Representation theory effective ergodic theorems, and applications

June 30, 2015

Representation Theory, Dynamics and Geometry

CIRM, Luminy, June 2015

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

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- Talk I : Averaging operators in dynamical systems and effective ergodic theorems

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- Talk III : Best possible spectral estimates, the automorphic representation of a lattice subgroup, and the duality principle on homogeneous spaces
- Talk IV : Fast equidistribution of dense lattice orbits, and best possible Diophantine approximation on homogeneous algebraic varieties

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  and study their norm convergence properties, ideally establishing operator-norm decay estimates,
- including for the rep's π_X^0 on $L_0^2(X)$ for an ergodic probability measure-preserving G-action on X.

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- It was established by Diximier (1969) that weak containment is equivalent to the norm inequality $\|\sigma(f)\| \leq \|\pi(f)\|$ for every $f \in L^1(G)$.

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- This condition is equivalent to $\|r_G(\beta)\| = 1$ for every probability measure β, and characterizes amenable groups (Reiter's criterion).
- Property T is characterized by the trivial rep' being separated from unitary reps' π without invariant unit vectors, or equivalently

$$\sup_{\pi} \|\pi(\beta)\| \leq \alpha(\beta) < 1$$

for every generating measure β.

Tempered representations

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- It is typically not a feasible task to compute, or even to estimate $\|\pi(\beta)\|$, for general G, π and β.
- But, remarkably, for the extensive class consisting of non-amenable algebraic groups it is possible to establish operator norm estimates which go far beyond the contraction property guaranteed by a spectral gap. In these norm estimates the regular representation plays a major role.

Integrability of matrix coefficients

- A unitary rep' π of G on $\mathcal{H}_\pi$ is called an L^q-representation if there exists a dense subspace $\mathcal{H}'_\pi \subset \mathcal{H}_\pi$ such that the matrix coefficients $c_{v,w}(g) = \langle \pi(g)v, w\rangle$ are in $L^q(G)$ for $v, w \in \mathcal{H}'_\pi$.

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- Note that if π is an L^q-rep' and σ is an L^s-rep' with $q, s \geq 2$, then $\mathcal{H}'_\pi \otimes \mathcal{H}'_\sigma$ is dense in $\mathcal{H}_\pi \otimes \mathcal{H}_\sigma$, and $\pi \otimes \sigma$ is an L^r-rep', with $\frac{1}{r} = \frac{1}{q} + \frac{1}{s}$. So if $q = s$ then $\pi \otimes \sigma$ is an $L^{\frac{q}{2}}$-representation.

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- Similarly, if $N \geq p^+(\sigma)/2$ then $\sigma^{\otimes N}$ has a dense subspace of matrix coefficients in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$, and then $\sigma^{\otimes N}$ is weakly contained in r_G (Cowling-Haagerup-Howe).

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- The general case was established by Cowling (1978) for simple Lie groups, using Herz' majorization principle to reduce the problem to that of estimating the operator norm in the boundary representation.
  For simple algebraic Chevalley groups over local fields the inequality is due to Veca (2002), using the same method.

Spectral transfer principle

- **Corollary:** for EVERY prob. meas. on G of the form $\beta = \chi_B / m_G(B)$, and for all $1 < s < 2$, (with $\frac{1}{s} + \frac{1}{s'} = 1$) :

$$\|r_G(\beta)\|_{L^2(G)\to L^2(G)} \le C_s \left\| \frac{\chi_{B_t}}{m_G(B_t)} \right\|_{L^s(G)} = C_s m_G(B)^{-1/s'} .$$

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- Theorem C. Spectral transfer principle. [N 98], [Gorodnik+N 2005]. For every unitary representation π of G with a spectral gap and no finite-dimensional invariant subspaces, and for every family of probability measures $\beta_t = \chi_{B_t}/m_G(B_t)$, the following norm decay estimate holds (for every $\epsilon > 0$)

$$\|\pi(\beta_t)\| \le \|r_G(\beta_t)\|^{\frac{1}{n_e(\pi)}} \le C_\varepsilon m(B_t)^{-\frac{1}{2n_e(\pi)}+\varepsilon} ,$$

with $n_e(\pi)$ the least even integer $\ge p^+(\pi)$.

