REPRESENTATIONS OF SYMMETRIC GROUPS

AND FREE CUMULANTS

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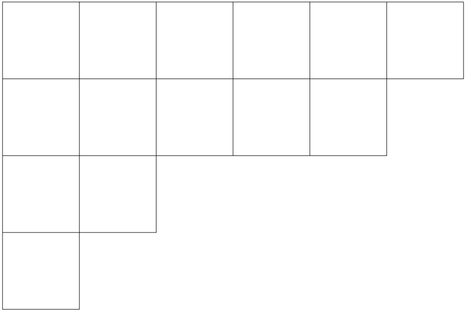
PARTITIONS

A partition is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.
Partitions label *irreducible representations of symmetric group* on $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ letters.



$$6 + 5 + 2 + 1 = 14$$

$$4 + 3 + 2 + 2 + 2 + 1 = 14$$

FRENCH CONVENTION

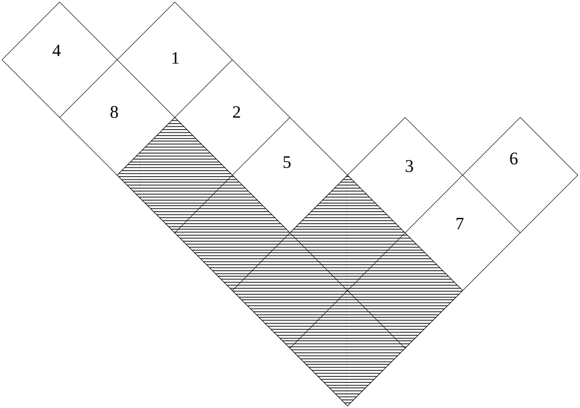
14			
12	13		
4	11		
3	8		
2	6	10	
1	5	7	9

1			
3	1		
4	2		
5	3		
7	5	1	
9	7	3	1

Dimension of a representation=number of Young tableaux.
Hook formula

$$\frac{n!}{\prod_{i,j} h_{ij}}$$

RUSSIAN CONVENTION



Restriction of a representation $S_{14} \downarrow S_6$.
The multiplicity is the number of ways to erase boxes.

LARGE SYMMETRIC GROUPS

Normalized characters $\chi_\lambda(\mu) = \frac{Tr(\rho_\lambda(\mu))}{\dim(\lambda)}$

$\mu =$ fixed conjugacy class of $S_\infty = \cup_n S_n$

$N = \sum_i \lambda_i \qquad \lambda_i/N \to \alpha_i \qquad \lambda_i'/N \to \beta_i$

$\chi_\lambda(\mu) \to \chi^\infty_{\alpha,\beta}(\mu)$ for a factor representation of S_∞.

Thoma/Vershik/Kerov theory $\to$ representation theory of S_∞ in terms of S_N for $N \to \infty$.

For "most" Young diagrams $\lambda_i = o(N)$ and $\chi_\lambda(\mu) \to 0$.
In this regime representation theory of symmetric groups is governed by *free probability*.

FREE COMPRESSION

$$X = UDU^*$$

D=diagonal $N \times N$ matrix, eigenvalues $D_1, ..., D_N$.
U=random Haar unitary $N \times N$ matrix.

$$\frac{1}{N} Tr(X^k) = \frac{1}{N} \sum_j D_j^k \to_{N \to \infty} \int x^k \mu(dx)$$

$0 < p < 1$, $X^{(p)}$=$pN \times pN$ upper corner of X.

$$\frac{1}{pN} Tr((X^{(p)})^k) \to_{N \to \infty} \int x^k \mu^{(p)}(dx)$$

$\mu^{(p)}$=free compression of μ, depends only on μ and p.

FREE CUMULANTS

$$G_\mu(z) = \int \frac{1}{z-x}\mu(dx) \quad = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} \int x^n \mu(dx)$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} M_n$$

$$K_{\mu_i}(G_\mu(z)) = G_\mu(K_\mu(z)) = z; \qquad K_\mu(z) = \frac{1}{z} + \sum_{n=0}^{\infty} R_n(\mu) z^n$$

$R_n(\mu)$ = **free cumulants** (D. Voiculescu, R. Speicher) of μ.
Free cumulants are polynomial functions of moments

$$M_n = \int x^n \mu(dx)$$

Conversely moments are polynomial functions of free cumulants.

