin) A space with walls is a ~~a set~~ pair (X, W) where ~~X~~ X is a set and W is a collection of partitions of X into two sets (= the walls) s.t. $\forall x,y \in X$: $w(x,y) = \#$ walls separating x from y is finite.

Observe that $w(x,y)$ is a pseudo-metric on X, called the wall distance.

Examples: 1) Trees: if $T=(V,E)$ is a tree, every edge defines two half-trees, and $d(x,y) = w(x,y)$

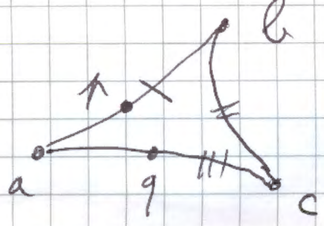
2) A cube complex is a collection of euclidean cubes glued along common faces:

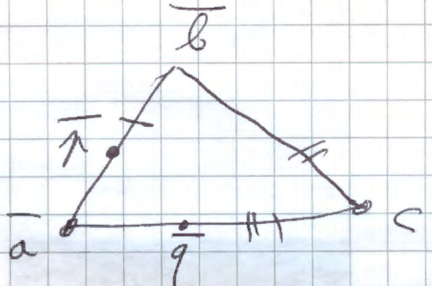
the distance is the path distance (infimum of lengths of curves).



A geodesic metric space is CAT(0) if $\forall$ geodesic triangle

$a, b, c \in X$, the comparison triangle ~~in~~ $\bar{a}, \bar{b}, \bar{c}$ in E^2 satisfies:



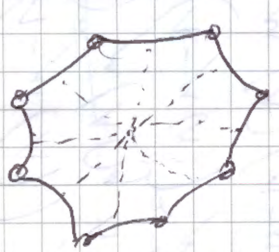


$d(p,q) \le d_{E^2}(\bar{p}, \bar{q})$
$\forall p \in [a,b]$,
$\forall q \in [a,c]$

Thm (Sageev 1995): In a CAT(0) cube complex, the hyperplanes separate in two connected components, and define a structure of space with walls on the set of vertices.

Chatterji - Niblo 2003, indep. Nica: there is an equivalence of categories between spaces with walls and CAT(0) cube complexes.

Example:



Consider the hyperbolic tiling consisting of ~~squ~~ regular octagons (with 8 octagons meeting at each vertex). Subdivide each octagon into 8 squares. Get a CAT(0) square complex on which the fundamental group of a surface of genus 2 acts properly.

Observation (Haglund - Paulin - V 1998) If G acts properly on a space with walls (i.e. $\lim_{g \to \infty} w(gx, x) = +\infty$) then G is a-(T)-menable.

Pf: Define a half-space in a space with wall (X, W) as one of the two classes of a wall. Let $\mathbb{H}$ be the set of half-spaces.

For $x \in X$: χ_x = char. function of the set of half-spaces through x. Set $c: X \times X \to \ell^2(\mathbb{H}): (x,y) \mapsto \chi_x - \chi_y \in \ell^2(\mathbb{H})$

Then: • $c(x,y) + c(y,z) = c(x,z)$

- $\pi c(gx, gy) = \pi(g)\, c(x,y) \quad \forall g \in G$
  (π = permutation rep. on $\ell^2(H)$).
- $\|c(x,y)\|^2 = 2\,w(x,y) \Rightarrow \lim_{g\to\infty} \|c(gx,x)\| = +\infty$ ☐

Mention Coxeter groups

Thompson's group F

Burger-Mozes [illegible]

$\mathcal{C}(\Delta)$-small

3. Characterizations of a-(T)-menability.

Let π be a ~~rep~~ unitary representation of G on $\mathcal{H}$.

Def: a) π almost has invariant vectors if $\forall \varepsilon > 0$, $\forall K$ compact in G, $\exists \xi \in \mathcal{H}, \|\xi\| = 1$: $\max_{k\in K} \|\pi(k)\xi - \xi\| < \varepsilon$.

b) π is a C_0-representation if $\forall \xi \in \mathcal{H}$:
$\lim_{g\to\infty} \langle \pi(g)\xi \mid \xi\rangle = 0$.

Examples: 1). λ the left regular representation of G on $L^2(G)$.

- λ is always C_0 (indeed, for $\xi \in C_c(G)$, the function $g \mapsto \langle \lambda(g)\xi \mid \xi\rangle$ has compact support. Then use density of $C_c(G)$ in $L^2(G)$).
- λ almost has invariant vectors iff G is amenable.

($\Leftarrow$: if G is amenable, and U is a (ε, K)-Følner set, then the normalized characteristic function of U is (ε, K)-invariant).

