The Entropy of Schur-Weyl measures

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July 1, 2015

Asymptotic representation theory

Goal: Study the asymptotic behavior of classical groups and their representations when the rank of the group goes to infinity.

- The groups of interest: S_n, $GL(N, \mathbb{C})$, $GL(N, \mathbb{F}_q)$, $U(N)$, etc.
- Types of questions asked:
  - Study the befavior of representations of S_n when $n \to \infty$.
  - Consider S_∞ - the inductive limit of S_n when $n \to \infty$. Study representation theory of S_∞.
  - Study connections between the representation theory of the finite rank objects and the representation theory of the limiting objects.

Tensor representations

Tensor representations of S_n

- **Group:** S_n - group of permutations of n symbols.
- **Vector Space:** $V = (\mathbb{C}^m)^{\otimes n}$.
- **Action:** S_n acts by permuting the factors in the tensor product : if $\pi \in S_n$ then

$$\pi \cdot \left(v_1 \otimes v_2 \otimes \ldots \otimes v_n\right) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \ldots \otimes v_{\pi^{-1}(n)}.$$

Example: if $n = 4$ and

$$\pi = (1342)\,,$$

then

$$\pi \cdot \left(v_1 \otimes v_2 \otimes v_3 \otimes v_4\right) = v_2 \otimes v_4 \otimes v_1 \otimes v_3.$$

Isotypic decomposition

- Consider a decomposition of a representation into a direct sum of irreducibles and collect isomorphic representations together:

$$V = (V_{1,1} \oplus V_{1,2} \oplus \cdots \oplus V_{1,m_1}) \oplus \cdots \oplus (V_{k,1} \oplus V_{k,2} \oplus \cdots \oplus V_{k,m_k})$$

  where $V_{i,j}$ are all irreducible, $V_{i_1,j_1} \simeq V_{i_2,j_2}$ iff $i_1 = i_2$.
- The components $(V_{l,1} \oplus V_{l,2} \oplus \cdots \oplus V_{l,i_l})$ are called *isotypic* components.
- While the decomposition is not unique, the multiplicities m_i and the isotypic components (up to permutations) are unique.

Question

What does the isotypic decomposition of $(\mathbb{C}^m)^{\otimes n}$ look like?

Irreducible representations of S_n and Young diagrams

- The number of irreducible representations of a finite group is the same as the number of conjugacy classes.
- For S_n there is an explicit bijaction between irreducible representations and conjugacy classes (integer partitions of n).
- Integer partitions of n can be represented as Young diagrams $\lambda \in \mathbb{Y}^n$ with n cells.

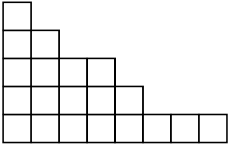

Figure : The Young diagram of the partition $\lambda = (8, 5, 4, 2, 1)$ of 20.

Isotypic decomposition

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$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\text{Some Young diagrams } \lambda} E_\lambda.$$

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Question

What can we say about the index set and the isotypic components E_λ?

Schur–Weyl duality.

- $(\mathbb{C}^m)^{\otimes n}$ is naturally a representation of $GL(m,\mathbb{C})$: if $A \in GL(m,\mathbb{C})$,

$$A \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_n) = Av_1 \otimes Av_2 \otimes \ldots, Av_n.$$

- The actions of S_n and $GL(m,\mathbb{C})$ commute and are dual.
- It follows from the duality that the isotypic decomposition of $(\mathbb{C}^m)^{\otimes n}$ with respect to $GL(m,\mathbb{C})$ coincides with the isotypic decomposition with respect to S_n.
- Irreducible representations of $GL(m,\mathbb{C})$ that appear in the decomposition are polynomial, so they are parametrized by Young diagrams with at most m rows.

Isotypic components

- Decompose the tensor representations of S_n into isotypic components:

$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Y}_m^n} E_\lambda.$$

  The index set $\mathbb{Y}_m^n$ is the set of all Young diagrams with n cells and at most m rows.