FREE COMPRESSION

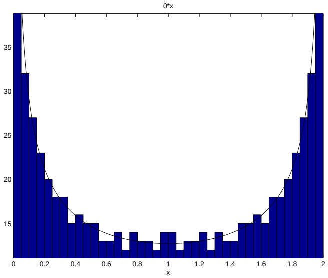
The free compression of a measure is obtained by the rule

$$R_n(\mu^{(p)}) = p^{n-1} R_n(\mu)$$

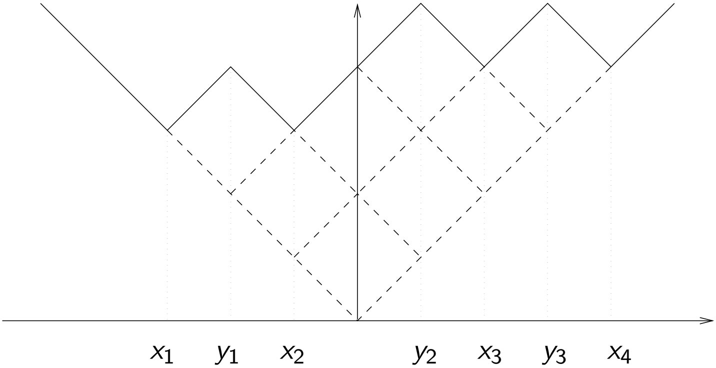
Since free cumulants determine the measure, this determines $\mu^{(p)}$.

Example: $\mu = \frac{1}{2}(\delta_0 + \delta_1)$
Random matrix model: compute the spectrum of $\Pi_1 \Pi_2 \Pi_1$ where $\Pi_1, \Pi_2 =$ orthogonal projections on random subspaces of dimensions $N/2$.

$$\mu^{(1/2)} = \frac{dx}{\pi\sqrt{x(1-x)}} \qquad \text{arcsine distribution}$$



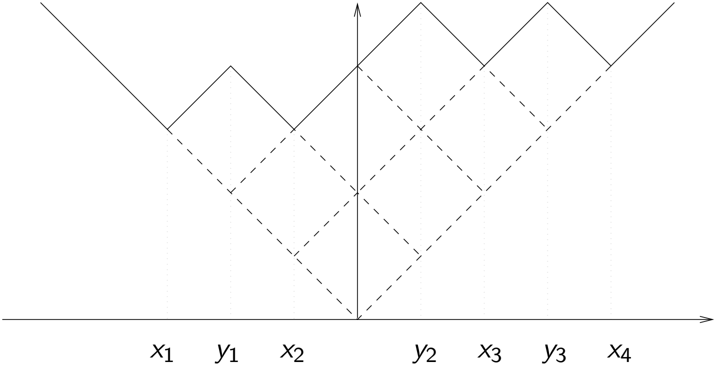
Histogram with a 400×400 random matrix.



A diagram may be identified with a function $\omega(x)$ such that

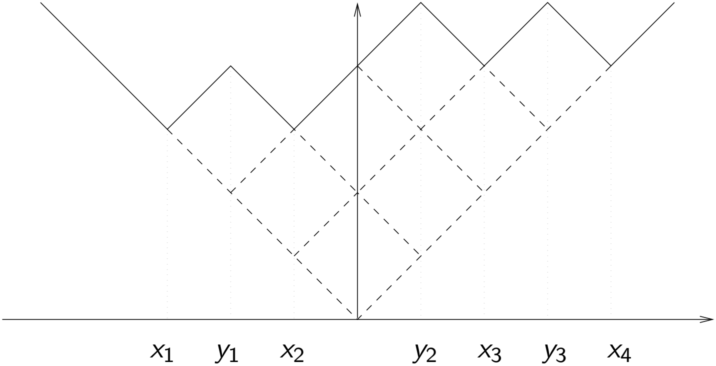
$$|\omega(x)| = |x| \text{ for } x >> 1 \qquad |\omega(x) - \omega(y)| \leq |x - y|.$$

TRANSITION MEASURE OF A DIAGRAM



(S.Kerov) there exists a unique probability measure m_ω such that

$$m_\omega = \sum_{k=1}^{n} \mu_k \delta_{x_k} \qquad \mu_k = \frac{\prod_{i=1}^{n-1}(x_k - y_i)}{\prod_{i \neq k}(x_k - x_i)}$$



m_ω gives the decomposition of $\omega \uparrow S_{n+1}$.

$$\sigma(u) = (\omega(u) - |u|)/2$$

$$\begin{aligned} G_{m_\omega}(z) &= \tfrac{1}{z}\exp\int \tfrac{1}{x-z}\sigma'(x)\,dx \\ &= \int \tfrac{1}{z-x} m_\omega(dx) \\ &= \tfrac{\prod_{i=1}^{n-1}(z-y_k)}{\prod_{i=1}^{n}(z-x_k)} \end{aligned}$$

$$K_\omega = G_\omega^{\langle -1\rangle}$$

$$K_\omega(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(\omega) z^{n-1}$$

$R_n(\omega)$= the free cumulants of the diagram.

