On Certain Tempered Unitary Representations of Gromov Hyperbolic Groups

Tim Steger

Università degli Studi di Sassari

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This is an overview of a project which I and others have been working on for the last 30 years. I was introduced to most of the basic ideas by my advisor:

Sandro Figà-Talamanca (Roma 1)

who had developed those ideas during a collaboration with:

Massimo Picardello (Roma 2)

Besides Sandro, my principal collaborator on this project has been:

Gabriella Kuhn (Milano Bicocca)

Other collaborators include:

Chris Bishop (NYU, Stony Brook)
Michael Cowling (UNSW, Sydney)
Waldek Hebisch (Wrocław)
Alessandra Iozzi (ETH, Zürich)
Sandra Saliani (Potenza)

Of course, relevant work has been done by people who aren't my direct collaborators, for example:

Bill Paschke (Kansas)

- Most of our work has been done in the case where Γ is a non-abelian free group with finitely many generators; $A \subseteq \Gamma$ is some set of free generators together with their inverses.
- One expects that many of the results generalize to the case where Γ is a finitely generated, discrete, non-elementary **Gromov hyperbolic** group; $A \subseteq \Gamma$ is any finite set of generators, closed under inverse.

[Ohshika, 1998, Chapter 2] is an excellent, concise exposition of lots of basic material about Gromov hyperbolic groups.

Let Ω denote the boundary of Γ. When Γ is free, this is the usual boundary of the tree which is the Cayley graph of (Γ, A). For more general hyperbolic groups, there is a definition of Ω which generalizes this.

- Ω is a second-countable compact Hausdorff space.
- $\Gamma \cup \Omega$ has a topology which makes it a compactification of the discrete space Γ.
- There is a Γ-action on Ω, which fits together with the left action of Γ on itself to give a Γ-action on $\Gamma \cup \Omega$.

Our basic object of interest is a **tempered irreducible unitary** representation of Γ:

$$\pi : \Gamma \to \mathcal{U}(\mathcal{H})$$

where $\mathcal{H} = \mathcal{H}_\pi$. **Irreducible** means that the only Γ-invariant subspaces of $\mathcal{H}$ are zero and $\mathcal{H}$ itself. **Tempered** means that π is weakly contained in the regular representation:

$$\pi_{\mathsf{reg}} : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$$

According to a little result in [Cecchini–Figà-Talamanca, 1974], π cannot be both irreducible and strongly contained in $\ell^2(\Gamma)$. This depends only on $|\Gamma| = \infty$.

Since Γ is a type $\mathbf{II}$ group, the usual program of classifying all tempered irreducible representations up to equivalence is hopeless!

Our goals are more modest:

- Construct concrete examples of tempered irreducible unitary representations.
- Find uniform constructions which produce large families of tempered irreducible unitary representations.
- Prove that the representations constructed are irreducible.
- Establish the equivalence/inequivalence of any two of the examples.

The other (and sexier) theme is the study of how a tempered unitary representation π can be realized in a nice way as acting on a space of the form $L^2(\Omega)$. Here:

- The measure on Ω is deliberately omitted; any quasi-invariant measure is acceptable.
- The L^2-functions might be scalar valued, but they might be vector-valued, even vector-valued with values in an infinite-dimensional Hilbert space. All these possibilities are considered equally acceptable.
- The action of $\pi(x)$ must be made up of (1) simple translation by x followed by (2) application of a multiplier, depending on x, which is a function of $\omega \in \Omega$, and which is operator-valued if the L^2-functions are vector-valued.

Let $C(\Omega)$ be the commutative C*-algebra of continuous, complex-valued functions on Ω. Given an identification of some Hilbert space $\mathcal{H}$ with some $L^2(\Omega)$, as above, we get a multiplication representation:

$$C(\Omega) \to \mathcal{L}(\mathcal{H})$$

Vice versa, the **spectral theorem** says that any such representation gives rise to an identification of $\mathcal{H}$ with some $L^2(\Omega)$. (By sloppiness, I omit the possibility that the L^2-space is vector-valued, with the vectors lying in spaces of variable dimension.) Moreover, any two identifications corresponding to the same representation of $C(\Omega)$ are equivalent in an obvious way.