- As a consequence of Schur–Weyl duality the isotypic components E_λ are of the form $E_\lambda = V_\lambda \otimes W_\lambda$, where V_λ and W_λ are the irreducible representations of S_n and $GL(m, \mathbb{C})$ corresponding to λ.

$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Y}_m^n} V_\lambda \otimes W_\lambda,$$

Questions

What is the behavior of the isotypic decomposition when $n, m \to \infty$? More simply, how do dimensions of isotypic components grow when $n, m \to \infty$?

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- We will consider the limit $n, m \to \infty$ when $\dfrac{\sqrt{n}}{m} \to c \geq 0$.
- **Question 1:** How do the maximal dimensions of isotypic components grow?

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What is the behavior of the isotypic decomposition when $n, m \to \infty$? More simply, how do dimensions of isotypic components grow when $n, m \to \infty$?

- We will consider the limit $n, m \to \infty$ when $\frac{\sqrt{n}}{m} \to c \geq 0$.
- **Question 1:** How do the maximal dimensions of isotypic components grow?
- Consider the measure $\mathbb{P}_m^n$ on $\mathbb{Y}_m^n$ given by the relative dimensions of the isotypic components in the Schur-Weyl decomposition:

$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Y}_m^n} E_\lambda.$$

$$\mathbb{P}_n^m(\lambda) := \frac{\dim(E_\lambda)}{m^n}.$$

These measures are called Schur-Weyl measures. $supp(\mathbb{P}_m^n) = \mathbb{Y}_m^n$.
Question 2: How do the dimensions of typical (with respect to $\mathbb{P}_n^m$) isotypic components grow?

Main theorems

$$\max_{\lambda \in \mathbb{Y}_m^n} \frac{\dim E_\lambda}{m^n}$$

Main theorems

$$-\frac{1}{\sqrt{n}} \ln \left(\max_{\lambda \in \mathbb{Y}_m^n} \frac{\dim E_\lambda}{m^n} \right)$$

Main theorems

Theorem (M.)

There exists $\beta > 0$ *and for any* $c \geq 0$, $\alpha_c > 0$ *such that for large enough* $n, m \in \mathbb{N}$, *if* $c = \frac{\sqrt{n}}{m}$, *then*

$$\alpha_c < -\frac{1}{\sqrt{n}} \ln \left(\max_{\lambda \in \mathbb{Y}_m^n} \frac{\dim E_\lambda}{m^n} \right) < \beta.$$

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There exists $\beta > 0$ *and for any* $c \geq 0$, $\alpha_c > 0$ *such that if* $\lim_{n\to\infty} \frac{\sqrt{n}}{m} = c$, *then*

$$\lim_{n\to\infty} \mathbb{P}_m^n \left\{ \lambda \in \mathbb{Y}_m^n : \alpha_c < -\frac{1}{\sqrt{n}} \ln \frac{\dim E_\lambda}{m^n} < \beta \right\} = 1.$$

Main theorems

Theorem (M.)

For any $c > 0$, $c \neq 1$ there exists a positive number H_c such that for any $\varepsilon > 0$ we have

$$\lim_{\substack{n\to\infty \\ m\to\infty \\ \frac{\sqrt{n}}{m}\to c}} \mathbb{P}^n_m \left\{ \lambda \in \mathbb{Y}^n_m : \left| -\frac{1}{\sqrt{n}} \ln \frac{\dim E_\lambda}{m^n} - H_c \right| < \varepsilon \right\} = 1.$$

Note

The theorems were conjectured by Grigori Olshanski.

Note

By analogy with the Shannon-McMillan-Breiman theorem, H_c should be interpreted as the entropy of the family of Schur-Weyl measures $\mathbb{P}^n_m$, $\sqrt{n} = cm$. H_c is the amount of information encoded in a Young diagram from $\mathbb{Y}^n_m$.

