Persistent homology and minimum spanning acycle for random simplicial complexes

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Persistent homology

- Homology theory first appeared around 1900 (Poincaré).
- Persistent homology theory appeared around 2000 independently by
  - Frosini, Ferri et al. in Bologna, Italy.
  - Robins, Colorado, Boulder,
  - Edelsbrunner, at Duke, North Carolina.
- Software for computing persistent homology has also been developed and used for data analysis such as material science, biology etc.

Persistent homology

$\approx$ time-dependent version of homology

Persistent homology for random object

$\approx$ Stochastic process in homology theory

Filtration and persistent homology

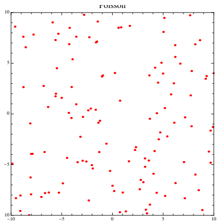
- Input 1: $\boxed{\text{Point cloud data}}$

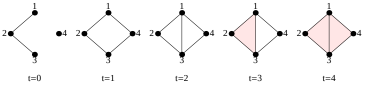
$$\Downarrow \text{ via Čech or Rips-Vietoris complex etc.}$$

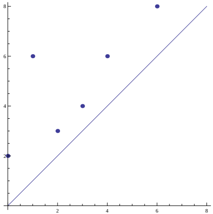
- Input 2: $\boxed{\text{Filtration} = \text{increasing sequence of simplicial complex}}$

$$\Downarrow$$

- Output: $\boxed{\text{Persistence diagram}}$







Random filtrations

Figure: 20 Poisson points and its filtration $\{X(t), t \geq 0\}$

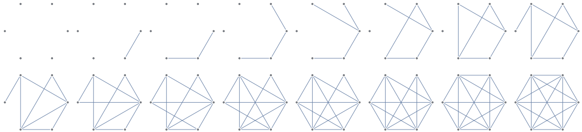


Figure: Erdös-Rényi random graph process $\{X(t), t \geq 0\}$ for $n = 6$

- Homology vs. Persistent homology:
  - $H_0(X(t))$ = connected components of $X(t)$.
  - $H_1(X(t))$ = cycles of $X(t)$.
  - $PH_0(\{X(t)\}_{t\geq 0})$ = "death" of connected components.
  - $PH_1(\{X(t)\}_{t\geq 0})$ = "birth and death" of cycles.

Atomic configuration of SiO2

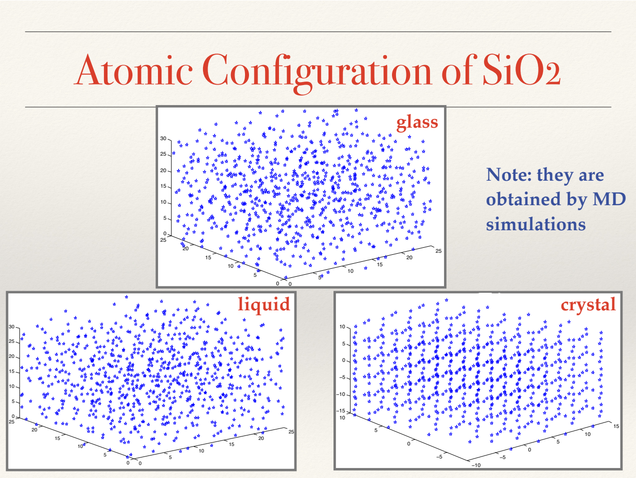


Figure: T. Nakamura, Y. Hiraoka, A. Hirata, et al.
http://arxiv.org/abs/1501.03611 and http://arxiv.org/abs/1502.07445

1-dim. Persistence diagram

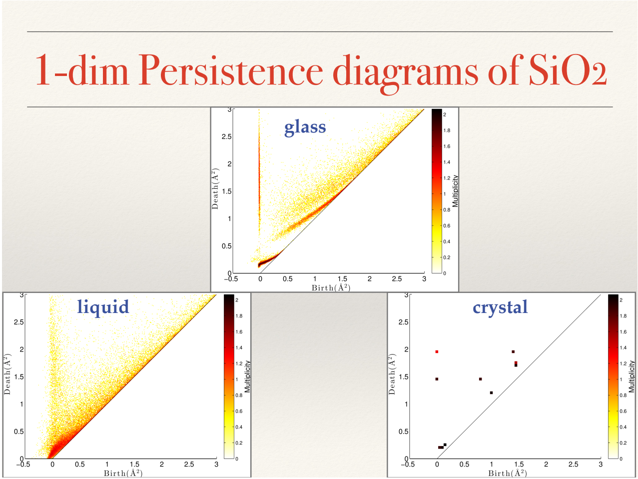


Figure: T. Nakamura, Y. Hiraoka, A. Hirata, et al.
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Minimum spanning tree (MST)