Moral: Identification with some $L^2(\Omega)$ corresponds to an action of $C(\Omega)$.

Now suppose $\mathcal{H} = \mathcal{H}_\pi$. If we want to identify $\mathcal{H}$ with some $L^2(\Omega)$ and if we want the action of $\pi(\Gamma)$ to be nice, then a certain relationship has to hold between the actions of Γ and of $C(\Omega)$ on $\mathcal{H}$.

- For convenience, use π to denote the action of $C(\Omega)$ as well as the action of Γ.
- Define the left translation action of Γ on $C(\Omega)$ by

$$(\lambda(x)F)(\omega) = F(x^{-1}\omega)$$

- The necessary relationship between the actions of Γ and $C(\Omega)$ is:

$$\pi(x)\pi(F)\pi(x^{-1}) = \pi(\lambda(x)F) \tag{1}$$

This corresponds to some of Mackey's big ideas, and is not that hard to see in the case of discrete groups.

Summary: a representation π of Γ on $\mathcal{H}$, together with some nice identification of $\mathcal{H}$ with some $L^2(\Omega)$, corresponds to a pair made up of

1. The original representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$;
2. another representation $\pi : C(\Omega) \to \mathcal{L}(\mathcal{H})$;

and this pair of representations must satisfy (1).

There is a certain C*-algebra, not difficult to construct, and denoted $\Gamma \ltimes_\lambda C(\Omega)$ whose representations correspond precisely to pairs of representations satisfying (1). This algebra is called a **crossed-product** C*-algebra.

I often use the phrase **boundary representation** to refer to a representation of $\Gamma \ltimes C(\Omega)$. A boundary representation corresponds to some representation of Γ on $\mathcal{H}$ together with some identification of $\mathcal{H}$ with some $L^2(\Omega)$.

Proposition: If π' is a representation of $\Gamma \ltimes C(\Omega)$ on $\mathcal{H}'$, then the Γ-representation

$$\pi' : \Gamma \to \mathcal{U}(\mathcal{H}')$$

is tempered.

The proof is in two references: [Adams, 1994] establishes that the action of Γ on Ω is topologically amenable for any hyperbolic group Γ and its (hyperbolic group) boundary. Moreover, he observes that topological amenability implies Zimmer amenability for any quasi-invariant measure on Ω. [Kuhn, 1994] shows that this universal Zimmer amenability implies that all nice representations of Γ on spaces $L^2(\Omega)$ are tempered, i.e. weakly contained in the regular representation.

Proposition: If π is a tempered irreducible representation of Γ on $\mathcal{H}_\Gamma$, then there exists some representation π' of $\Gamma \ltimes C(\Omega)$ on $\mathcal{H}'$ and some isometric Γ-inclusion $\iota : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$.

In other words, every tempered irreducible representation of Γ is a subrepresentation of some boundary representation. This proposition says nothing about the uniqueness of π' and ι.

Proof: Choose any unit vector $u \in \mathcal{H}_\Gamma$. By irreducibility, u is cyclic. Let ϕ be the matrix coefficient of u:

$$\phi(x) = \langle u, \pi(x)u \rangle$$

Then ϕ is positive-definite. Starting with ϕ, the Gelfand–Naimark construction reproduces the triple $(\pi, \mathcal{H}_\pi, u)$.

Since π is weakly contained in the regular representation, we can find vectors $u_n \in \ell^2(\Gamma)$ so that the positive definite functions

$$\phi_n(x) = \langle u_n, \pi_{\mathsf{reg}}(x) u_n \rangle$$

tend to ϕ pointwise.