Dimensions of irreducible representations

Consider the groups S_n. How do dimensions of irreducible representations grow when $n \to \infty$?

- **Question 1:** How does the dimension of the highest dimensional irreducible representations grow?
- **Question 2:** How do the dimensions of typical irreducible representations grow?

Vershik and Kerov [1985] gave two sided, logarithmically order-sharp asymptotic bounds for both the maximal and the typical dimensions.

Bufetov [2010] proved that after appropriate scaling, the typical dimensions converge to a constant.

Note: Typical here is with respect to the Plancherel measure.

The Plancherel measure

The theorems above are related to the theorems of Vershik and Kerov, and Bufetov regarding the irreducible representations of S_n.

- The regular representation of S_n decomposes into irreducibles as follows:

$$\mathbb{C}S_n = \sum_{\lambda \in \mathbb{Y}^n} V_\lambda \otimes V_\lambda^*$$

- Looking at dimensions gives

$$n! = \sum_{\lambda \in \mathbb{Y}^n} (\dim V_\lambda)^2$$

- The Plancherel measure is the probability measure on Young diagrams

$$\mathbb{P}l^n(\lambda) = \frac{(\dim V_\lambda)^2}{n!}.$$

- $\mathbb{P}l^n(\lambda)$ is the relative dimension of the isotypic component in the regular representation of S_n corresponding to λ .

Dimensions of irreducible representations of S_n

Theorem (Vershik-Kerov, 1985)

There exist positive constants α, β *such that for all* $n \in \mathbb{Z}_{>0}$

$$\alpha < -\frac{1}{\sqrt{n}} \ln \left(\max_{\lambda \in \mathbb{Y}^n} \frac{(\dim V_\lambda)^2}{n!} \right) < \beta.$$

and

$$\lim_{n \to \infty} \mathbb{P}l^n \left\{ \lambda \in \mathbb{Y}^n : \alpha < -\frac{1}{\sqrt{n}} \ln \frac{(\dim V_\lambda)^2}{n!} < \beta \right\} = 1.$$

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Theorem (Bufetov, 2010)

There exists a constant $H > 0$ such that for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mathbb{P}l^n \left\{ \lambda \in \mathbb{Y}^n : \left| -\frac{\ln \mathbb{P}l^n(\lambda)}{\sqrt{n}} - H \right| \leq \varepsilon \right\} = 1.$$

These correspond to the case $c = 0$ in the theorems above.

The lower bound for dimensions

- The upper estimate β, which corresponds to a lower bound on the dimensions is easy to obtain.
- The Hardy-Ramanujan asymptotic formula for the number of Young diagrams gives:

$$|\mathbb{Y}^n| \approx \frac{1}{4\sqrt{3}n} e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}.$$

- "There are not enough Young diagrams for the typical measure to be too small":

$$\mathbb{P}_m^n \left\{ \lambda : -\frac{\ln \frac{\dim E_\lambda}{m^n}}{\sqrt{n}} > \beta \right\} = \mathbb{P}_m^n \left\{ \lambda : \mathbb{P}_m^n(\lambda) < e^{-\beta\sqrt{n}} \right\} \leq e^{-\beta\sqrt{n}} \frac{e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}}{4\sqrt{3}n}.$$

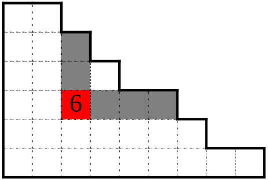
- Setting $\beta = \frac{2\pi}{\sqrt{6}}$ gives

$$\lim_{n\to\infty} \mathbb{P}_m^n \left\{ \lambda \in \mathbb{Y}_m^n : -\frac{1}{\sqrt{n}} \ln \frac{\dim E_\lambda}{m^n} < \beta \right\} = 1$$

- The lower bound for the maximal dimension follows trivially.

Hook formula

- Given a cell in a Young diagram its hook length is defined to be the number of cells above and to the right of it, including the cell itself.