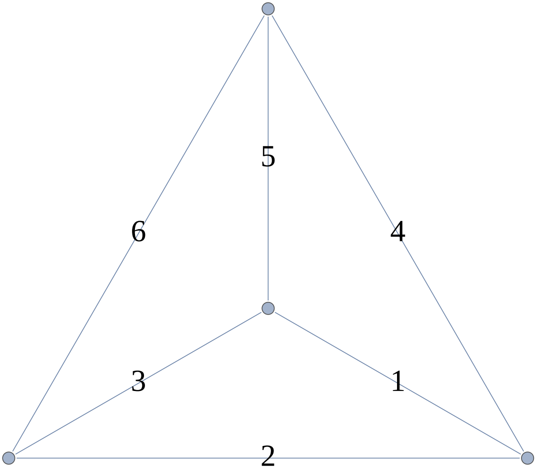


Figure: Weighted graph K_4

Minimum spanning tree (MST)

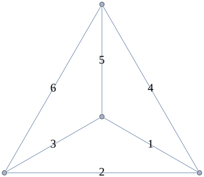


Figure: Weighted graph K_4

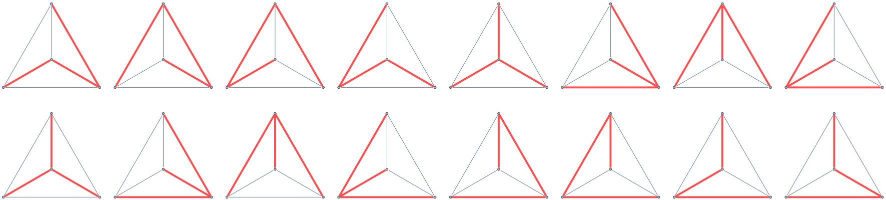


Figure: All possible 16 spanning trees on K_4

Minimum spanning tree (MST)

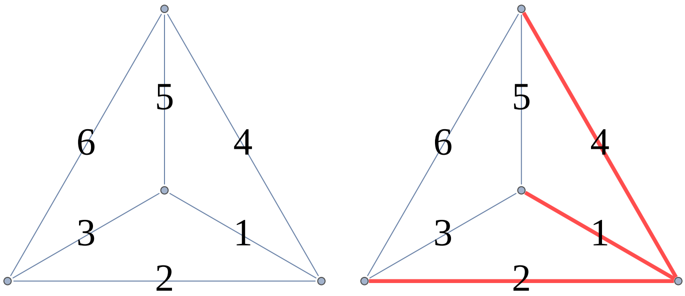


Figure: Minimum spanning tree (MST) on a weighted graph K_4

Frieze' result on MST

- Let $K_n = (V_n, E_n)$ be the complete graph with n vertices, and for each edge $e \in E_n$, a uniform random variable $t(e)$ on $[0,1]$ is assigned independently.
- Minimum spanning tree is the spanning tree T which minimizes the weight

$$wt(T) = \sum_{e \in T} t(e), \quad W_n := \min_{T \in \mathcal{S}_n} wt(T),$$

where $\mathcal{S}_n$ is the set of spanning trees in K_n. Remark that

$$|\mathcal{S}_n| = n^{n-2}$$

by Cayley's theorem (1889).
- Frieze (1985): As $n \to \infty$,

$$\mathbb{E}[W_n] \to \zeta(3) = 1.20206\ldots.$$

- Janson (1995): the CLT:

$$\sqrt{n}(W_n - \zeta(3)) \to N(0, \sigma^2), \quad \sigma^2 = 6\zeta(4) - 4\zeta(3) \approx 1.68571....$$

A generalization of the problem of MST

- The purpose of this talk is to extend Frieze's result to higher dimensions.
  - (random) graph $\Longrightarrow$ (random) simplicial complex
  - spanning tree $\Longrightarrow$ spanning acycle
- We interpret the weight of the minimum spanning tree in terms of persistent homology.
- Kruskal's algorithm of finding MST $\Longrightarrow$ Erdös-Rényi graph process.

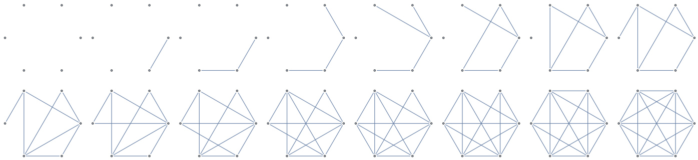


Figure: Erdös-Rényi random graph process for $n = 6$

Simplicial complex

V : a finite set

X : a collection of non-empty subsets of V.