2) Let $(X, \mathcal{B}, \mu)$ be a standard probability space, assume that $G \curvearrowright X$ in a prob. measure preserving way. Let π be the representation of G on $L^2_0(X,\mu) = \{ f \in L^2(X,\mu) : \int_X f\, d\mu = 0\}$

- π is $C_0 \Leftrightarrow$ The action $G \curvearrowright X$ is mixing (i.e. $\forall A, B \in \mathcal{B}$: $\lim_{g\to\infty} \mu(gA \cap B) = \mu(A)\,\mu(B)$, i.e. A, B are asymptotically ~~invariant~~ independent). If the action is mixing, set $\xi_A(x) = \begin{cases} 1-\mu(A) & \text{if } x \in A \\ -\mu(A) & \text{if } x \notin A \end{cases}$ so $\langle \pi(g)\xi_A \mid \xi_A \rangle \xrightarrow[g\to\infty]{} 0$.

Since the span of the ξ_A's is dense in $L^2_0(X,\mu)$: π is mixing.

π almost has invariant vectors $\Leftrightarrow$ $G \curvearrowright X$ has almost invariant sets, i.e. $\exists (A_n)_{n\geq 1} \subset X$, $\mu(A_n) = \frac{1}{2}$: $\forall K \subseteq G$ compact:
$\lim_{n\to\infty} \max_{g\in K} \mu(gA_n \Delta A_n) = 0$.
(indeed, if $(A_n)_{n\geq 1}$ are almost invariant: $(\sqrt{2}\,\xi_{A_n})_{n\geq 1}$ is a sequence of almost invariant vectors).

Def: G has the Haagerup property if G admits a C_0-representation almost having invariant vectors.

Example: • Amenable grps have the Haagerup property.
• $\mathbb{F}_n$ has the Haagerup prop. (Haagerup 1979).

Thm Let G be locally compact σ-compact. TFAE:
i) G is a-(T)-menable
ii) G has Haagerup property
iii) G admits an action on a standard prob space $(X, \mathcal{B}, \mu)$, which is mixing and has almost invariant sets
(similar to amenable $\Rightarrow$ a-(T)-menable)

Pf: (iii)$\Rightarrow$(i) If (A_n) is a sequence of almost invariant sets: write $G = \bigcup_n K_n$ (K_n compact), may assume $\max_{g\in K_n} \mu(gA_n\Delta A_n) < 2^{-n}$, set $b_n(g) = n\,(\pi(g)\xi_{A_n} - \xi_{A_n})$, $\sigma = \bigoplus_n \pi = \infty\pi$,
$b(g) = \bigoplus_n b_n(g)$ (converges uniformly on compacta).
Now $\|b(g)\|^2 \leq R^2 \Rightarrow \forall n: \|b_n(g)\|^2 \leq R^2$
$\Rightarrow \|\pi(g)\xi_{A_n} - \xi_{A_n}\|^2 \leq \frac{R^2}{n^2}$.
Now observe that $\|\pi(h)\xi_{A_n} - \xi_{A_n}\|^2 = 1 - 2\langle \pi(h)\xi_{A_n} | \xi_{A_n}\rangle$
$\longrightarrow 1$ unif. on compact sets.
for $n\to\infty$ So take $n \gg 0$ to have $\frac{R^2}{n^2} < 1$: then $\{g\in G:$
$\|b_n(g)\|^2 \leq R^2\}$ is compact $\Rightarrow b$ is a proper cocycle.

(i)$\Rightarrow$(ii) We appeal to:

<u>Thm</u> (Schönberg 1930). Let $\psi : G \to \mathbb{R}^+$ be a continuous function, with $\psi(e) = 0$ and $\psi(g^{-1}) = \psi(g)$. TFAE

a) $\exists\ \pi$ and $b \in Z^1(G, \pi)$ s.t. $\psi(g) = \|b(g)\|^2$

b) $\forall\, t > 0$: $\exists\ \pi_t$ cyclic rep. of G and ξ_t cyclic vector s.t. $e^{-t\psi(g)} = \langle \pi_t(g)\xi_t \mid \xi_t \rangle$.

Here, let $b \in Z^1(G, \pi)$ be a proper 1-cocycle, set $\psi(g) = \|b(g)\|^2$, and take π_n s.t. $\langle \pi_n(g)\xi_n \mid \xi_n \rangle = e^{-\frac{\psi(g)}{n}}$.

Then π_n is C_0-rep, hence $\sigma = \bigoplus_n \pi_n$ is a C_0-rep.

Since $\|\pi_n(g)\xi_n - \xi_n\|^2 = 2 - 2e^{-\frac{\psi(g)}{n}} \xrightarrow[n\to\infty]{} 0$ unif. on compact subsets, σ almost has invariant vectors.