- Let $\mathfrak{C}(\lambda)$ be the set of cells in the Young diagram λ and let $h(\mathfrak{c})$ denote the hook length of $\mathfrak{c} \in \mathfrak{C}(\lambda)$.
- If the cell $\mathfrak{c}$ is in the i'th row and j'th column, the content of the cell is defined to be $\mathcal{C}(\mathfrak{c}) := j - i$, i.e. the signed distance from the diagonal.

Dimension formula for the isotypic component

- Hook formula for the dimensions of irreducible representations of S_n:

$$\dim V_\lambda = \frac{n!}{\prod_{\mathfrak{c}\in\mathfrak{C}(\lambda)} h(\mathfrak{c})}.$$

- Formula for the dimensions of irreducible representations of $GL(m,\mathbb{C})$:

$$\dim W_\lambda = \frac{\prod_{\mathfrak{c}\in\mathfrak{C}(\lambda)} (m+\mathcal{C}(\mathfrak{c}))}{\prod_{\mathfrak{c}\in\mathfrak{C}(\lambda)} h(\mathfrak{c})}.$$

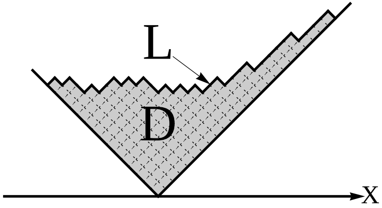
- Dimensions of isotypic components:

$$\dim E_\lambda = \frac{n! \prod_{\mathfrak{c}\in\mathfrak{C}(\lambda)} (m+\mathcal{C}(\mathfrak{c}))}{\prod_{\mathfrak{c}\in\mathfrak{C}(\lambda)} h(\mathfrak{c})^2}.$$

- We are interested in $\ln \dim E_\lambda$.

The boundary function

- Rotate the diagrams by 45° and scale down by $\sqrt{n/2}$ in both directions.

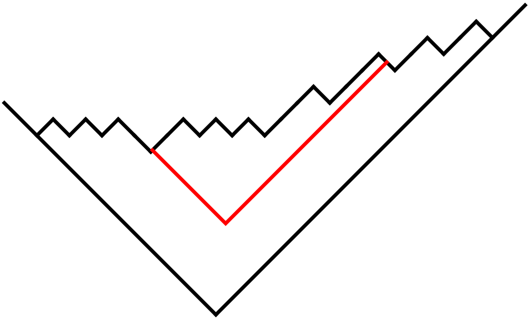


- Denote the function giving the top boundary by $L_\lambda(x)$.
- $L_\lambda(x)$ is a piecewise linear function with slopes ± 1 such that $L_\lambda(x) = |x|$ for $|x| \gg 1$.

Continuous hook length

Continuous analogue of hook length:

- Let $L(x)$ be a Lipschitz function with Lipschitz constant 1 and such that $L(x) = |x|$ for $|x|$ large enough.
- Denote by D_L the region bounded between the graphs of the functions $|x|$ and $L(x)$.
- Given a point $(x, y) \in D_L$ define its hook length:



The variational formula for the Schur-Weyl Measures

Proposition (M.)

$$-\frac{\ln \mathbb{P}_m^n(\lambda)}{\sqrt{n}} = \sqrt{n}(\theta(L_\lambda) - \rho(L_\lambda)) + \hat{\theta}(\lambda) - \hat{\rho}(\lambda) - \varepsilon_n$$

where

$$\theta(L) = 1 + 2 \iint_{(x,y)\in D_L} \ln h_L(x,y) dxdy$$

is the so called Hook integral,

$$\rho(L) = 2 \iint_{(x,y)\in D_L} \ln \left(1 + \frac{\sqrt{2n}}{m}(x-y)\right) dxdy,$$