Definition

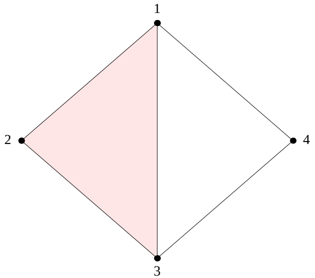
X is said to be a (n abstract) simplicial complex on V if

1. $\{v\} \subset X$ *for all* $v \in V$.
2. X *is closed under the operation of taking nonempty subsets, i.e.,*

$$\sigma \in X, \ \emptyset \neq \tau \subset \sigma \Longrightarrow \tau \in X.$$

- $\sigma \subset V$ with $|\sigma| = k + 1$ is called k-simplex or k-face.
- We write $\dim(\sigma) = k$ if σ is a k-face.
- When $d = \max_{\sigma \in K} \dim(\sigma)$, we say that X is a d-dimensional simplicial complex or just d-complex.

Example of simplicial complex



- $X = \{1, 2, 3, 4, 12, 13, \underline{14}, 23, \underline{34}, \underline{123}\}$ is a 2-dimensional simplicial complex.
- X is determined by facets $\{14, 34, 123\}$, i.e. maximal faces w.r.t. inclusion.

$$X = \{\underbrace{\underbrace{1, 2, 3, 4}_{X_0}, \underbrace{12, 13, 14, 23, 34}_{X_1}}_{X^{(1)}\text{: a graph}}, \underbrace{123}_{X_2}\}$$

- $X^{(i)} = \{\sigma \in X : \dim(\sigma) \le i\}$: i-skeleton.
- $X_i = \{\sigma \in X : \dim(\sigma) = i\}$: i-faces of X.

Homology group

- Chain group: for oriented i-faces $\langle \sigma \rangle = \langle v_0 v_1 \dots v_i \rangle$, we set

$$C_i(X,\mathbb{Z}) := \{ \sum_{\sigma : i\text{-faces of } X} a_\sigma \langle \sigma \rangle : a_\sigma \in \mathbb{Z} \}$$

- Boundary operator: $\partial_i : C_i(X,\mathbb{Z}) \to C_{i-1}(X,\mathbb{Z})$

$$\partial_i \langle v_0 v_1 \dots v_i \rangle := \sum_{j=0}^{i} (-1)^j \langle v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_i \rangle.$$

- Chain complex: $\partial_i \circ \partial_{i+1} = 0$ and

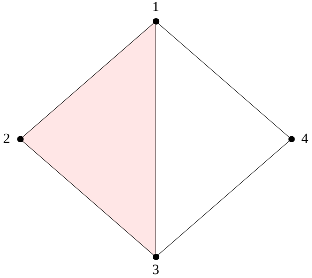
$$\cdots \stackrel{\partial_{i+2}}{\to} C_{i+1}(X,\mathbb{Z}) \stackrel{\partial_{i+1}}{\to} C_i(X,\mathbb{Z}) \stackrel{\partial_i}{\to} C_{i-1}(X,\mathbb{Z}) \stackrel{\partial_{i-1}}{\to} \cdots$$

- Structure theorem for homology groups:

$$H_i(X,\mathbb{Z}) := Z_i(X,\mathbb{Z})/B_i(X,\mathbb{Z}) = \ker \partial_i / \mathrm{Im}\partial_{i+1} \cong \mathbb{Z}^{\beta_i} \oplus \bigoplus_{j=1}^{p} \mathbb{Z}_{\gamma_j},$$

where $\beta_i = \beta_i(X)$ is the i-th Betti number.

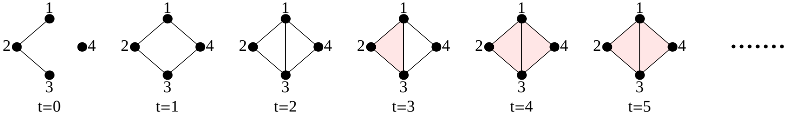
Matrix representation of boundary operator



$$\partial_1 = \begin{array}{c c} X_0 \backslash X_1 & \begin{array}{ccccc} 12 & 13 & 14 & 23 & 34 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}, \quad \partial_2 = \begin{array}{c c} X_1 \backslash X_2 & 123 \\ \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \end{array} & \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

Filtration

- $\mathbb{X} = (X(t))_{t \geq 0}$: an increasing sequence of simplicial complexes.



