Convergence of pseudospectra, constant resolvent norm and Schrödinger operators with complex potentials

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Based on

[1] S. Bögli and P. Siegl: *Remarks on the convergence of pseudospectra*, Integral Equations and Operator Theory 80, 2014, 303-321, arXiv:1408.3431.

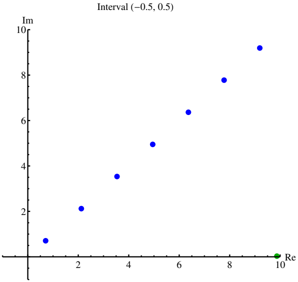
[2] S. Bögli, P. Siegl, and C. Tretter: *Approximations of spectra of Schrödinger operators with complex potentials on* $\mathbb{R}^d$, 32 pp., arXiv:1512.01826

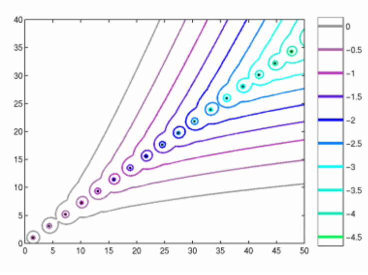
1. Motivation and introduction
2. Convergence of pseudospectra
3. Constant resolvent norm
4. Domain truncation for Schrödinger operators with complex potentials

Motivation - domain truncation for rotated (Davies) oscillator

Rotated oscillator[1]: $A = -\partial_x^2 + \mathrm{i}x^2$ in $L^2(\mathbb{R})$

- spectrum: $\sigma(A) = \left\{ e^{\mathrm{i}\pi/4}(2k+1) : \ k = 0, 1, 2, \dots \right\}$





Domain truncation[2]

$$A_n = -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2((-n,n)) \ + \text{ Dirichlet BC at } \pm n$$

- does $\sigma(A_n) \to \sigma(A)$ or $\sigma_\varepsilon(A_n) \to \sigma_\varepsilon(A)$?

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[1] L. Boulton. *J. Operator Theory* 47 (2002), pp. 413–429; E. B. Davies. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455 (1999), pp. 585–599; P. Exner. *J. Math. Phys.* 24 (1983), pp. 1129–1135; K. Pravda-Starov. *J. London Math. Soc.* 73 (2006), pp. 745–761.

[2] K. Beauchard et al. *ESAIM Control Optim. Calc. Var.* 21 (2015), pp. 487–512.

Definition of pseudospectra[3]

Let A be a closed operator in a Banach space $\mathcal{X}$ and let $\varepsilon > 0$. The ε-pseudospectrum of A is the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ z \in \mathbb{C} \ : \|(A-z)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

Brief history

- the notion (various names and approaches) introduced by several authors
  - 1972 Arnold, 1957 Vishik & Lyusternik: quasimodes in mathematical physics
  - 1967 Ph.D. thesis of Varah: r-approximate eigenvalues in “computer science”
  - 1975 H. Landau: ε-approximate eigenvalues
  - 1986 Wilkinson: spectral instability
  - 60-80’s Godunov et. al. (numerical analysis), 80’s Demmel, 80’s Chatelin, ...
  - 90’s Trefethen: ε-pseudospectrum
  - 1999 Davies: pseudospectra for differential operators, many generalizations

Why to study $\sigma_\varepsilon(A)$?

- high contrast in properties of normal and non-normal operators
- conclusions (stability, decay rates,...) based solely on spectrum can be misleading

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[3] L. N. Trefethen and M. Embree. Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators. Princeton University Press, 2005.

Pseudomodes

$z \in \sigma_\varepsilon(A) \iff z \in \sigma(A)$ or z is a pseudoeigenvalue, *i.e.* there is $\psi \in \operatorname{Dom}(A)$ such that

$$\|(A - z)\psi\| < \varepsilon\|\psi\|$$

Spectral (in)stability

$$\sigma_\varepsilon(A) = \bigcup_{\|B\|<\varepsilon} \sigma(A + B)$$

Some basic properties

- $\sigma_\varepsilon(A) \neq \emptyset$ for any $\varepsilon > 0$
- any bounded component of $\sigma_\varepsilon(A)$ contains some point of $\sigma(A)$
- $\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A)$

Pseudospectrum of normal operators

- if $AA^* = A^*A$ or $A = A^*$

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \operatorname{dist}(z, \sigma(A)) < \varepsilon\}$$

  since $\|(A - z)^{-1}\| = \operatorname{dist}(z, \sigma(A))^{-1}$

- otherwise only

$$\{z \in \mathbb{C} : \operatorname{dist}(z, \sigma(A)) < \varepsilon\} \subset \sigma_\varepsilon(A)$$