$\varepsilon_n = o\left(\frac{\ln n}{\sqrt{n}}\right)$ *is independent of* λ*, and* $\hat{\theta}(\lambda)$ *and* $\hat{\rho}(\lambda)$ *are as follows:*

The variational formula for the Schur-Weyl Measures cont.

$$\hat{\theta}(\lambda) = \frac{1}{\sqrt{n}} \sum_{\mathfrak{c} \in \mathfrak{C}(\lambda)} M(h(\mathfrak{c})),$$

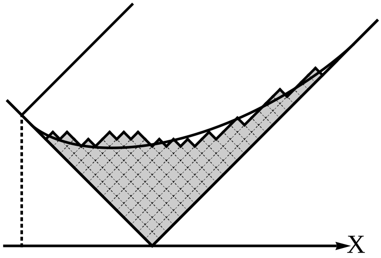
$$\hat{\rho}(\lambda) = \frac{1}{2\sqrt{n}} \sum_{\mathfrak{c} \in \mathfrak{C}(\lambda)} M(m + \mathcal{C}(\mathfrak{c})).$$

$$M(x) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} \frac{1}{x^{2k}}.$$

As before, $\mathfrak{C}(\lambda)$ is the set of cells in the Young diagram λ, $h(\mathfrak{c})$ is the hook length of the cell $\mathfrak{c}$ and $\mathcal{C}(\mathfrak{c}) \in \mathbb{Z}$ is the content of the cell $\mathfrak{c}$.

The variational problem

- We prove that the functional $\theta(L) - \rho(L)$ has a unique minimizer.
- We give an explicit formula for the quadratic variation.
- The Young diagrams in $\mathbb{Y}_m^n$ converge to a limit shape when $n, m \to \infty$ and $\frac{\sqrt{n}}{m} \to c \geq 0$, and the minimizer is the limit shape.



The limit shape

Theorem (Biane, 2001)

Let $m = m(n)$ be such that $\lim_{n\to\infty} \frac{\sqrt{n}}{m} = c$. Then, for any fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}^n_m(\lambda \in \mathbb{Y}^n_m : |L_\lambda(x) - \Omega_c(x)| < \varepsilon) = 1,$$

where $\Omega_c(x)$ is a differentiable function such that $\Omega_c(x) = |x|$ for $|x|$ large enough and

$$\Omega'_c(x) = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{c+x}{2\sqrt{1+xc}}\right), & x \in [c-2, c+2] \\ \pm 1 & x \notin [c-2, c+2] \end{cases}.$$

Note

Biane's proof is by methods of free probability. By showing that the limit shape is the solution of our variational problem, we obtain a new proof of Biane's theorem.

The Vershik-Kerov, Logan-Shepp Limit shape

$\Omega_c(X)$ is a deformation (depending on c) of the limit shape found by Vershik and Kerov and simultaneously and independently by Logan and Shepp in 1977. $\lim_{c\to 0}\Omega_c(x) = \Omega(x)$.

Theorem (Vershik-Kerov77, Logan-Shepp77)

For any fixed $\varepsilon > 0$

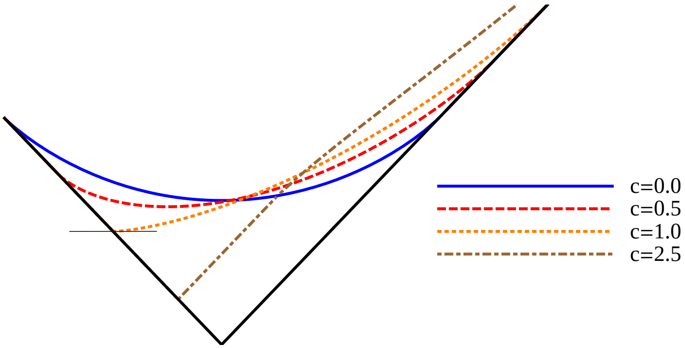
$$\lim_{n\to\infty} \mathbb{P}l^n(\lambda \in \mathbb{Y}_n : |L_\lambda(x) - \Omega(x)| < \varepsilon) = 1,$$

where $\Omega(x)$ *is a differentiable function such that* $\Omega(x) = |x|$ *for* $|x|$ *large enough and*

$$\Omega'(x) = \begin{cases} \frac{2}{\pi} ArcSin(x), & |x| \leq 1 \\ Sign(x), & |x| \geq 1 \end{cases}.$$