- $X(0) = \{1, 2, 3, 4, 12, 23\}$, $\quad T(1) = \cdots = T(23) = 0$,
- $X(1) = \{1, 2, 3, 4, 12, 23, 14, 34\}$ $\quad T(14) = T(34) = 1$,
- $X(2) = \{1, 2, 3, 4, 12, 23, 14, 34, 13\}$ $\quad T(13) = 2$,
- $X(3) = \{1, 2, 3, 4, 12, 23, 14, 34, 13, 123\}$ $\quad T(123) = 3$,
- $X(4) = \{1, 2, 3, 4, 12, 23, 14, 34, 13, 123, 134\}$ $\quad T(134) = 4$.

- $T(\sigma)$ denotes the *birth time* of σ.

Persistent homology group (I)

- Given a filtration $\mathbb{X} := \{X(t)\}_{t \geq 0}$, and then the birth times $\{T(\sigma)\}$. Suppose that the filtration is *saturated* in the sense that

$$\exists \tau \geq 0 \text{ s.t. } X(t) = X(\tau) \quad \forall t \geq \tau$$

- ℓ-th chain group as a graded module on the polynomial ring $\mathbb{F}[x]$:

$$C_\ell(\mathbb{X}) := \bigoplus_{t \geq 0} C_\ell(X(t), \mathbb{F}) = \{(c_t)_{t \geq 0} : c_t \in C_\ell(X(t), \mathbb{F})\}$$

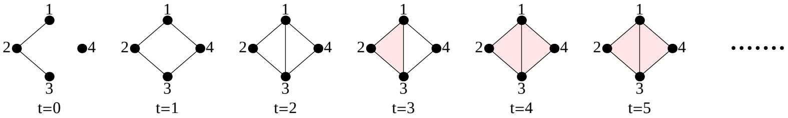
with the right-shift actions

$$x^s \cdot (c_0, c_1, \dots) := (\underbrace{0, \dots, 0}_{s\text{-times}}, c_0, c_1, \dots)$$

- Boundary operator $\partial_\ell(x) : C_\ell(\mathbb{X}) \to C_{\ell-1}(\mathbb{X})$:

$$\partial_\ell(x)\langle\langle v_0 v_1 \dots v_\ell \rangle\rangle = \sum_{j=0}^{\ell} (-1)^j x^{T(\sigma) - T(\sigma_i)} \langle\langle v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_\ell \rangle\rangle.$$

Matrix representation of boundary operator



$$\partial_1(x) = \begin{array}{c|c} X_0 \backslash X_1 & \begin{array}{ccccc} 12 & 13 & 14 & 23 & 34 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -1 & -x^2 & -x & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & x^2 & 0 & 1 & -x \\ 0 & 0 & x & 0 & x \end{bmatrix} \end{array}, \quad \partial_2(x) = \begin{array}{c|c} X_1 \backslash X_2 & \begin{array}{cc} 123 & 134 \end{array} \\ \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 34 \end{array} & \begin{bmatrix} x^3 & 0 \\ -x & x^2 \\ 0 & -x^3 \\ x^3 & 0 \\ 0 & x^3 \end{bmatrix} \end{array}$$

- We can see that $\partial_\ell(x) \circ \partial_{\ell+1}(x) = 0$ and

$$\cdots \overset{\partial_{\ell+2}}{\to} C_{\ell+1}(\mathbb{X}) \overset{\partial_{\ell+1}}{\to} C_\ell(\mathbb{X}) \overset{\partial_\ell}{\to} C_{\ell-1}(\mathbb{X}) \overset{\partial_{\ell-1}}{\to} \cdots$$

Persistent homology group (II)

- Structure theorem for persistent homology for $\mathbb{X} = \{X(t)\}_{t\geq 0}$:

$$PH_k(\mathbb{X}) := Z_k(\mathbb{X})/B_k(\mathbb{X}) \cong \bigoplus_{i=1}^{s}(x^{b_i})/(x^{d_i}) \oplus \bigoplus_{i=s+1}^{s+r}(x^{b_i})$$

- $(x^{b_i})/(x^{d_i}) \Longleftrightarrow$ persistent interval $[b_i, d_i)$ for an i-th homology class
- b_i : birth time, d_i : death time, $\ell_i := d_i - b_i$: lifetime.