The effective mean ergodic theorem

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- Thm. D. Effective mean ergodic theorem. [N 98], [Gorodnik+N 2005]. For any weak mixing action of a simple algebraic group G which has a spectral gap, and for any family $B_t \subset G$ with $m_G(B_t) \to \infty$, the convergence of the time averages $\pi_X(\beta_t)$ to the space average takes place at a definite rate :

$$\left\| \pi(\beta_t)f - \int_X f d\mu \right\|_{L^2(X)} \leq C_\theta \left(m_G(B_t)\right)^{-\theta} \|f\|_2 \ ,$$

for every $0 < \theta < \frac{1}{2n_e(\pi_X^0)}$.

The effective mean ergodic theorem

- In particular, we can bound the norm of the averaging operators of $\pi_X(\beta_t)$ acting on $L^2_0(X)$, when the action is ergodic and weak mixing, namely has no finite-dimensional invariant subspaces.

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- Note that the only requirement needed to obtain the effective mean ergodic Thm' is simply that $m(B_t) \to \infty$, and the geometry of the sets is not relevant at all. This fact allows a great deal of flexibility in its application.

Some comments on the spectral transfer principle

- The norm estimate of the operator $\pi(\beta)$ in a general rep' π, has been reduced to a norm estimate for the convolution operator $r_G(\beta)$ in the regular rep' r_G. This establishes for simple groups an analog of the transfer(ence) principle for amenable groups.

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- When the sets B_t are bi-K-invariant for a (suitable) maximal compact sbgp K, a better estimate holds, namely

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- We now turn to consider our first application of the effective mean ergodic theorem for averages on simple algebraic groups, namely the solution of the lattice point counting problem in domains B_t in G.

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- 1) Haar measure $m(B_t)$ is the main term in the asymptotic,
- 2) There is an error estimate of the form

$$\frac{|\Gamma \cap B_t|}{m(B_t)} = 1 + O\left(m(B_t)^{-\delta}\right)$$

where $\delta > 0$ and is as large as possible, and is given in an explicit form,

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- 4) the solution should establish whether the error estimate can be taken as uniform over all (or some) finite index subgroups $\Lambda \subset \Gamma$, and over all their cosets, namely:

$$\frac{|\gamma\Lambda \cap B_T|}{m(B_T)} = \frac{1}{[\Gamma : \Lambda]} + O\left(m(B_T)^{-\delta}\right)$$

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- Main point: A general solution obeying the 4 requirements above can be given for lattices in simple algebraic groups and general domains B_t, using a method based on the effective mean ergodic theorem for G.

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- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$, t_0 and ε_0 such that for all $t \geq t_0$ and $\varepsilon < \varepsilon_0$ we have :

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

$$m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(B_t). \tag{2}$$

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- An increasing family of bounded Borel subset B_t, $t > 0$, of G will be called admissible if there exists $c > 0$, t_0 and ε_0 such that for all $t \geq t_0$ and $\varepsilon < \varepsilon_0$ we have :

$$\mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}, \tag{1}$$

$$m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon) \cdot m_G(B_t). \tag{2}$$

- When B_t are admissible, pointwise almost sure convergence holds for the averages β_t, with a prescribed rate of convergence (N. '98, Margulis+N+Stein '99, N. '04, Gorodnik+N, '06)

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- Assume that the error term in the mean ergodic theorem for β_t in $L^2(m_{G/\Gamma})$ satisfies

$$\left\| \pi(\beta_t) f - \int_{G/\Gamma} f \, dm_{G/\Gamma} \right\|_{L^2} \leq C m(B_t)^{-\theta} \left\| f \right\|_{L^2}$$

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$$\frac{|\Gamma_t|}{m_G(B_t)} = 1 + O\left(m(B_t)^{-\theta/(\dim G + 1)} \right).$$

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- Clearly ϕ is a bounded function on G/Γ with compact support,

$$\int_G \chi_\varepsilon \, dm_G = 1, \quad \text{and} \quad \int_{G/\Gamma} \phi_\varepsilon \, d\mu_{G/\Gamma} = 1.$$

