On matrix coefficients of unitary representations of semisimple Lie groups

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Thanks

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Introduction

- Structure of semisimple Lie groups
- Representations of semisimple Lie groups
- Decay of matrix coefficients of irreducible representations
- Better control of the decay of matrix coefficients.

Semisimple Lie groups

For me, "semisimple Lie group" means a connected real Lie group G whose Lie algebra $\mathfrak{g}$ is a sum of simple ideals, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(p, q)$, $\mathrm{SU}(p, q)$, $\mathrm{Sp}(p, q)$, E_8. However, because the methods are quite abstract, many of the ideas should also apply in the p-adic case.

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- a maximal compact subgroup K,
- a maximal simply connected abelian subgroup A,
- a Cartan decomposition $G = KA^+K$, where A^+ is a cone in A.

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The Lie algebra $\mathfrak{a}$ is a vector space with a canonical inner product, and it is possible to identify $\mathfrak{a}$ and $\mathfrak{a}^*$.

An element λ of $\mathfrak{a}^*$ or $\mathfrak{a}^*_{\mathbb{C}}$ gives a homomorphism from A to $\mathbb{C}$:

$$\exp H \mapsto \exp(\lambda H).$$

Roots

We may write
$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha,$$
where $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}$ and all $X \in \mathfrak{g}_\alpha$. The hyperplanes $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, where $\alpha \in \Sigma$, divide $\mathfrak{a}$ into cones. We call one of these the positive cone, $\mathfrak{a}^+$.

We order $\mathfrak{a}^*$: $\quad \beta \le \gamma \quad \iff \quad \beta(H) \le \gamma(H) \quad \forall H \in \mathfrak{a}^+$.

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha$; then $\rho \in (\mathfrak{a}^*)^+$, the cone in $\mathfrak{a}^*$ corresponding to $\mathfrak{a}^+$ under the identification of $\mathfrak{a}$ and $\mathfrak{a}^*$.

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The Weyl group W is the group of transformations of $\mathfrak{a}$ generated by the reflections in the hyperplanes $\{H \in \mathfrak{a} : \alpha(H) = 0\}$.

Haar measure on G

$$\int_G f(x)\,dx = \int_K \int_{\mathfrak{a}^+} \int_K f(k\exp(H)k')\,w(H)\,dk\,dH\,dk',$$

where $w(H) = \sum_j \exp(\beta_j H)$: the dominant term is $\exp(2\rho H)$.

A special function on G, and $L^{q+}(G)$ decay.

Suppose that $\beta \in \mathfrak{a}^*$. Define $\mathrm{B}_\beta : G \to \mathbb{R}^+$ by

$$\mathrm{B}_\beta(k \exp(H) k') = (1 + |H|)^N \exp((\beta - \rho)H)$$

for all $k, k' \in K$ and all $H \in \mathfrak{a}^+$, where $N \in \mathbb{N}$ depends on G.

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$$\| \mathrm{B}_\beta \|_q^q = \int_{\mathfrak{a}^+} (1 + |H|)^{Nq} \exp(q(\beta - \rho)H) \exp(2\rho H) \, dH < \infty$$

if and only if $q(\beta - \rho) < -2\rho$, that is, if and only if $q > q_0$, say.

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Write $f \in L^{q+}(G)$ if $f \in L^{q+\varepsilon}(G)$ for all $\varepsilon \in \mathbb{R}^+$. The statement $f \in L^{q+}(G)$ gives information about the decay of f, and finding the minimal q gives sharper information.

Note that, when the rank of G is more than 1, different β give rise to the same q.

Unitary representations and $B(G)$

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$$u(x) = \langle \pi(x)\xi, \eta\rangle \quad \forall x \in G,$$

where $\pi \in \bar{G}$ and $\xi, \eta \in \mathcal{H}_\pi$.

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$$B(G) = \{u \in C(G) : u = \langle \pi(\cdot)\xi, \eta \rangle\,,\ \pi \in \bar{G},\ \xi, \eta \in \mathcal{H}_\pi\};$$

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the same function u may arise in different ways. For $u \in B(G)$,

$$\|u\|_B = \inf\{\|\xi\|\,\|\eta\| : u = \langle \pi(\cdot)\xi, \eta \rangle\,,\ \pi \in \bar{G},\ \xi, \eta \in \mathcal{H}_\pi\}.$$

Restricting unitary representations to K

If $\pi \in \bar{G}$, then $\pi\big|_K = \bigoplus_{\tau \in \hat{K}} n_\tau \tau$, and $\mathcal{H}_\pi = \bigoplus_{\tau \in \hat{K}} n_\tau \mathcal{H}_\tau$.