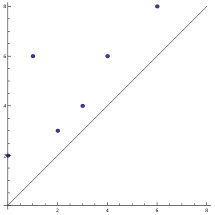


Figure: Persistence diagram

- For simplicity, we suppose $r = 0$, i.e., *finite* lifetimes.
- The sum of lifetimes of k-dimensional homology classes:

$$L_k = \sum_{i=1}^{s} \ell_i.$$

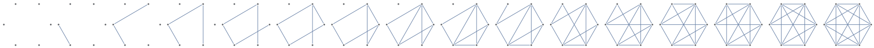
Persistent homology for random s.c.

deterministic

- $\mathbb{X} = (X(t))_{t \geq 0} \Rightarrow k$-th persistence diagram $\Rightarrow$ the sum of lifetimes L_k

stochastic

- Let $\mathbb{X} = (X(t))_{t \geq 0}$ be an increasing stochastic process of simplicial complexes, e.g., the Erdös-Rényi graph process



$$\mathbb{X} = (X(t))_{t \geq 0} \Longrightarrow \text{random } k\text{-th persistence diagram, } k = 0, 1, 2, \ldots$$
$$\Longleftrightarrow \text{point process } \xi \text{ on } \Delta = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$$
$$\Longrightarrow \text{the sum of lifetimes } L_k(\xi)$$

Proposition

When $\mathbb{X} = (X(t))_{0 \leq t \leq 1}$ is the Erdös-Rényi graph process,

$$L_0(\xi) = \textit{the weight of the minimum spanning tree}$$

Persistence diagram

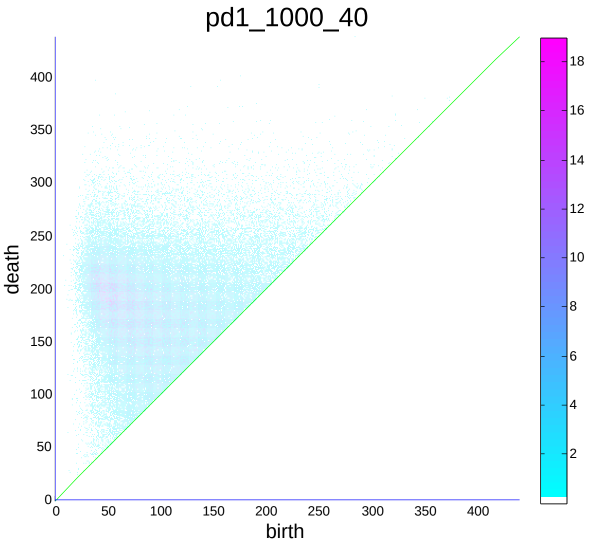


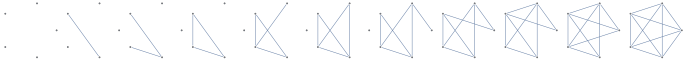
Figure: Persistence diagram for Erdös-Rényi clique complex $n = 40$

ℓ-Linial-Meshulam process

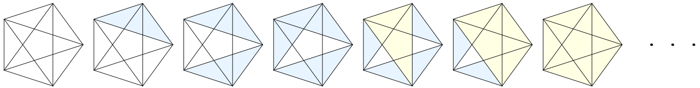
- $[n] = \{1, 2, \ldots, n\}$
- $\mathcal{F}_\ell := \binom{[n]}{\ell+1}$: the set of ℓ-faces or $(\ell+1)$-subsets of $[n]$.
- $\{t(\sigma) : \sigma \in \mathcal{F}_\ell\}$: i.i.d. uniform r.v.'s on $[0, 1]$: birth times.

$$X^{(\ell)}(t) = \underbrace{\bigcup_{j=0}^{\ell-1} \mathcal{F}_j}_{\text{all } j(<\ell)\text{-faces}} \cup \underbrace{\{\sigma \in \mathcal{F}_\ell : t(\sigma) \le t\}}_{\ell\text{-faces born before } t}.$$

- $(\ell = 1)$ $X^{(1)}(t)$ is the Erdös-Rényi graph process ($X^{(1)}(t) \overset{d}{=} G(n, t)$).



- $(\ell = 2)$ $X^{(2)}(t)$ starts from K_n at time 0, a 2-face is attached at random birth time, and ends up with complete 2-skeleton.



Threshold for connectivity for Erdös-Rényi graph

$X^{(1)}(t)$: Erdös-Rényi graph.

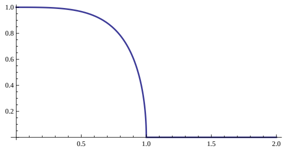
- $(\ell = 1)$ Erdös-Rényi (1960)

$$\mathbb{P}(G(n,t) \text{ is connected}) = \mathbb{P}(\tilde{H}_0(X^{(1)}(t)) = 0) \to \begin{cases} 0 & t = \frac{\log n - \omega_n}{n} \\ 1 & t = \frac{\log n + \omega_n}{n} \end{cases}$$

- $(\ell = 1)$ Pittel (1988)

$$\mathbb{P}(G(n,t) \text{ has no cycle}) = \mathbb{P}(\tilde{H}_1(X^{(1)}(t)) = 0)$$

$$\to \begin{cases} \sqrt{1-c}\exp(\frac{c}{2} + \frac{c^2}{4}) & t = \frac{c}{n}\ (0 < c < 1) \\ 0 & t = \frac{c}{n}\ (c > 1) \end{cases}$$