(ii) $\Rightarrow$ (iii) Use Gaussian Hilbert space construction: for each orthogonal rep. π of G, $\exists\ (X, \mathcal{B}, \mu)$ standard Borel space with $G \curvearrowright X$, s.t. $L^2_0(X, \mu) \simeq \widehat{\bigoplus_{n=1}^{\infty}} S^n\pi$, where $S^n\pi$ is the n-th symmetric power of π.

- If π almost has invariant vectors, so is $\widehat{\bigoplus_{n=1}^{\infty}} S^n\pi$
- If π is C^0, so is $\widehat{\bigoplus_{n=1}^{\infty}} S^n\pi$

$\Rightarrow G \curvearrowright X$ is C^0 and almost has invariant vectors. $\boxtimes$

⚠ An <u>uncountable</u> discrete abelian group has the Haagerup property, but is <u>not</u> a-(T)-menable (as the existence of a proper cocycle on G implies that G is σ-compact).

<u>3. Haagerup property: what for?</u>

<u>1) von Neumann algebras</u>:

For G a discrete group, let $\mathbb{C}G$ be the group ring.
Represent faithfully $\mathbb{C}G$ on $\ell^2(G)$ by the left regular

representation: $\lambda : \mathbb{C}G \hookrightarrow \mathcal{B}(\ell^2 G) : f \mapsto \lambda(f)$ where $\lambda(f)\xi = f * \xi$. The reduced C^*-algebra of G is $C^*_r G = \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|}$, the von Neumann algebra of G is $L(G) = \overline{\mathbb{C}G}^{w}$.

Def: a von Neumann algebra is a weakly closed, unital, $*$-subalgebra of $\mathcal{B}(\mathcal{H})$; a von Neumann algebra M is a II_1-factor if it is infinite-dimensional, $Z(M) = \mathbb{C}1$, and there exists a positive trace τ on M; i.e. $\tau : M \to \mathbb{C}$,
$\tau(ab) = \tau(ba)$, $\tau(a^*a) \geq 0$, $\tau(1) = 1$

Example: On $L(G)$, $\tau(a) = \langle a\delta_e | \delta_e \rangle$ is a positive trace $(\tau(a^*a) = \|a(\delta_e)\|^2)$; $L(G)$ is a II_1-factor iff G has infinite conjugacy classes.

(Choda 1983; Jolissaint 2002)

Def: A II_1-factor M has Haagerup property if there exists a net $(\varphi_i) : M \to M$ of completely positive, unital maps s.t.:
- $\tau \circ \varphi_i = \tau$ (trace-preserving)
- φ_i extends to a compact operator on $L^2(M, \tau)$ (= completion of M for $\langle x | y \rangle = \tau(x^*y)$)
- $\forall x \in M : \|\varphi_i(x) - x\|_2 \xrightarrow[i \to \infty]{} 0$.

Thm: G has Haagerup property $\Leftrightarrow$ $(L(G), \tau)$ has Haagerup property.

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2) The Baum-Connes conjecture.

Assume G discrete and torsion-free. The K-theory of the C^*-algebra $C^*_r(G)$, $K_i(C^*_r G)$ $(i = 0, 1)$ ~~describes~~ encodes projective finite type modules over $C^*_r G$ and isomorphisms between them. (analytical object)

10 | Let BG be the classifying space of G (a CW-complex whose fundamental group is G, and with $\widetilde{BG}$ contractible).
BG is unique up to homotopy
The K-homology $K_i(BG)$ $(i=0,1)$ is a variant of ordinary ~~co~~homology: $K_i(BG)\otimes\mathbb{Q} \simeq \bigoplus_{j=0}^{\infty} H_{i+2j}(BG,\mathbb{Q}) = \bigoplus_{j=0}^{\infty} H_{i+2j}(G,\mathbb{Q})$ (=group homology). $K_i(BG)$ is a topological, or geometric object.

Around 1981, P. Baum and A. Connes defined an index, or assembly map $\mu_G : K_i(BG) \to K_i(C_r^* G)$ (homom. of abelian groups) and conjectured that this map is always an isomorphism. It is known:

μ_G onto $\Rightarrow$ Kaplansky-Kadison conjecture (C_r^*G has no non-trivial idempotent) $\Rightarrow$ Kaplansky conjecture ($\mathbb{C}G$ has no non-trivial idempotent)

μ_G injective $\Rightarrow$ $\boxed{\text{Novikov conjecture on homotopy invariance of higher signatures for } G}$