- Let us apply the mean ergodic theorem to the function ϕ_ε. It follows from Chebycheff's inequality that for every $\delta > 0$,

$$m_{G/\Gamma}(\{h\Gamma \in G/\Gamma : |\pi_{G/\Gamma}(\beta_t)\phi_\varepsilon(h\Gamma) - 1| > \delta\}) \longrightarrow 0$$

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Claim I. Given $\varepsilon, \delta > 0$, for t sufficiently large, there exists $g_t \in \mathcal{O}_\varepsilon$ satisfying

$$1 - \delta \le \frac{1}{m_G(B_t)} \int_{B_t} \phi_\varepsilon(g^{-1} g_t \Gamma) dm_G \le 1 + \delta\,.$$

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- Claim II. For sufficiently large t and any $h \in \mathcal{O}_\varepsilon$,

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- Hence,

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- Now taking t sufficiently large, $h = g_t$ and using Claims I and II

$$|\Gamma_t| \leq (1+\delta)m(B_{t+\varepsilon}) \leq$$

$$\leq (1+\delta)(1+c\varepsilon)m(B_t),$$

by admissibility. The lower estimate is proved similarly.

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- By the mean ergodic theorem and Chebycheff's inequality :

$$m_{G/\Gamma}(\{x \in G/\Gamma : |\pi_{G/\Gamma}(\beta_t)\phi_\varepsilon(x) - 1| > \delta\})$$

$$\leq C\delta^{-2}\varepsilon^{-n}m(B_t)^{-2\theta}.$$

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- Taking $\delta \sim \varepsilon \sim m(B_t)^{-\theta/(n+1)}$ to balance the two significant parts of the error appearing in the estimate $(1+\delta)(1+c\varepsilon)$, the result follows.

Uniformity in the lattice point counting problem

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- This holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L^2_0(G/\Lambda)$ has a positive lower bound.

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- Indeed all that is needed is that the averaging operators $\pi(\beta_t)$ satisfy the same norm decay estimate in the space $L^2(G/\Lambda)$.
- This holds when the set of finite index subgroups satisfy property τ, namely when the spectral gap appearing in the representations $L^2_0(G/\Lambda)$ has a positive lower bound.
- Property τ has been shown to hold for the set of congruence subgroups of any arithmetic lattice in a semisimple Lie group (Burger-Sarnak, Lubotzky, Clozel..... generalizing the Selberg property).

Counting unimodular integral matrices

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• Then :

$$N'_T = C_n T^{n^2-n} + O_n\left(T^{n^2-n-\frac{1}{2n(n+1)^2}}\right)$$

$$= N_T\left(1 + O_n\left(T^{-\frac{1}{2n(n+1)^2}}\right)\right)$$

Some previous results

- The classical non-Euclidean counting problem asks for the number of lattice points in a Riemannian ball in hyperbolic space $\mathbb{H}^d$. For this specific case, the best known bound is still the one due to Selberg (1940's) or Lax-Phillips (1970's). Bruggeman, Gruenwald and Miatello have established similar quality results for lattice points in Riemannian balls in products of $SL_2(\mathbb{R})$'s (2008).

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- Eskin-McMullen (1990) devised the mixing method, which applies to general (well-rounded) sets, but have not produced an error estimate. Maucourant (2005) has obtained an error estimate using an effective form of the mixing method, and so have Benoist-Oh in the S-algebraic case (2012). The resulting estimates are weaker than those of Thm. E.

Counting rational points

- Counting rational points on algebraic varieties homogeneous under a simple algebraic group G defined over $\mathbb{Q}$ has been considered by Shalika, Takloo-Bighash and Tschinkel (2000's). They used direct spectral expansion of the height zeta function in the automorphic representation (using, in particular, regularization estimates for Eisenstein series).

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- Gorodnik, Maucourant and Oh (2008) have used the mixing method in the problem of counting rational points.

- It is possible to consider the corresponding group G over the ring of adéles, in which the group of rational points is embedded as a lattice. Generalizing the operators norm estimates from $G(F)$ for all field completions F to the group of adéles, it is possible to use the method based on the effective mean ergodic theorem here too. The error estimate of Thm. E is better that the estimate that both other methods produce.