Remark $\omega \mapsto m_\omega$ can be extended to 1-Lipschitz maps.

ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with q boxes, $\lambda \sim \sqrt{q}\omega$.

Number of rows and columns $= O(\sqrt{q})$.

$\chi_\lambda =$ normalized character of λ.

$$\chi_\lambda(\sigma) = q^{-|\sigma|/2}\Big(\prod_{c|\sigma} R_{|c|+2}(\omega) + O(q^{-1})\Big)$$

$|\sigma|=$ length of σ w.r.t generating set of all transpositions, the product is over cycles of σ.

ASYMPTOTIC OF RESTRICTION

$\omega=$ continuous diagram, $0 < t < 1$,
define ω_t by

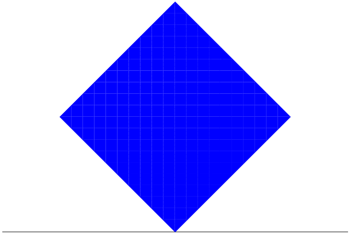
$$R_n(\omega_t) = t^{n-1} R_n(\omega)$$

The restriction of λ to $S_p \times S_{q-p} \subset S_q$ splits into irreducible

$$\bigoplus c^{\lambda}_{\mu\nu} [\mu] \otimes [\nu] \quad \text{(Littlewood-Richarson rule).}$$

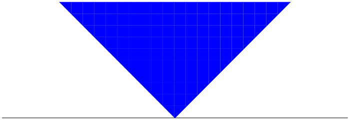
Give a weight $c^{\lambda}_{\mu\nu}\, dim(\mu)\, dim(\nu)$ to the pair (μ, ν).
Then as $q \to \infty$ and $p/q \to t$, almost all pairs (μ, ν) (rescaled by $\sqrt{q}$),
become close to (ω_t, ω_{1-t}).

EXAMPLE



Square diagram

$$m_\omega = \frac{1}{2}(\delta_{-1} + \delta_1)$$



1/2 Compression of the square diagram

$$m_\omega^{(1/2)} = \frac{dx}{\pi\sqrt{x(1-x)}}$$

Asymptotic of induction of representations

$$S_p \times S_q \uparrow S_{p+q}$$

can be interpreted in terms of *sums* of independent random matrices and *free convolution*.

FROBENIUS FORMULA FOR CHARACTERS OF CYCLES

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$=partition of n,

$$\varphi(z) = \prod_i (z - \lambda_i - n + i)$$

$$z\varphi(z-1)/\varphi(z) = 1/G_\lambda(z+n-1) = H_\lambda(z+n-1)$$

Frobenius' formula is
(c_k=cycle of order k, $\chi_\lambda(\sigma) = \frac{Tr(\rho_\lambda(\sigma))}{Tr(\rho_\lambda(e))}$)

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, z(z-1)\dots(z-k+1)\varphi(z-k)/\varphi(z).$$

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z+n-1)\dots H_\lambda(z+n-k)$$

Using the invariance of the residue under translation of the variable one gets

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z)\dots H_\lambda(z-k+1).$$

KEROV POLYNOMIALS

Consider the formal power series

$$H(z) = z - \sum_{j=2}^{\infty} B_j z^{1-j}.$$

Define

$$\Sigma_k = -\frac{1}{k}[z^{-1}]H(z)\dots H(z-k+1)$$

$$R_{k+1} = -\frac{1}{k}[z^{-1}]\, H(z)^k$$

The expression of Σ_k in terms of the R_j's is given by Kerov's polynomials.

