Representation theory effective ergodic theorems, and applications

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Representation Theory, Dynamics and Geometry

CIRM, Luminy

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Based on joint work with Alex Gorodnik, and on joint work with Anish Ghosh and Alex Gorodnik

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- Talk I : Averaging operators in dynamical systems and effective ergodic theorems

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- Talk IV : Fast equidistribution of dense lattice orbits, and best possible Diophantine approximation on homogeneous algebraic varieties

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- Do the time averages converge ? If so, what is their limit ?

Ergodicity and the ergodic hypothesis

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- Bolzmann's Ergodic Hypothesis : for an ergodic flow, the time averages of an observable f converge to the space average of f on phase space, namely to $\int_M f \, d\mathrm{vol}_M$.

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- For the proof, von-Neumann utilized his recently established spectral theorem for unitary operators.

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• To conclude the proof that $\beta_t f \to \mathcal{E}_I f$ for every f, note first that if f is invariant, then $\beta_t f = f = \mathcal{E}_I f$ for all t,

• and finally that the span of $\{a_s h - h\,;\ s \in \mathbb{R}, h \in \mathcal{H}\}$ is dense in the orthogonal complement of the space of invariants.

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- von-Neumann established this class as a common generalization of compact groups and Abelian groups by proving the existence of Haar measure for compact groups, and the existence of invariant means (Banach limits) for general Abelian groups.

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- Følner showed that the existence of an asymptotically invariant sequence is equivalent to the existence of an invariant mean, namely it characterizes meanable groups, subsequently renamed amenable groups.

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- We will briefly mention some of the results, but first let us introduce the general set-up of ergodic theorems and the averaging operators which will be our main subject below.

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- Basic problem : study the averaging operators

$$\pi_X(\beta_t)f(x) = \frac{1}{|B_t|}\int_{B_t} f(g^{-1}x)dm_G(g)$$

and their convergence properties

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- The transference principle which reduces the maximal inequality for the operators $\pi_X(\beta_t)$ in a general action to the maximal inequality for the convolution operators $r_G(\beta_t)$ in the regular action of the group on itself by translations, (Wiener 1939, Calderon 1952, Herz 1968, Coifman-Weiss 1974......)

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- Existence of invariant measures for continuous actions of on compact metric spaces, and the resulting theory of equidistribution (Krylov-Bogliubov, 1950......)

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- In every properly ergodic action of G on a probability space X, there exists an asymptotically invariant sequence of unit vectors $f_k \in L^2_0$ (with zero mean), namely a sequence such that $\|\pi_X(g)f_k - f_k\| \to 0$ for all $g \in G$.

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- Corollary : in properly ergodic actions of amenable groups no rate of convergence to the ergodic mean can be established, in the operator norm.

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- We now turn to a systematic study of averaging operators which are strict contractions.

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  for all $f \in L^2(X)$ and a fixed $\eta(\beta) > 0$.

- Here generating means that the support of $\beta^* * \beta$ generates G as a semigroup.

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- In fact, an even more remarkable property holds, namely the following strong uniform estimate.
- G has property T if and only if for every absolutely continuous generating measure β there exists $\alpha(\beta) < 1$, such that in every ergodic action of G on X, the following uniform operator norm estimate holds : $\|\pi_X(\beta)\|_{L^2_0(X)} \le \alpha(\beta)$.

Results on spectral gaps and ergodic theorems

- The possibility of ergodic theorems with rate of convergence for actions of free groups with a spectral gap was first realized by the Lubotzky-Phillips-Sarnak construction of a dense free group of isometries of $\mathbb{S}^2$ which has an optimal (!) spectral gap (1985-6).

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- For groups acting simply transitively on the vertices of an $\tilde{A}_2$ building Cartwright-Mlotkowsky-Steger (1993) have established property T directly and computed the norms of natural averages on the group explicitly. As a consequence one can obtains an effective ergodic theorem for these averages in general ergodic actions of the group.

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- For higher rank connected simple Lie groups with finite center the effective pointwise ergodic theorem for radial averages was established by Margulis+N+Stein 1999.

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The mean and pointwise ergodic theorem

Thm. A. Effective ergodic theorems for lattice subgroups.
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where $\theta_p = \theta_p(X) > 0$.

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- We emphasize that this result holds for all Γ-actions. The only connection to the original embedding of Γ in the group G is in the definition of the sets Γ_t.

Further comments

- In particular, if Γ has property T, then the ultimate effective mean and pointwise ergodic theorems hold in every ergodic measure-preserving action, namely convergence takes place with a fixed uniform rate $\theta_p = \theta_p(G) > 0$ independent of X.

Further comments

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$$\left\| \lambda_t f - \int_X f d\mu \right\|_2 \le C m(B_t)^{-\theta_2} \left\| f \right\|_2 ,$$
- Clearly, this estimate goes well beyond the contraction property guaranteed by the spectral gap, and furthermore it holds uniformly over families of finite-index subgroups provided the group satisfies property T, or more generally the subgroups satisfy Lubotzky-Zimmer property τ.

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- Tempelman has proved mean ergodic theorems for averages on semisimple Lie group using spectral theory, namely the Howe-Moore vanishing of matrix coefficients theorem (1980's).

Ergodic theorems for general lattice actions

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- Both Thm. A and Thm. B have more general versions applying to lattice in general semisimple S-algebraic groups over local fields.