Representation theory effective ergodic theorems, and applications

July 2, 2015

Representation Theory, Dynamics and Geometry

CIRM, Luminy, June 2015

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

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- Talk I : Averaging operators in dynamical systems and effective ergodic theorems

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- Talk III : Best possible spectral estimates, the automorphic representation of a lattice subgroup, and the duality principle on homogeneous spaces
- Talk IV : Fast equidistribution of dense lattice orbits, and best possible Diophantine approximation on homogeneous algebraic varieties

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- It follows that $\pi\mid_L$ is weakly contained in the regular rep' r_L !!!!!

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- H is a tempered subgroup of G if EVERY unitary rep' π of G without inv' unit vectors has a tempered restriction to H.

Examples of subgroup temperedness

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- 2) When G is simple with property T, there are universal pointwise bounds on the K-finite matrix coefficients of G in general unitary representations (Cowling 1980, Howe 1980, Howe-Moore 1976, How-Tan 1992, Li 1994, Oh 1998.....). These bounds can be restricted to a simple subgroup H and are often in $L^{2+\eta}(H)$ so that every restricted rep' of H is tempered.

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- 3) Margulis 1995 observed that this holds for (the images of) all the irreducible linear representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This observation can be greatly generalized.

Subgroup temperedness, continued

- 4) Unitary rep's of simple groups have matrix coefficients in $L^{2k}(G)$ for some k. Restricting a rep's of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is $(G^k, G, \pi^0_{G^k/\Gamma})$ tempered.

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- 5) For some lattices and their low level congruence subgroups the Selberg eigenvalue conjecture is known to hold, so that $L^2_0(G/\Gamma)$ is known to be a tempered representation of G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.

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• Let $\|g\|$ denote a natural gauge on G, namely a continuous, non-negative and proper function from G to $\mathbb{R}_+$.

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• with $\gamma \in \Gamma$ satisfying $\|\gamma\| < \left(\frac{1}{\epsilon}\right)^{\zeta}$, and $\zeta = \zeta(x, x_0) < \infty$.

Thus $\zeta(x, x_0)$ gives a rate of approximation of a general point $x_0 \in G/H$ by the Γ-orbit of x.

Basic problems

• Problem I : finiteness. Determine when does there exist a finite constant $\zeta(\Gamma, G/H)$ which bounds the rate of approximation by almost every lattice orbit. Determine when does there exist a finite uniform bound for every lattice orbit, without exception.

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• Problem III : Optimality. Give a simple, easily verifiable and widely applicable criterion for when the upper and lower bounds coincide, giving rise to the optimal rate of approximation by lattice orbits on the homogeneous space.

Scope of the problem : some instances

• $G(\mathbb{R})$ a real algebraic group defined over $\mathbb{Q}$, $H(\mathbb{R})$ an algebraic subgroup, $\Gamma = G(\mathbb{Z})$ the lattice of integral points. This includes natural Diophantine approximation problems on homogeneous affine varieties, as well as on homogeneous projective varieties.

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• G is an S-algebraic $\mathbb{Q}$-group, H a closed subgroup, $\Gamma = G(\mathbb{Z}[S^{-1}])$. This includes for example $G = G(\mathbb{R}) \times G(\mathbb{Q}_p)$ and $H = G(\mathbb{Q}_p)$, namely approximation in the connected group $G(\mathbb{R})$ by the dense subgroup $G(\mathbb{Z}[\frac{1}{p}])$.

• $G(\mathbb{A})$ and $H(\mathbb{A})$ are groups of rational adéles, $\Gamma = G(\mathbb{Q})$. This includes the problem of intrinsic diophantine approximation, namely by rational points lying on the algebraic variety $G(\mathbb{R})/H(\mathbb{R})$ itself.

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• We now turn to describe a general approach to the problem of establishing explicit bounds, and give a sufficient condition for optimality, under certain conditions.

• A main assumption in our approach is that G is non-amenable.

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• Kleinbock and Merrill (2013) have established the best possible exponent for rational approximation on the unit spheres in any dimension $n \geq 2$, together with an analog of Khinchine's theorem (and even sharper results). More recently [FKMS] considered general quadratic varieties.

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- Fix a norm on $\mathbb{R}^n$ and $\mathbb{C}^n$, and on $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$,

- and in the local field case, take the standard valuation on the field, and the standard maximum norm on the linear space F^n, and on $M_n(F)$.

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• $\kappa(x, x_0)$ is a $\Gamma \times \Gamma$-invariant function, hence almost surely a constant κ when the action is ergodic. κ depends on G, Γ and V, but not on the norms chosen on F^n and $M_n(F)$.

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• Taking the resulting two inhomogeneous equations mod 1, we conclude that for every $x_0 = (u_0, v_0) \in \mathbb{T}^2$, for almost every $x = (u, v) \in \mathbb{T}^2$, and for every ϵ sufficiently small, there are integers a, b, c, d with

$$\|(au + bv, cu + dv) - (u_0, v_0)\| < \epsilon$$

such that $ad - bc = 1$, and

$$\max\{|a|, |b|, |c|, |d|\} < \frac{B}{\epsilon} \cdot \log^{2+\eta}\left(\frac{1}{\epsilon}\right)$$

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• For the corresponding approximation result using algebraic integers in other imaginary quadratic number fields, it is possible to give upper estimates for the exponent κ, but its exact value remains an open problem.