Threshold for connectivity for the Linial-Meshulam process

- $(\ell = 1)$ Erdös-Rényi (1960)

$$\mathbb{P}(G(n,t) \text{ is connected}) = \mathbb{P}(\tilde{H}_0(X^{(1)}(t)) = 0) \to \begin{cases} 0 & t = \frac{\log n - \omega_n}{n} \\ 1 & t = \frac{\log n + \omega_n}{n} \end{cases}$$

- $(\ell = 2, p = 2)$ Linial-Meshulam(2006)
  $(\ell \geq 2, \forall p)$ Meshulam-Wallach (2009)

$$\mathbb{P}(\tilde{H}_{\ell-1}(X^{(\ell)}(t), \mathbb{Z}_p) = 0) \to \begin{cases} 0 & t = \frac{\ell \log n - \omega_n}{n} \\ 1 & t = \frac{\ell \log n + \omega_n}{n} \end{cases}$$

- Remark that the threshold for $\tilde{H}_{\ell-1}(X^{(\ell)}(t), \mathbb{Z}) = 0$ is still open. Hoffman-Kahle-Paquette obtained a partial result.
- They obtained the same threshold for "$\pi_1(X^{(2)}(t))$ has property (T)".

Betti number $\beta_1(X^{(2)}(t))$ in the case $\ell = 2$

$$\tilde{\beta}_1(X^{(2)}(0)) = \binom{n-1}{2} \searrow 0 = \tilde{\beta}_1(X^{(2)}(1))$$

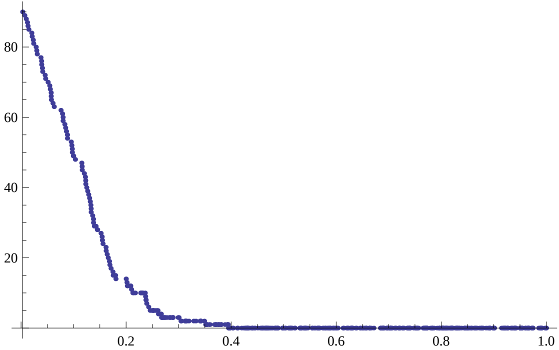


Figure: $\beta_1(X^{(2)}(t))$ for 2-Linial-Meshulam process when $n = 15$.

Spanning ℓ-acycle

- Spanning tree T in the complete graph K_n satisfies the following:
  1. $|T| = n-1 = \binom{n-1}{1}$.
  2. $\tilde{H}_0(T) = 0 \iff$ connected.
  3. $\tilde{H}_1(T) = 0 \iff$ no cycle.

Definition

A subset T of ℓ-faces is a spanning ℓ-acycle over $[n]$ if

1. $|T| = \binom{n-1}{\ell}$.
2. $\tilde{H}_{\ell-1}(X_T) =$ *finite group*
3. $\tilde{H}_\ell(X_T) = 0 \iff$ *no ℓ-dim. cycles.*

$$X_T = \Delta_{n-1}^{(\ell-1)} \sqcup T$$

- This definition was given by G. Kalai(1983).
- Any two of the above three conditions implies the third.

Spanning ℓ-acycle in general simplicial complex

Definition

Let X be a simplicial complex. For $\ell \leq \dim X$, a subset T of ℓ-faces in X is a spanning ℓ-acycle if

1. $\tilde{H}_{\ell-1}(X_T) =$ *finite group*
2. $\tilde{H}_{\ell}(X_T) = 0 \iff$ *no ℓ-dim. cycles.*

where

$$X_T = X^{(\ell-1)} \sqcup T.$$

In this case, $|T| = |X_\ell| - \tilde{\beta}_\ell(X)$.

- Any two of the above three conditions implies the third.
- k-spanning acycle exists only if $|\tilde{H}_{k-1}(X^{(k)})| < \infty$.

1. A triangulation of 2-sphere minus one 2-face is a spanning 2-acycle.
2. For a triangulation X_g of a compact oriented surface of $g \geq 1$, since $\tilde{H}_1(X_g) = 2g$, there is no spanning 2-acycle in our definition.