The limit shape

The graphs of the functions $\Omega_c(x)$ intersect $|x|$ at two points. All the intersections are tangential except the intersections on the left side for $c \geq 1$. At the left intersection point $\Omega_1(x)$ has slope 0, while $\Omega_c(x)$ when $c > 1$ has slope 1.



The solution of the variational problem

Proposition (M.)

Let $c = \frac{\sqrt{n}}{m} > 0$. We have

$$-\frac{\ln \mathbb{P}_m^n(\lambda)}{\sqrt{n}} = \frac{\sqrt{n}}{8}\|f_\lambda\|_{\frac{1}{2}}^2 + \frac{\sqrt{n}}{2}\int_{|x-c|>2} G_c(x) f_\lambda(x)dx + \hat{\theta}(\lambda) - \hat{\rho}(\lambda) - \varepsilon_n,$$

where $f_\lambda(x) = L_\lambda(x) - \Omega_c(x)$,

$$\|f\|_{\frac{1}{2}}^2 = \iint \left(\frac{f(s)-f(t)}{s-t}\right)^2 dsdt$$

is the $\frac{1}{2}$*–Sobolev norm in the space of piecewise-smooth functions, and*

$$G_c(x) = \left(\operatorname{arccosh}\left|\frac{x-c}{2}\right| + \operatorname{sign}(1-c)\operatorname{arccosh}\left|\frac{3c-c^3+(1+c^2)x}{2(1+cx)}\right|\right).$$

The associated point process

Associate with each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m$ the point configuration

$$\mathcal{P}(\lambda) := \{\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_m - m\} \subset \mathbb{Z}.$$

Under this correspondence the pushforward of $\mathbb{P}^n_m$ is a random m-point process on $\mathbb{Z}$.

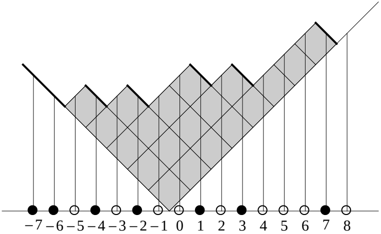


Figure : Black dots are particles in the configuration while white dots are empty.

Given an integer vector $\vec{m} = (m_1, \ldots, m_r)$ and a subset $X \subset \mathbb{Z}$, let $c_{\vec{m}}(\lambda)$ be

$$c_{\vec{m}}(\lambda) = \delta_{m_i \in \mathcal{P}(\lambda), \forall i}.$$

Express the logarithm of the measure in terms of local patterns

- We express the terms in $-\frac{\ln \mathbb{P}_m^n(\lambda)}{\sqrt{n}}$ in terms of local patterns. For example

$$\hat{\theta}(\lambda) = \sum_{k=1}^{\infty} \frac{h_k(\lambda)}{\sqrt{n}} M(k),$$

  where $h_k(\lambda)$ is the number of cells in λ with hook length k, and we have

$$h_k(\lambda) = \sum_{i=-\infty}^{\infty} (c_i(\lambda) - c_i(\lambda)c_{i-k}(\lambda)) = \sum_{i=-\infty}^{\infty} (c_i(\lambda) - c_{i,i-k}(\lambda)).$$

- All the other terms in $-\frac{\ln \mathbb{P}_m^n(\lambda)}{\sqrt{n}}$ can be expressed in terms of weighted sums of the local statistics $c_{\vec{m}}(\lambda)$.