Let P_τ be the orthogonal projection of $\mathcal{H}_\pi$ onto $n_\tau \mathcal{H}_\tau$. We say that $\xi \in \mathcal{H}_\pi$ is τ-isotypic if $P_\tau \xi = \xi$, and K-finite if it is a finite linear combination of isotypic vectors.

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Theorem

For all $\pi \in \hat{G}$ and all $\tau \in \hat{K}$,

$$n_\tau \leq \dim(\mathcal{H}_\tau).$$

Restricting attention to A

Let $\pi \in \bar{G}$, $\sigma, \tau \in \hat{K}$. Define Φ in $C(A^+, \mathrm{Hom}(\sigma, \tau))$ by

$$\Phi(a) = P_\tau \pi(a) P_\sigma \quad \forall a \in A.$$

Note that Φ depends on π, σ, and τ.

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If ξ is σ-isotypic and η is τ-isotypic, then

$$\begin{aligned} \langle \pi(kak')\xi, \eta \rangle &= \langle \pi(a)\pi(k')\xi, \pi(k)^*\eta \rangle \\ &= \langle \Phi(a)\pi(k')\xi, \pi(k^{-1})\eta \rangle \end{aligned}$$

for all $k, k' \in K$ and all $a \in A$. Thus the matrix-valued functions Φ encapsulate the behaviour of π.

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If $\pi \in \hat{G}$, then $\Phi(a)$ is finite-dimensional.

Asymptotic behaviour of matrix coefficients

Theorem (Harish-Chandra)

Suppose that $\pi \in \hat{G}$, and $\sigma, \tau \in \hat{K}$. Then $\Phi = \sum_j \Phi_j$, and for each j there exist $\alpha_j \in \mathfrak{a}^*_{\mathbb{C}}$ and a polynomial p_j, independent of σ and τ, and $\varphi_j \in \mathrm{Hom}(\sigma, \tau)$ such that

$$\Phi_j(\exp(H)) \asymp p_j(H) \exp((\alpha_j - \rho)H)\, \varphi_j \quad \text{as } H \to \infty \in \mathfrak{a}^+.$$

The indices j may be chosen such that $\mathrm{Re}\, \alpha_1 \geq \mathrm{Re}\, \alpha_j$ when $j \neq 1$, and $\deg p_j \leq N$; the integer N depends only on G. The number of terms in the sum is bounded by $|W|$.

Asymptotic behaviour of matrix coefficients

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Corollary

If $\pi \in \hat{G}$*, then there exists* $\beta \in \mathfrak{a}^*$*, such that for all* K*-finite vectors* $\xi, \eta \in \mathcal{H}_\pi$*,*

$$|\langle \pi(\cdot)\xi, \eta \rangle| \leq C(\pi, \xi, \eta)\, \mathrm{B}_\beta.$$

Classification of irreducible unitary representations

The description of all irreducible unitary representations is a hard problem. It may be divided into the classification of the elements of $\hat{G}_{\mathrm{red}}$, the representations that appear in the Plancherel formula, and the representation of the other representations, the so-called complementary series.

Two essentially equivalent classifications, due to Langlands and to Vogan, describe the complementary series representations by two parameters. One is the index α_1, which controls the decay at infinity, and the other is either a representation of a reductive subgroup M (not necessarily compact) of G or a minimal K-type.

Examples

- if $\pi \in \hat{G}_{\mathrm{disc}}$, then $|\langle \pi(\cdot)\xi, \eta\rangle| \leq C(\pi, \xi, \eta)\, \mathrm{B}_\beta$, where $\beta < 0$, and the matrix coefficients lie in $L^{q+}(G)$ for some $q < 2$. However, *not all* matrix coefficients lie in $L^{q+}(G)$.
- if $\pi \in \hat{G}_{\mathrm{red}}$, then $|\langle \pi(\cdot)\xi, \eta\rangle| \leq C(\pi, \xi, \eta)\, \mathrm{B}_0$, and the matrix coefficients lie in $L^{2+}(G)$. Actually, *all* matrix coefficients lie in $L^{2+}(G)$.
- if $\pi \in \hat{G}_{\mathrm{comp}}$, then $|\langle \pi(\cdot)\xi, \eta\rangle| \leq C(\pi, \xi, \eta)\, \mathrm{B}_\beta$, where $\beta > 0$, and the matrix coefficients lie in $L^{q+}(G)$ for some $q > 2$. In every case that we know about, *all* matrix coefficients lie in $L^{q+}(G)$.