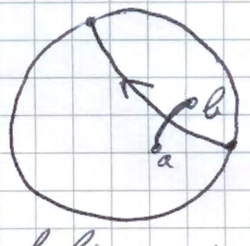
$\hookrightarrow$ Let M be a closed manifold with a map $f: M \to BG$ ~~with $\pi_1(M) = G$. Let $f: M \to BG$ be the classifying map ($\widetilde{M}$ = pullback of~~

$$\begin{tikzcd} \widetilde{M} \arrow[r] \arrow[d] & \widetilde{BG} \arrow[d] \\ M \arrow[r, "f"] & BG \end{tikzcd}$$

~~)~~. For $x \in H^*(BG,\mathbb{Q})$, consider the higher signature $\sigma_x(M,f) = \langle f^*(x) \cup L(M), [M] \rangle \in \mathbb{Q}$ where $L(M)$ is the L-genus (a polynomial in the Pontryagin classes, depending on the smooth structure of M). The conjecture is that these numbers are homotopy invariant (and so do not depend on the smooth structure): if $h: N \to M$ is a homotopy equivalence, then $\sigma_x(M,f) = \sigma_x(N, f\circ h)$.

Thm (HIGSON-KASPAROV 1997): BC holds for a-(T)-menable gps.
So: you can prove BC without knowing what it is!

11 4. Spaces with measured walls



Consider $\mathbb{H}^2(\mathbb{R})$, real hyperbolic space.
The space of oriented lines in $\mathbb{H}^2(\mathbb{R})$ identifies with $(S^1 \times S^1) \setminus \Delta$, and carries an $SL_2(\mathbb{R})$-invariant measure μ. The set of lines intersecting a given geodesic segment $[a\, b]$ is relatively compact, hence of finite measure. Moreover, we have Crofton's formula:

$$\mu\{\text{lines intersecting } [a,b]\} = \lambda\, d(a,b)$$

(where $\lambda > 0$ only depends on normalization of measure).
Moreover, we feel it is a kind of wall spaces, as lines divide $\mathbb{H}^2(\mathbb{R})$ into two halves.

<u>Def</u>: Let X be a set. Let 2^X be the power set of X, with product topology. For $x \in X$, denote $\mathcal{A}_x = \{A \subset X \mid x \in A\}$: a clopen subset of 2^X. A <u>measured wall structure</u> on X is a pair (X, μ) where μ is a Borel measure on 2^X with $\forall x, y \in X$: $d_\mu(x,y) = \mu(\mathcal{A}_x \Delta \mathcal{A}_y) < +\infty$.

<u>Example</u>: Let $\mathbb{H}^n(\mathbb{R})$ be real hyperbolic n-space. Let $\mathcal{H}$ be the set of closed half-spaces of $\mathbb{H}^n(\mathbb{R})$. The isometry group $O^{\#}(n,1)$ acts transitively on $\mathcal{H}$, with an invariant measure ν such that $\nu\{\text{half-spaces separating } x \text{ from } y\} = \lambda\, d(x,y)$.
Let $i: \mathcal{H} \to 2^{\mathbb{H}^n(\mathbb{R})}$ be inclusion; then $\mu = i_* \nu$ is a measured wall structure.

<u>Example</u>: Every space with walls $(X, \mathcal{W})$ is a measured wall structure.
Let $\mathbb{H}$ be the set of half-spaces of X (i.e. one class of a wall).
~~Set μ~~ For $B \subset 2^X$, set $\mu(B) = \frac{1}{2} \#(B \cap \mathbb{H})$ (counting measure)
Then $d_\mu(x,y) = w(x,y)$.

Prop: Every group acting properly on a measured wall space is a-(T)-menable.

Example: $SO(n,1)$ has Haagerup.

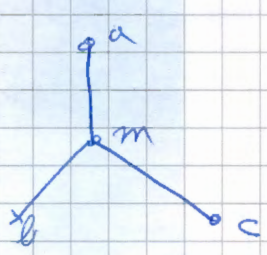
This time there is a converse

Thm (Chatterji-Drutu-Haglund) G is a-(T)-menable
$\Leftrightarrow$ G acts properly on measured wall structure
$\Leftrightarrow$ G acts properly isometrically on some subset of $L^p[0,1]$
$(1 \leq p \leq 2)$.

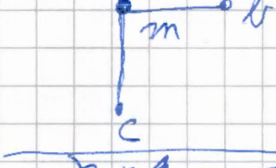
The key concept is the one of median space.

Def: A metric space (X,d) is median if $\forall a,b,c \in X$, $\exists !$ unique median point $m = m(a,b,c)$, i.e. a unique point m such that $d(a,m)+d(m,b) = d(a,b)$ (and same for $\{b,c\}$ and $\{c,a\}$).

Examples: a) Trees



b) $\mathbb{R}^n$ with ℓ^1-metric



c) L^1.

Thm (CDH) a) Any measured wall structure embeds isometrically into a canonical median space

b) Any median space has a canonical structure of measured wall structure.

c) Any median space embeds isometrically into L^1.