Representation theory effective ergodic theorems, and applications

July 2, 2015

Representation Theory, Dynamics and Geometry

CIRM, Luminy, June 2015

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

Plan

- Talk I : Averaging operators in dynamical systems and effective ergodic theorems

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- Talk III : Best possible spectral estimates, the automorphic representation of a lattice subgroup, and the duality principle on homogeneous spaces
- Talk IV : Fast equidistribution of dense lattice orbits, and best possible Diophantine approximation on homogeneous algebraic varieties

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$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

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- 2) $\left\|\pi(\beta_t)f - \int_{G/\Gamma} f \, dm_{G/\Gamma}\right\|_{L^2} \leq Cm(B_t)^{-\theta} \left\|f\right\|_{L^2}$ implies

$$\frac{|\Gamma_t|}{m_G(B_t)} = 1 + O\left(m(B_t)^{-\theta/(\dim G+1)}\right).$$

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- ϕ is a bounded function on G/Γ with compact support,

$$\int_G \chi_\varepsilon \, dm_G = 1, \quad \text{and} \quad \int_{G/\Gamma} \phi_\varepsilon \, d\mu_{G/\Gamma} = 1.$$

- Let us apply the mean ergodic theorem to the function ϕ_ε. It follows from Chebycheff's inequality that for every $\delta > 0$,

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Claim I. Given $\varepsilon, \delta > 0$, for t sufficiently large, there exists $g_t \in \mathcal{O}_\varepsilon$ satisfying

$$1 - \delta \leq \frac{1}{m_G(B_t)} \int_{B_t} \phi_\varepsilon(g^{-1} g_t \Gamma) dm_G \leq 1 + \delta\,.$$

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and if for $\gamma \notin \Gamma_t$ the integrand is zero, then we can estimate

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- But If $\chi_\varepsilon(g^{-1}h\gamma) \neq 0$ for some $g \in B_{t-c\varepsilon}$ and $h \in \mathcal{O}_\varepsilon$, then clearly $g^{-1}h\gamma \in \operatorname{supp} \chi_\varepsilon = \mathcal{O}_\varepsilon$, and so

$$\gamma \in h^{-1} \cdot B_{t-c\varepsilon} \cdot (\operatorname{supp} \chi_\varepsilon) \subset B_t.$$

  by admissibility.

- on the other hand, since $\chi_\varepsilon \geq 0$ and $\int_G \chi_\varepsilon dm = 1$:

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- Now taking t sufficiently large, $h = g_t$ and using Claims I and II

$$|\Gamma_t| \leq (1+\delta)m(B_{t+\varepsilon}) \leq$$

$$\leq (1+\delta)(1+c\varepsilon)m(B_t),$$

by admissibility. The lower estimate is proved similarly.

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- By the mean ergodic theorem and Chebycheff's inequality :

$$m_{G/\Gamma}(\{x \in G/\Gamma : |\pi_{G/\Gamma}(\beta_t)\phi_\varepsilon(x) - 1| > \delta\})$$

$$\leq C\delta^{-2}\varepsilon^{-n}m(B_t)^{-2\theta}.$$

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- Taking $\delta \sim \varepsilon \sim m(B_t)^{-\theta/(n+1)}$ balances the two significant parts of the error and meets the condition above, and the result follows.

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- This holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L_0^2(G/\Lambda)$ has a positive lower bound.

- Property τ has been shown to hold for the set of congruence subgroups of any arithmetic lattice in a semisimple Lie group (Burger-Sarnak, Lubotzky, Clozel.....), generalizing the Selberg property.

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this specific case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar quality results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

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- Riemannian and other bi-K-invariant balls in certain higher rank simple Lie groups were considered by Duke-Rudnik-Sarnak (1991). In this case, they have obtained the best error estimate to date, which is matched by Thm. E (when adjusted for radial averages).
- Eskin-McMullen (1991) devised the mixing method, which applies to general (well-rounded) sets, but have not produced an error estimate. Maucourant (2005) has obtained an error estimate using an effective form of the mixing method, and so have Benoist-Oh in the S-algebraic case (2012). The resulting estimates are weaker than those of Thm. E.

Counting rational points

- Counting rational points of bounded height on algebraic varieties homogeneous under a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

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- It is possible to consider the corresponding group G over the ring of rational adéles, in which the group of rational points is embedded as a lattice. Generalizing the operators norm estimates from $G(F)$ (for all field completions F) to the group of adéles, it is possible to use the method based on the effective mean ergodic theorem here too. The error estimate of Thm. E is better that the estimate that both other methods produce.