Kerov's polynomials express normalized characters of cycles in terms of free cumulants of Young diagrams.

$$\Sigma_1 = R_2$$

$$\Sigma_2 = R_3$$

$$\Sigma_3 = R_4 + R_2$$

$$\Sigma_4 = R_5 + 5R_3$$

$$\Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2$$

$$\Sigma_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

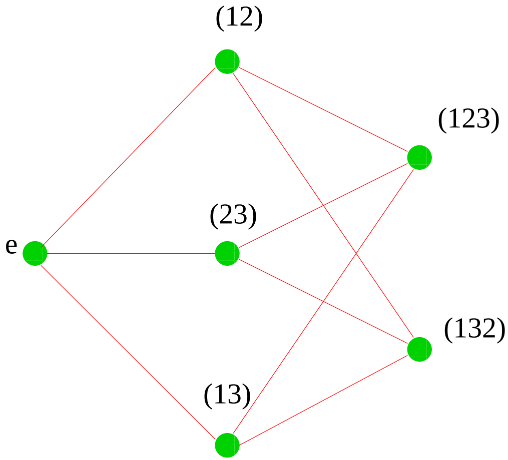
$$\Sigma_7 = R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2$$

$$\begin{aligned}\Sigma_8 &= R_9 + 126R_7 + 169R_5R_2 + 252R_4R_3 + 30R_3R_2^2 \\ &\quad + 1869R_5 + 3392R_3R_2 + 3044R_3\end{aligned}$$

GEOMETRY OF SYMMETRIC GROUPS

Cayley graph of S_n: (π_1, π_2) edge if and only if $\pi_1 \pi_2^{-1} =$ transposition.

$$d(\sigma_1, \sigma_2) = |\sigma_1 \sigma_2^{-1}| = n - |\{\text{cycles of } \sigma_1 \sigma_2^{-1}\}|$$

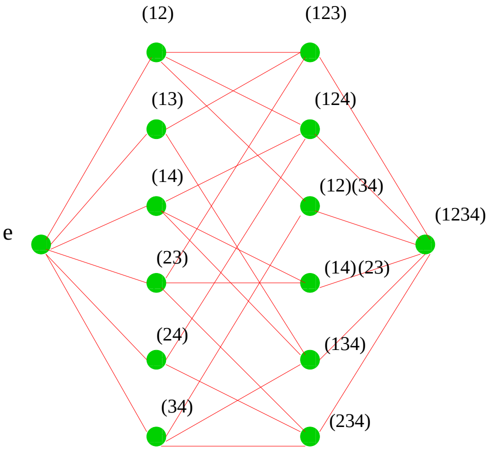


Cayley Graph of S3

INTERVALS IN THE SYMMETRIC GROUPS

An interval in the Cayley graph

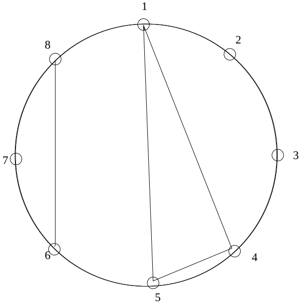
$$[\pi_1, \pi_2] = \{\sigma \mid d(\pi_1, \sigma) + d(\sigma, \pi_2) = d(\pi_1, \pi_2)\}$$



The interval [e,(1234)]

NONCROSSING PARTITIONS AND FREE CUMULANTS

$[e, (1234...n)] \sim NC(n)$= lattice of noncrossing partitions of $\{1, 2, ..., n\}$.

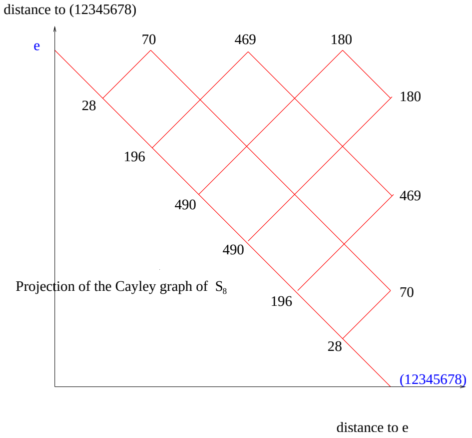


Moments and free cumulants are related by (Speicher)

$$M_n = \sum_{\pi \in NC(n)} R_\pi \qquad R_n = \sum_{\pi \in NC(n)} \mu([\pi, c_n]) M_\pi$$

$$R_\pi = \prod_{p \in \pi} R_{|p|} \qquad M_\pi = \prod_{p \in \pi} M_{|p|}$$

$$\Sigma_7 = R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2$$



The coefficient of R_{k+1-2l} in Σ_k is equal to the number of cycles $c \in S_k$, of length k, such that $(1\,2\ldots k)\,c^{-1}$ has $k-2l$ cycles

In general coefficients of Kerov polynomials count certain factorizations in symmetric groups (Dolega, Féray, Sniady)

This implies Kerov's conjecture: all coefficients of Kerov polynomials are nonegative.