Decay of general matrix coefficients

Even if π is not irreducible, we can often show that

$$|\langle \pi(\cdot)\xi, \eta\rangle| \leq C(\pi, \xi, \eta)\, \mathsf{B}_\beta$$

for all ξ and η in a dense subset $\mathcal{H}^0_\pi$ of $\mathcal{H}_\pi$, for instance,

- if π is the quasi-regular representation of G on $L^2(G/H)$, where H is a closed subgroup of G;
- if $\pi = \upsilon\big|_G$, where $G \subset H$ and $\upsilon \in \hat{H}$.

The second follows because we have estimates for the decay of υ on H. The first, for reductive H, is essentially in recent work of Benoist and Kobayashi, who show that

$$|\langle \pi(\exp Y)\xi, \eta\rangle| \leq C(\pi, \xi, \eta) \exp(-\rho_{\mathfrak{q}}^{\min}(Y))$$

for many vectors ξ and η, and that this is best possible. They then deduce L^{q+} estimates for all matrix coefficients of π, where q is an even integer.

It would be nice to do better

If

$$|\langle \pi(\cdot)\xi, \eta\rangle| \leq C(\pi, \xi, \eta)\, \mathsf{B}_\beta$$

for all ξ and η in a dense subset $\mathcal{H}_\pi^0$ of $\mathcal{H}_\pi$, and we also knew that

$$C(\pi, \xi, \eta) \leq C\, \|\langle \pi(\cdot)\xi, \eta\rangle\|_B\,,$$

then we could extend the inequality to all vectors in $\mathcal{H}_\pi$ by density.

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then we could extend the inequality to all vectors in $\mathcal{H}_\pi$ by density.

If $\chi \preceq \pi$, then the matrix entries $\langle \chi\theta, \zeta\rangle$ of χ are limits, uniformly on compacta, of nets of matrix entries $\langle \pi(\cdot)\xi_n, \eta_n\rangle$ of π, with

$$\|\langle \pi(\cdot)\xi_n, \eta_n\rangle\|_B \leq \|\langle \chi\theta, \zeta\rangle\|_B\,;$$

estimates for π would pass to χ. This would give us information about the direct integral decomposition of π.

Can we do better?

Unfortunately, estimates of the form

$$|\langle \pi(\cdot)\xi, \eta\rangle| \leq C \, \|\langle \pi(\cdot)\xi, \eta\rangle\|_B \, \mathsf{B}_\beta$$

are impossible—just consider translates.

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We know that L^{q+} estimates are translation-invariant. Moreover, we have the following result.

Theorem (CHH)

Suppose that $\pi \in \bar{G}$ and $\langle \pi(\cdot)\xi, \eta\rangle \in L^{q+}(G)$ for all ξ and η in a dense subset in $\mathcal{H}_\pi$, where $q > 0$. If $k = \lceil q/2 \rceil$, then

$$\|\langle \pi(\cdot)\xi, \eta\rangle\|_{2k+\varepsilon} \le C(\varepsilon) \, \|\langle \pi(\cdot)\xi, \eta\rangle\|_B$$

for all matrix entries $\langle \pi(\cdot)\xi, \eta\rangle$ of π.

Strengths and weaknesses

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To do better, we let

$$\mathcal{A}u(x) = \left(\int_K \int_K \left| u(kxk') \right|^2 \, dk \, dk' \right)^{1/2} .$$

Good news

I can show in many cases that, if

$$\mathcal{A}\,\langle \pi(\cdot)\xi,\eta\rangle \le C(\pi,\xi,\eta)\,\mathsf{B}_\beta$$

for all ξ and η in a dense subspace of $\mathcal{H}_\pi$, then

$$\mathcal{A}\,\langle \pi(\cdot)\xi,\eta\rangle \le C\,\|\langle \pi(\cdot)\xi,\eta\rangle\|_B\,\mathsf{B}_\beta$$

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I can also show that, if

$$\mathcal{A}\left\langle \pi(\cdot)\xi, \eta \right\rangle \leq C(\pi, \sigma, \tau) \left\| \xi \right\| \left\| \eta \right\| \mathsf{B}_\beta$$

for all ξ in a dense subspace of $\mathcal{H}_\sigma$ and all η in a dense subspace of $\mathcal{H}_\tau$, and all $\sigma, \tau \in \hat{K}$, then

$$\mathcal{A}\left\langle \pi(\cdot)\xi, \eta \right\rangle \leq C \left\| \left\langle \pi(\cdot)\xi, \eta \right\rangle \right\|_B \mathsf{B}_\beta$$

for all ξ in a dense subspace of $\mathcal{H}_\sigma$ and all η in a dense subspace of $\mathcal{H}_\tau$, and all $\sigma, \tau \in \hat{K}$, which is nearly as useful.