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Ergodic theorems for lattice groups : Induced actions

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- Denote by $Y = G/\Gamma \times X = \frac{G \times X}{\Gamma}$, with the measure $m_{G/\Gamma} \times \mu$, the action of G induced from the Γ-action on X.

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- It is defined as the space $Y = \frac{G \times X}{\Gamma}$ of Γ-orbits in $G \times X$, where Γ acts via $(h, x)\gamma = (h\gamma, \gamma^{-1}x)$. G acts on $G \times X$ via $g \cdot (h, x) = (gh, x)$, an action which commutes with the Γ-action and is therefore well defined on Y. The measure $m_{G/\Gamma} \times \mu$ is G-invariant.

Ergodic theorems for lattice groups : proof overview

- The essence of the matter is to estimate the ergodic averages $\pi_X(\lambda_t)\phi(x)$ given by

$$\frac{1}{|\Gamma \cap B_t|} \sum_{\gamma \in \Gamma \cap B_t} \phi(\gamma^{-1}x)\,, \quad \phi \in L^p(X),$$

above and below by the ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon(y)$, namely by

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$$\frac{1}{m_G(B_{t \pm C})} \int_{g \in B_{t \pm C}} F_\varepsilon(g^{-1}y) dm_G(g) \; .$$

- The link between the two expressions is given by setting $y = (h, x)\Gamma \in (G \times X)/\Gamma = Y$ and

$$F_\varepsilon((h, x)\Gamma) = \sum_{\gamma \in \Gamma} \chi_\varepsilon(h\gamma)\phi(\gamma^{-1}x) \; , \quad F \in L^p(Y),$$

where χ_ε is the normalized characteristic function of an identity neighborhood $\mathcal{O}_\varepsilon$. Assume from now on $\phi \geq 0$.

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

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- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .

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- A favorable lower bound arises if $\chi_\varepsilon(g^{-1}h\gamma)\neq 0$ and $g\in B_{t-C}$ imply that $\gamma\in B_t$, and a favorable upper bound arises if for $\gamma\in\Gamma\cap B_t$ the support of $\chi_\varepsilon(g^{-1}h\gamma)$ contained in B_{t+C}.

- The ergodic averages $\pi_Y(\beta_{t\pm C})F_\varepsilon$ can be rewritten in full as

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{m_G(B_{t\pm C})}\int_{g\in B_{t\pm C}}\chi_\varepsilon(g^{-1}h\gamma)\right)\phi(\gamma^{-1}x)\ ,$$

- We would like the expression in parentheses to be equal to one when (say) $\gamma\in\Gamma\cap B_{t-C}$ and equal to zero when (say) $\gamma\notin\Gamma\cap B_{t+C}$, in order to be able to compare it to $\pi_X(\lambda_t)\phi$. .
- A favorable lower bound arises if $\chi_\varepsilon(g^{-1}h\gamma)\neq 0$ and $g\in B_{t-C}$ imply that $\gamma\in B_t$, and a favorable upper bound arises if for $\gamma\in\Gamma\cap B_t$ the support of $\chi_\varepsilon(g^{-1}h\gamma)$ contained in B_{t+C}.
- Thus favorable lower and upper estimates depend only on the regularity properties of the sets B_t, specifically on the stability property under perturbations by elements h in a fixed neighborhood, and volume regularity of $m_G(B_t)$.

- Therefore taking some fixed $\mathcal{O}_\varepsilon$ and C, the usual strong maximal inequality for averaging over λ_t follows from the ordinary strong maximal inequality for averaging over β_t. It follows that for lattice actions, as for actions of the group G, the maximal inequality holds in great generality and requires only a coarse form of admissibility.

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- The effective uniform volume estimate appearing in the definition of admissibility is utilized, and is matched against the unavoidable quantity $m_G(\mathcal{O}_\varepsilon)^{-1}$ which the approximation procedure introduces.

- The effective mean ergodic theorem requires in addition an effective estimate on the decay of the operator norms $\left\|\pi_Y^0(\beta_t)\right\|_{L_0^p(Y)}$.

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- As a byproduct of the proof, we obtain an effective decay estimate on the norms $\left\|\pi_X^0(\lambda_t)\right\|_{L_0^p(X)}$, for $1 < p < \infty$.