Since $\Gamma \subset \Gamma \cup \Omega$, $\ell^2(\Gamma)$ has a natural extension to a representation of $\Gamma \ltimes C(\Gamma \cup \Omega)$. Consequently, each u_n gives rise to a positive definite functional (or state):

$$\psi_n : \Gamma \ltimes C(\Gamma \cup \Omega) \to \mathbf{C}$$

Passing to a subsequence, we may assume that $\psi_n \to \psi$ weakly for some ψ. The "restriction" of ψ_n to Γ is ϕ_n, so the "restriction" of ψ to Γ is ϕ.

Let π' be the representation of $\Gamma \ltimes C(\Gamma \cup \Omega)$ obtained from ψ by the Gelfand–Naimark construction, and let u' be the corresponding special vector. The matrix coefficient of u' is $\psi|_\Gamma = \phi$, while ϕ is the matrix coefficient of u. Consequently there exists a Γ-isometry $\iota : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ determined by $\iota(u) = u'$.

Finally, by irreducibility, all of $\mathcal{H}_{\pi'}$ must lie over Γ, or all of it must lie over Ω. The first possibility is excluded by irreducibility and the aforementioned result from [Cecchini–Figà-Talamanca, 1974]. $\square$

Actually, a similar but simpler proof proves a much more general result:

Proposition: Let Γ be any finitely generated discrete group, let π be any tempered unitary representation of Γ, and let Ω be any non-empty second countable compact space on which we have a Γ-action. Then there exists a representation π' of $\Gamma \ltimes C(\Omega)$ and an isometric Γ-inclusion $\iota : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$.

Proof: Fix some $\omega \in \Omega$, and let Γ_0 be its stabilizer. The orbit of ω is a copy of Γ/Γ_0. One can identify $\ell^2(\Gamma)$, the representation space of the regular representation, with a representation of $\Gamma \ltimes C(\Gamma/\Gamma_0)$, hence with a representation of $\Gamma \ltimes C(\Omega)$. Now use the limit procedure as above to pass from the regular representation to any representation weakly contained in the regular representation. $\square$

These rather general considerations show that an irreducible unitary representation π of Γ can be expressed as a subrepresentation of a boundary representation if and only if it is tempered. So suppose that it is tempered.

Big Question: In how many different ways can it be so expressed?

One could always add extra, irrelevant terms to the boundary representation π'. To avoid that, assume we are looking for:

- a boundary representation $\pi' : \Gamma \ltimes C(\Omega) \to \mathcal{L}(\mathcal{H}')$,
- an isometric Γ-map $\iota : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$,

so that

- $\pi'(C(\Omega))\iota(\mathcal{H}_\pi)$ is dense in $\mathcal{H}_{\pi'}$.

Based on the study of many examples and families of examples, we have found that at least three things can happen, depending on which tempered irreducible representation π we are working with:

- There is (up to the obvious equivalence) only one possibility for (π', ι). Moreover, $\iota : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ is bijective. — "Monotony"

- There are precisely two possibilities for (π', ι), and for each possibility ι is bijective. — "Duplicity"

- There is only one possibility for (π', ι), but ι is not bijective. Instead, $\mathcal{H}_{\pi'}$ breaks up into two irreducible Γ-stable subspaces, which are inequivalent as Γ-representations. Evidently these two subspaces are $\iota(\mathcal{H}_\pi)$ and the orthogonal complement of $\iota(\mathcal{H}_\pi)$. — "Oddity"

Even for a single example, one must identify one or two possibilities for (π', ι) and then prove that there are no others. Most of the examples for which this has been carried through are for the free group.

Another doable case arises when Γ is a lattice subgroup of a rank 1 Lie group G, and π is the restriction of an irreducible representation of G. All three possibilities can arise when G is, for example, $SL(2, \mathbf{R})$.

Based solely upon the study of examples, we

Conjecture: For any tempered irreducible representation π of Γ, the possibilities for (π', ι) conform to one of the three cases on the last slide.