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• The group $G = SL_3(F) \times SL_3(F)$ acts transitively on V, via $(g_1, g_2)X = g_1 X g_2^{-1}$. The stability group of the point $X = Id$ is the subgroup $H = (\{h, h)\,;\, h \in SL_3(F)\}$.

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• The group $G = SL_3(F) \times SL_3(F)$ acts transitively on V, via $(g_1, g_2)X = g_1 X g_2^{-1}$. The stability group of the point $X = Id$ is the subgroup $H = (\{h, h)\,;\, h \in SL_3(F)\}$.

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• The best possible exponent for irreducible lattices in $SL_2(F) \times SL_2(F)$ is $\kappa = 3/2$. Whether it is achieved by any irreducible lattice remains an open problem.

Lower bound for the Diophantine exponent

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- Define the empirical growth parameter for such points :

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• Thm F. [Ghosh-Gorodnik-N 2013] $\kappa \geq \frac{d}{a}$, namely it is impossible to approximate points on $V = G/H$ as above by points in lattice orbits any faster, namely using matrices of smaller norm.

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• Consider the intersection of norm balls with the stability group H, namely $H_T = \{h \in H\,;\ \|h\| < T\}$.

• Consider the invariant probability measure $m_{\Gamma\setminus G}$ on $Y = \Gamma \setminus G$ and the averaging operators $\pi_Y(\beta_T) : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)$, given by

The effective mean ergodic theorem

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h \in H_T} f(yh)dm_H(h) \ , \ y \in \Gamma \setminus G \, .$$

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• there exists $\theta > 0$ such that

$$\|\pi_Y(\beta_T)f - \int_Y fdm\|_{L^2(\Gamma\setminus G)} \le C(\eta)m_H(H_T)^{-\theta+\eta}\|f\|_{L^2(\Gamma\setminus G)}$$

for every $\eta > 0$, suitable $C(\eta)$, and $t \ge t_\eta$.

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• Conclusion : if $2\theta = 1$ then the lower and upper bounds for the Diophantine exponent coincide !

Best possible rate of approximation

- Corollary 1. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L_0^2(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $V = G/H$ is best possible, and the Diophantine exponent is given by $\kappa = \frac{d}{a}$, the a-priori pigeon-hole bound.

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• Corollary 2. If the stability group H is semi simple and non-compact, and the restriction of the automorphic representation $\pi^0_{G/\Gamma}$ to H is a tempered representation of H, then the Diophantine exponent of the irreducible lattice Γ of G in its action on G/H is best possible, and is given by $\kappa = \frac{d}{a}$.

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• So understanding the exact extent of the class of tempered triples (G, H, Γ) is a very intriguing problem !

The duality principle

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• A general quantitative duality principle has been developed in joint work with Alex Gorodnik. It yields conclusions which are considerably more precise than just the existence of a rate of approximation by Γ-orbits.

• For example, it is possible to prove quantitative mean and pointwise ergodic theorems for the discrete averages supported on orbit points when ordered by a norm, although the optimality of the rate is compromised.

Exploting duality in the Diophantine exponent problem

- The gist of the matter is that given $x_0 \in G/H$, we place it in an ε-neighbourhood $x_0 \in \mathcal{V}_\varepsilon \subset G/H$ (so that $m_{G/H}(\mathcal{O}_\varepsilon) \sim \varepsilon^{\dim G/H}$.

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- Let χ_ε be the normalized characteristic function of $\mathcal{O}_\varepsilon$. Let us periodize χ_ε under Γ, forming $\phi_\varepsilon(\Gamma g) = \sum_{\gamma\in\Gamma} \chi_\varepsilon(\gamma g)$
- ϕ_ε being in $L^2(\Gamma \setminus G)$, we consider the averaging operators (supported on $B_t \subset H$) :

$$\pi_{\Gamma\setminus G}(\beta_t)\phi_\varepsilon(\Gamma u) = \frac{1}{m_H(B_t)} \int_{h\in B_t} \phi_\varepsilon(\Gamma u h) dm_H(h)$$

Duality and approximation, cont'd

- and deduce from the effective mean ergodic theorem for H in $L_0^2(\Gamma \setminus G)$ that

$$\left\| \pi_{\Gamma\setminus G}(\beta_t)\phi_\varepsilon - \int_{\Gamma\setminus G} \phi_\varepsilon dm_{\Gamma\setminus G} \right\|_{L^2} \leq C m_H(B_t)^{-\theta} \left\| \phi_\varepsilon \right\|_{L^2}$$

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- equivalently, the family of functions $\pi_{\Gamma\setminus G}(\beta_t)\phi_\varepsilon$ converge at a definite rate to the constant (non-zero) function

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- Since ϕ_ε is a Γ-periodization, we conclude that for all $t \geq t_\varepsilon$ sufficiently large, for some u close to e, we have

$$\pi_{\Gamma\setminus G}(\beta_t)\phi_\varepsilon(\Gamma uh) = \sum_{\gamma\in\Gamma} \int_{h\in B_t} \chi_\varepsilon(\gamma uh) dm_H(h) \neq 0$$

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- or equivalently, $\gamma u H \in \mathcal{O}_\varepsilon H$.