Structure of the proof

- We need to understand properties of Schur-Weyl-random Young diagrams at the microscopic scale.
- Poissonize.
- Show that you obtain a determinantal point process.
- Take the limit of the process and show it converges to the sine process.
- Depoissonize using the technique of Borodin, Okounkov, Olshanski.

Beta Random Matrix Ensembles

Consider probability distributions on $\mathbb{R}^N$ with density w.r.t. Leb of the form

$$P_N^{V,\beta}(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N(\beta)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-\beta N V(\lambda_i)/2},$$

where the potential V is a real analytic function satisfying some growth condition at infinity.

Hermite beta ensemble: Take quadratic potential V:

$$P_N^{Her,\beta}(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N^{Her}(\beta)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-\frac{\beta N}{4}\lambda_i^2}.$$

For $\beta = 1, 2,$ and 4, this is the distribution of the eigenvalues of a random matrix from the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE), and Gaussian Symplectic Ensemble (GSE) respectively.

Entropy of Beta Random Matrix Ensembles

Theorem (Bufetov,M.,Scherbina,Soshnikov)

(Asymptotic equipartition property) If the potential V is "nice", the random variables

$$-\frac{\ln P_N^{V,\beta}(\bar{\lambda}_N)}{N}$$

converge almost surely to a constant $E_\beta(V)$.

Theorem (Bufetov,M.,Scherbina,Soshnikov)

(Central Limit Theorem) If the potential V is "nice", the random variables

$$\frac{\ln P_N^{V,\beta}(\bar{\lambda}) + NE_\beta(V)}{N^{1/2}}$$

converge in distribution as $N \to \infty$ to the Gaussian random variable with expected value 0 and variance $\frac{\beta}{2} - \frac{\beta^2}{4}\psi'\left(1 + \frac{\beta}{2}\right)$, where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$.

Circular, Laguerre, and Jacobi ensemble

Our results also hold for the following classical ensembles.

- *Circular beta:* $\lambda_1, \ldots, \lambda_N \in [0, 2\pi]$ with joint density w.r.t Leb:

$$P_N^{Cir,\beta}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N^{Cir}(\beta)} \prod_{k<j} |e^{i\lambda_k} - e^{i\lambda_j}|^\beta.$$

- *Laguerre beta:* $\alpha > 0$, $\lambda_1, \ldots, \lambda_N \in [0, \infty)$ with joint density w.r.t. Leb:

$$P_N^{Lag,\beta}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N^{Lag}(\beta)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{j=1}^{N} \lambda_j^{\alpha-1} e^{-\beta N \lambda_j}.$$

- *Jacobi beta:* $\mu, \nu > 0$, $\lambda_1, \ldots, \lambda_N \in [-1, 1]$ with joint density w.r.t. Leb:

$$P_N^{Jac,\beta}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N^{Jac}(\beta)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{j=1}^{N} (1 - \lambda_j)^{\mu-1} (1 + \lambda_j)^{\nu-1}.$$

These ensembles do not formally belong to the class described above since the particles are distributed, respectively, on the unit circle, positive half-line, and the interval $[-1, 1]$.

The entropy

Combining the various estimates, we obtain the following formula for the entropy:

$$H_c = \sum_{k=1}^{\infty} \left(\mathfrak{m}(k) \int_{c-2}^{c+2} \mathbb{E}_{\mathcal{S}(\phi_a)} c_{\{0\}} - \mathbb{E}_{\mathcal{S}(\phi_a)} c_{\{0,k\}} da \right)$$
$$+ \frac{1}{4} \int\limits_{c-2}^{c+2} \int\limits_{0}^{1} \int\limits_{0}^{\infty} \mathbb{E}_{\mathcal{S}(\phi_a)} \left(\frac{\mathcal{L}_\lambda(s+h) - \mathcal{L}_\lambda(s)}{h} - \frac{2}{\pi} \arcsin \left(\frac{c+a}{2\sqrt{1+ac}} \right) \right)^2 dhdsda.$$

Thank you for your attention.