How does one construct tempered irreducible representations of Γ? Often one constructs irreducible representations of $\Gamma \ltimes C(\Omega)$.

These are automatically tempered. They are **not** automatically irreducible as representations of Γ. However there are various circumstances under which the irreducibility as a representation of $\Gamma \ltimes C(\Omega)$ can be leveraged to prove irreducibility as a representation of Γ.

Beyond this general observation, there are certainly many possibilities. For instance one, can construct some quasi-invariant measure ν on Ω and consider the quasi-regular representation on $L^2(\Omega, d\nu)$.

In [Kuhn–Steger, 2004] we present another rather general construction for the case of the free group. In ongoing work (Iozzi, Kuhn and Steger), we have generalized that construction to arbitrary hyperbolic groups, but much, much less has been proved. Indeed, merely the construction of the representation presents all sorts of new difficulties.

Here is an outline of the method:

- One fixes a certain finite collection of finite matrices. These are parameters for the entire construction.
- Based on the parameters, one constructs a collection of "elementary" functions on Γ. The collection of "elementary" functions is stable under left translation.
- One lets $\mathcal{H}^\infty$ be the space of finite linear combinations of "elementary" functions.

Note that the "elementary" functions take their values in a certain finite vector space, as do the functions in $\mathcal{H}^\infty$. Sadly, I must omit entirely any discussion of what the "elementary" functions are.

We wish to put a translation-invariant positive semidefinite inner product on $\mathcal{H}^\infty$. One uses the definition:

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \to 0+} \epsilon \sum_{x \in \Gamma} e^{-\epsilon d(e,x)} \langle f_1(x), f_2(x) \rangle$$

In order for these limits to exist and not be identically zero, one needs the functions in $\mathcal{H}^\infty$ to be almost, but not quite, in $\ell^2(\Gamma)$. Given the form of our "elementary" functions, it turns out that this will be true so long as the parameters satisfy a single real-valued condition. Specifically a certain Perron–Frobenius eigenvalue must equal 1.

Suppose we left-translate f_1 and f_2 by $z \in \Gamma$. The only change in the above formula is $e^{-\epsilon d(e,x)} \rightsquigarrow e^{-\epsilon d(e,zx)}$. Since

$$e^{-\epsilon d(e,z)} \leq \frac{e^{-\epsilon d(e,zx)}}{e^{-\epsilon d(e,x)}} \leq e^{\epsilon d(e,z)}$$

the inner product $\langle f_1, f_2 \rangle$ remains the same.

Now define $\mathcal{H} = \mathcal{H}_\pi$ as the quotient-completion of $\mathcal{H}^\infty$, and define π as the extension by continuity of the left-translation action of Γ on $\mathcal{H}^\infty$.

How do we define the action of $C(\Omega)$ on $\mathcal{H}$?

- First consider a function of the form $e^{-\epsilon d(e,x)/2} f(x)$ for $f \in H^\infty$. This belongs to $\ell^2(\Gamma)$.
- As such, it is part of a representation on which $\ell^\infty(\Gamma)$ acts by multiplication. In particular, that representation admits an action by $C(\Gamma \cup \Omega)$.
- For each ϵ, this enables us to find a positive definite functional ψ_ϵ on the C*-algebra $\Gamma \ltimes C(\Gamma \cup \Omega)$.

- One can find a subsequence $(\epsilon_j)_j$ tending to zero so that ψ_{ϵ_j} converges weakly to some ψ.
- After considerable work, one verifies that
  - The representation corresponding to the limit positive definite function ψ factors through the quotient map $\Gamma \ltimes C(\Gamma \cup \Omega) \to \Gamma \ltimes C(\Omega)$.
  - The Γ-part of the limit representation is canonically identified with the representation π constructed above.
  - The limit is actually independent of the choice of subsequence $(\epsilon_j)_j$.

The action of $C(\Omega)$ which comes from the limit representation is the one we are looking for.

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