Positive results

Theorem

Suppose that $\pi \in \bar{G}$*, and* $\mathcal{H}$ *is a dense subspace of* $\mathcal{H}_\pi$*. If* $\beta \geq 0$ *and*

$$|\langle \pi(x)\xi, \eta \rangle| \leq C(\pi, \xi, \eta)\, \mathrm{B}_\beta(x) \quad \forall x \in G$$

for all $\xi, \eta \in \mathcal{H}$*, then*

$$|\langle \pi(x)\xi, \eta \rangle| \leq C\, \|\langle \pi(\cdot)\xi, \eta \rangle\|_B\, \mathrm{B}_\beta(x) \quad \forall x \in G$$

for all $\xi, \eta \in \mathcal{H}_\pi^K$*.*

Sketch of proof

We work with positive definite functions, that is, take $\xi = \eta$.

The main fact is that, for all $\chi \in \hat{G} \setminus \hat{G}_{red}$, and all $\theta \in \mathcal{H}_\chi^K$,

$$\langle \chi(\exp H)\theta, \theta \rangle \asymp p(H) \, \|\theta\|^2 \exp((\alpha - \rho)H) \quad \text{as } H \to \infty \text{ in } \mathfrak{a}^+,$$

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where p is positive.

The positive definite K-biinvariant matrix entries of π are integrals of positive definite K-biinvariant matrix entries of χ as above, where the parameter α varies over some set. For any H in $\mathfrak{a}^+$, if $\operatorname{Re} \alpha(H) > 0$, then $\alpha(H)$ is real; this stops cancellation. The biggest α must be "seen" by some K-biinvariant matrix entry $\langle \pi(\cdot)\xi, \xi \rangle$ where $\xi \in \mathcal{H}$. $\square$

Corollary

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Suppose that $\frac{1}{2}\rho \leq \beta \leq \rho$. *If*

$$\mathcal{A}\langle\pi(\cdot)\xi,\eta\rangle \leq C(\pi,\xi,\eta)\,\mathsf{B}_\beta$$

for all ξ *and* η *in a dense subspace of* $\mathcal{H}_\pi$, *then*

$$\mathcal{A}\langle\pi(\cdot)\xi,\eta\rangle \leq C\,\|\langle\pi(\cdot)\xi,\eta\rangle\|_B\,\mathsf{B}_\beta$$

for all ξ *and* η *in* $\mathcal{H}_\pi$.

Proof of Corollary

Observe that

$$v : x \mapsto \int_K \int_K \left|\langle \pi(kxk')\xi, \eta \rangle\right|^2 \, dk \, dk'$$

is a matrix coefficient of $\pi \otimes \bar{\pi}$, and is K-invariant on both the left and the right.

Proof of Corollary

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From the hypotheses on $\langle \pi(\cdot)\xi, \eta \rangle$, there is a dense subspace $\mathcal{H}$ of $\mathcal{H}^K_{\pi\otimes\bar{\pi}}$, the space of K-invariant vectors in $\mathcal{H}_{\pi\otimes\bar{\pi}}$, such that

$$|\langle \pi(x)\theta, \zeta \rangle| \leq C(\pi, \theta, \zeta) \, \mathrm{B}^2_\beta(x) \quad \forall x \in G.$$

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We apply the theorem and use the information of the theorem about v to deduce the desired information about $\langle \pi(\cdot)\xi, \eta\rangle$.
$\square$

Another corollary

Corollary

Suppose that G acts on a measure space X, and π is the "quasi-regular" representation of G on $L^2(X)$. If

$$\left|\left\langle \pi(kxk')\xi, \eta \right\rangle\right| \leq C(\pi, \xi, \eta)\, \mathrm{B}_\beta(x)$$

for all x in G, and all ξ and η in a dense subspace of $\mathcal{H}_\pi^K$, then

$$\mathcal{A} \left\langle \pi(\cdot)\xi, \eta \right\rangle \leq C \left\| \left\langle \pi(\cdot)\xi, \eta \right\rangle \right\|_B \mathrm{B}_\beta$$

for all ξ and η in $\mathcal{H}_\pi$.

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for all ξ and η in $\mathcal{H}_\pi$.

The proof uses a generalisation of an inequality of Herz. □

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