Extension of Cayley's theorem

Theorem (Kalai '83)

Let $\mathcal{S}^{(\ell)}(n)$ be the set of spanning ℓ-acycle with $(\ell-1)$-complete skeleton on n-vertices. Then,

$$\sum_{T \in \mathcal{S}^{(\ell)}(n)} |\tilde{H}_{\ell-1}(T)|^2 = n^{\binom{n-2}{\ell}}.$$

- For $\ell = 2$, since $\tilde{H}_1(T)$ is trivial for $n = 4, 5$,

$$|\mathcal{S}^{(2)}(4)| = 4^{\binom{4-2}{2}} = 4, \quad |\mathcal{S}^{(2)}(5)| = 5^{\binom{5-2}{2}} = 125$$

- For $\ell = 2$ and $n = 6$, since $\tilde{H}_1(T) \cong \mathbb{Z}_2$ for 12 spanning 2-acycles,

$$|\mathcal{S}^{(2)}(6)| = 46620 \neq 6^{\binom{6-2}{2}} = 6^6 = 46656$$

  Such a 2-complex is a triangulation of the projective plane.

  e.g. $\{123, 124, 135, 146, 156, 236, 245, 256, 345, 346\}$

Determinantal formula

X: a simplicial complex with $\tilde{H}_{\ell-2}(X^{(\ell-1)}) = 0$.

- L: an $(\ell-1)$-acycle, S: ℓ-faces with $|S| = |K|$.

$$\partial_\ell = \begin{array}{c|c} X_{\ell-1}\backslash X_\ell & X_\ell \\ \begin{array}{c} K \\ L \end{array} & \begin{bmatrix} \partial_K \\ * \end{bmatrix} \end{array} = \begin{array}{c|c} X_{\ell-1}\backslash X_\ell & \begin{array}{cc} S & S^c \end{array} \\ \begin{array}{c} K \\ L \end{array} & \begin{bmatrix} \partial_{KS} & * \\ * & * \end{bmatrix} \end{array}$$

- $\det \partial_{KS} \neq 0$ iff S is an ℓ-acycle.

Proposition (Matrix-Tree type theorem)

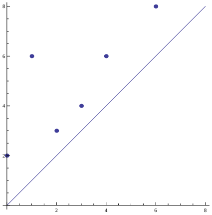
Let $L \in \mathcal{S}^{(\ell-1)}$ and set $K = X_{\ell-1} \setminus L$. Then,

$$\det(\partial_K \partial_K^T) = \sum_{S \in \mathcal{S}^{(\ell)}} (\det \partial_{KS})^2 = \sum_{S \in \mathcal{S}^{(\ell)}} |\tilde{H}_{\ell-1}(X_S)|^2$$

When $\ell = 1$, $|\tilde{H}_{\ell-1}(X_S)| = 1$, this gives us $\det(\partial_K \partial_K^T) = |\mathcal{S}^{(1)}|$.

Sum of lifetimes of homology generators

- $(\ell_i)_i = d_i - b_i$: lifetimes.
- $L_{\ell-1} := \sum_i \ell_i$



Theorem

1. *Let $\tilde{\beta}_{\ell-1}(t)$ be the (reduced) Betti number at time t. Then,*

$$L_{\ell-1} = \int_0^\infty \tilde{\beta}_{\ell-1}(t)dt.$$

2. *Let $T_{min}^{(\ell)}$ be the minimum spanning ℓ-acycle. Then,*

$$L_{\ell-1} = wt(T_{min}^{(\ell)}) - wt(X_{\ell-1} \setminus T_{min}^{(\ell-1)}),$$

Betti number $\tilde{\beta}_1(t)$ in the case $\ell = 2$

- For the 2-Linial-Meshulam process, $L_1 = \sum_{\Delta \in T^{(2)}_{min}} t(\Delta)$.

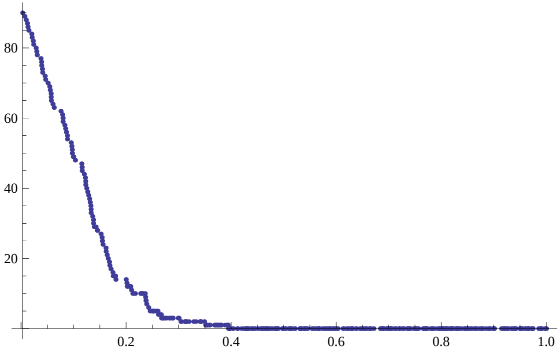


Figure: $\tilde{\beta}_1(t)$ for 2-Linial-Meshulam process when $n = 15$.

$$L_1 = \int_0^1 \beta_1(t)dt = wt(T^{(2)}_{\mathrm{min}})$$

Main result

Theorem (Y.Hiraoka-T.S.)