Pointwise convergence and the invariance principle

- A fundamental point in the completion of the proof of the pointwise ergodic theorem is an invariance principle, asserting that for any given Borel function F on Y, the pointwise ergodic theorem for $\pi_Y(\beta_t)$ holds for a set of points which contain a strictly G-invariant conull set.

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- Since the induced action $Y = (G \times X)/\Gamma$ is a G-equivariant bundle over G/Γ, this implies that for every single point $y\Gamma$, the set of points $x \in X$ where the pointwise ergodic theorem holds is conull in X. This allows us to deduce that the set of points where $\pi_X(\lambda_t)\phi(x)$ converges is also conull in X, by looking at the fiber over the point $e\Gamma \in G/\Gamma$.

Some examples of ergodic actions

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ergodically and with a spectral gap. There exists an explicit $\theta_p > 0$ such that for $f \in L^p(\mathbb{T}^n)$, $p > 1$ and almost every x,

$$\left| \frac{1}{|\Gamma_t|} \sum_{\gamma \in \Gamma_t} f(\gamma^{-1}x) - \int_{\mathbb{T}^n} f d\mu \right| \leq C_p(x,f) e^{-\theta_p t} .$$

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Example 2 : Action on the space of lattices.

Let $\mathcal{L}_d$ denote the space of unimodular lattice in $\mathbb{R}^d$, taken with its $SL_d(\mathbb{R})$-invariant probability m . Let $G \subset SL_d(\mathbb{R})$ be a semisimple group, and $\Gamma \subset G$ be any lattice subgroup. Then Γ acts ergodically and with a spectral gap, and thus for almost every lattice L, its Γ-orbit in the space of lattices satisfy, for $\theta_p = \theta_p(\Gamma, d) > 0$, $f \in L^p$, $p > 1$

$$\left| \frac{1}{|\Gamma_t|} \sum_{\gamma \in \Gamma_t} f(\gamma^{-1}L) - \int_{\mathcal{L}_d} f dm \right| \leq C_p(L,f) e^{-\theta_p t} .$$

Equidistribution in isometric actions

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- If the Γ-action has a spectral gap, and the measure satisfies $\mu(B_\varepsilon) \geq C\varepsilon^d$ (for example if it has dimension n), then for every Hölder continuous function f, and for every point x

$$\left| \frac{1}{|\Gamma_t|} \sum_{\gamma\in\Gamma_t} f(\gamma^{-1}x) - \int_{\mathbb{T}^n} f d\mu \right| \leq C \|f\| e^{-\kappa t}\,,$$

  for an explicit rate $\kappa > 0$ (depending only on the Hölder parameter of f).

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- When Γ has property T and acts isometrically, $\lambda_t f(x)$ for f continuous converges to $\int_X f d\mu$ for every point x, and furthermore converges exponentially fast (in t) to the limit for almost every point x.

- In every action of Γ on a finite homogeneous space X, we have the following norm bound for the averaging operators

$$\left\| \lambda_t f - \int_X f d\mu \right\|_2 \leq C m(B_t)^{-\theta_2} \|f\|_2 ,$$

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- π is tempered if $\pi \le_w r_G$, and then $\|\pi(\beta)\| \le \|r_G(\beta)\|$ for all abs' cont' prob' measures β, which is the best possible estimate.

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- It follows that $\pi \mid_L$ is weakly contained in the regular rep' r_L !!!!!

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- It is therefore natural to define the following for general G and (non-amenable) H.
- (G, H, π) is tempered if the restriction of π to H is a tempered rep' of H.
- H is a tempered subgroup of G if EVERY unitary rep' π of G without inv' unit vectors has a tempered restriction to H.

Examples of subgroup temperedness

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- 2) When G is simple with property T, there are universal pointwise bounds on the K-finite matrix coefficients of G in general unitary representations (Cowling 1980, Howe 1980, Howe-Moore 1976, How-Tan 1992, Li 1994, Oh 1998.....). These bounds can be restricted to a simple subgroup H and are often in $L^{2+\eta}(H)$ so that every restricted rep' of H is tempered.

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- 3) Margulis 1995 observed that this holds for (the images of) all the irreducible linear representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$. This observation can be greatly generalized.

Subgroup temperedness, continued

- 4) Unitary rep's of simple groups have matrix coefficients in $L^{2k}(G)$ for some k. Restricting a rep's of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is tempered.

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- 5) For some lattices and their low level congruence subgroups the Selberg eigenvalue conjecture is known to hold, so that $L^2_0(G/\Gamma)$ is known to be a tempered representation of G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.