Let $L_{\ell-1}(n)$ be the lifetime sum of $(\ell-1)$-st persistent homology for the ℓ-Linial-Meshulam process on n vertices,

$$\mathbb{E}[L_{\ell-1}(n)] = \mathbb{E}[\min_{T \in \mathcal{S}^{(\ell)}(n)} wt(T)] = O(n^{\ell-1})$$

as $n \to \infty$, where

$$wt(T) := \sum_{\sigma \in T} t(\sigma).$$

See http://arxiv.org/abs/1503.05669

- Remark: For $\ell = 1$, $\mathbb{E}[L_0(n)] \to \zeta(3)$.
  We will give a remark on the limiting value

$$\lim_{n\to\infty} \frac{1}{n^{\ell-1}} \mathbb{E}[L_{\ell-1}(n)].$$

Conjecture for the limiting values

Let $L_{\ell-1}(n)$ be the sum of lifetimes of $(\ell-1)$-th persistent homology for the ℓ-Linial-Meshulam process on n vertices,

$$\mathbb{E}[L_{\ell-1}(n)] = \mathbb{E}[\min_{T \in \mathcal{S}^{(\ell)}(n)} wt(T)] = O(n^{\ell-1}) \quad \text{as } n \to \infty.$$

Conjecture

$$\lim_{n\to\infty} \frac{1}{n^{\ell-1}} \mathbb{E}[L_{\ell-1}(n)] = \frac{1}{\ell!} \int_0^\infty h_{\ell-1}(c)dc =: I_{\ell-1},$$

where

$$h_d(c) = ct_c(1-t_c)^d + \frac{c}{d+1}(1-t_c)^{d+1} + t_c - \frac{c}{d+1}$$

and t_c is the minimum positive solution to $t = e^{-c(1-t)^d}$ for $c > c_d^*$ and $t_c = 1$ for $c < c_d^*$.

- Remark. For $\ell = 1$, $I_0 = \zeta(3)$, which recovers Frieze's result.

Clique (Flag) complex process

- The clique complex $\mathrm{Cl}(G)$ associated with a graph G is the maximal simplicial complex having G as a 1-dimensional skeleton.
- $X^{(1)}(t)$: Erdös-Rényi graph process

$$\mathcal{C}(t) = \mathrm{Cl}(X^{(1)}(t)), \quad 0 \le t \le 1,$$

- The process starts from the 0-skeleton, i.e., n isolated vertices, and ends up with Δ_{n-1}. Namely,

$$\Delta_{n-1}^{(0)} = \mathcal{C}(0) \subset \mathcal{C}(t) \subset \mathcal{C}(1) = \Delta_{n-1}.$$

- The main difference from the Linial-Meshulam case is absence of monotonicity of Betti numbers.

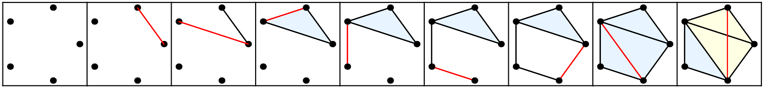


Figure: Clique complex process on 5-vertices

Partial result for clique complex process

Proposition

Let $L_{\ell-1}(n)$ be the lifetime sum for the clique complex process on n-vertices. Then, for $\ell = 1, 2$,

$$cn^{\ell-1} \le \mathbb{E}[L_{\ell-1}(n)] \le Cn^{\ell-1} \log n$$

and for $\ell \ge 3$,

$$cn^{\frac{(\ell+2)(\ell-1)}{2\ell}} \le \mathbb{E}[L_{\ell-1}(n)] \le Cn^{\ell-1}$$

- When $\ell = 1$, L_0 is equal to the lifetime sum of the Erdös-Rényi graph process. Since $\mathbb{E}[L_0] = O(1)$, the *lower* bound is correct.
- When $\ell = 2$, the *upper* bound seems to be more appropriate than the lower one by simulation.

Further directions

- Prove the conjecture for $\mathbb{E}[L_{\ell-1}(n)]$ as $n \to \infty$.
- Give an asymptotic expansion of $\mathbb{E}[L_{\ell-1}(n)]$.
- Give the exact asymptotics of $\mathbb{E}[L_{\ell-1}(n)]$ for clique complex.
- Limit theorem for scaled persistence diagrams (or barcodes, persistent landspaces etc.).
- Random geometric graph version of this problems.

$$\text{Point process on } \mathbb{R}^d \Longrightarrow k\text{-persistence diagram}$$

- Anatomy of ℓ-Linial-Meshulam complex as was done for the original Erdös-Rényi random graph. (cf. recent results by Linial-Peled.)
- Extension of Wilson's algorithm of generating a uniform spanning tree.

Filtration and persistent homology

- Input 1: $\boxed{\text{Point process}}$

  $\Downarrow$ via Čech or Rips-Vietoris complex etc.

- Input 2: $\boxed{\text{Random Filtration}}$

  $\Downarrow$

- Output: $\boxed{\text{Random persistence diagram as point process}}$