Pseudospectrum, similarity, basis properties

- if A is similar to a normal operator B, $A = \Omega^{-1} B \Omega$ with $\Omega, \Omega^{-1} \in \mathscr{B}(\mathcal{H})$, then

$$\sigma_\varepsilon(A) \subset \{z \in \mathbb{C} : \operatorname{dist}(z, \sigma(A)) < \kappa\varepsilon\}$$

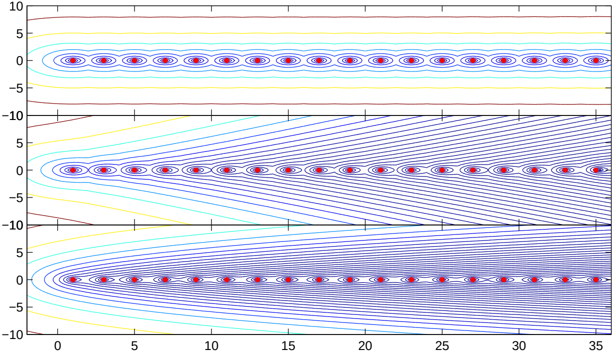
- for A with discrete spectrum:
  A is similar to a normal operator $\Leftrightarrow$ eigenvectors of A form a Riesz basis

In general...

$\sigma_\varepsilon(H)$ may be MUCH LARGER than ε-neighborhood of $\sigma(H)$

Rotated and shifted oscillator

- $H_\theta = e^{-i\theta}(-\partial_x^2 + e^{2i\theta}x^2)$ $\qquad$ $H_s = -\partial_x^2 + (x+i)^2$
- $\sigma(H_\theta) = \sigma(H_s) = \sigma(H_0) = \{2n+1, n \in \mathbb{N}_0\}$



- e.g. resolvent estimate[4] for A_3:

$$\|(H_s - z)^{-1}\| \geq \frac{1}{C} e^{\sqrt{\operatorname{Re} z}/C} \quad \text{for } z \text{ with } |\operatorname{Im} z| \leq 2(1-\varepsilon)\sqrt{\operatorname{Re} z}$$

[4] D. Krejčiřík et al. *J. Math. Phys.* 56 (2015), p. 103513.

A restriction on the behavior pseudospectrum[5]

Let A be the generator of a one-parameter semigroup e^{tA} on $\mathcal{X}$ with

$$\|e^{tA}\| \leq Me^{at} \quad \text{for all } t \geq 0.$$

Then $\sigma(A) \subset \{z \in \mathbb{C} \,:\, \operatorname{Re} z \leq a\}$ and

$$\|(A-z)^{-1}\| \leq \frac{M}{\operatorname{Re} z - a} \quad \text{for all } z \text{ with } \operatorname{Re} z > a.$$

Long time behavior (Gearhart-Prüss thm[6])

Let A be a densely defined closed operator in $\mathcal{H}$ such that $-A$ generates a contraction semigroup. Then

$$\lim_{t\to\infty} \frac{\log \|e^{-tA}\|}{t} = -\lim_{\varepsilon\to 0+} \inf_{z \in \sigma_\varepsilon(A)} \operatorname{Re} z$$

[5] E. B. Davies. Linear operators and their spectra. Cambridge University Press, 2007.

[6] B. Helffer. Spectral theory and its applications. Cambridge University Press, 2013.

Theorem [S. Bögli & PS, 2014]

Let

- $\mathcal{H}$ and $\mathcal{H}_n$, $n \in \mathbb{N}$, subspaces of a Hilbert space $\mathcal{H}_0$
- $A \in \mathcal{C}(\mathcal{H})$, $A_n \in \mathcal{C}(\mathcal{H}_n)$ densely defined
- $K \subset \mathbb{C}$ compact and $\varepsilon > 0$

If

(a) $\exists z_0 \in \cap_{n\in\mathbb{N}}\rho(A_n) \cap \rho(A)$:

$$\|(A_n - z_0)^{-1} P_{\mathcal{H}_n} - (A - z_0)^{-1} P_{\mathcal{H}}\| \to 0$$

(b) $z \mapsto \|(A - z)^{-1}\|$ is non-constant on any open subset of $\rho(A)$

(c) $\overline{\sigma_\varepsilon(A) \cap K} = \overline{\sigma_\varepsilon(A)} \cap K \neq \emptyset$

then

$$d_{\mathrm{H}}\left(\overline{\sigma_\varepsilon(A_n)} \cap K, \overline{\sigma_\varepsilon(A)} \cap K\right) \to 0, \quad n \to \infty.$$

Remarks

- Hausdorff distance: $M, N \subset \mathbb{C}$ non-empty and compact

$$d_{\mathrm{H}}(M, N) := \max\left\{ \max_{z\in M} \operatorname{dist}(z, N), \max_{w\in N} \operatorname{dist}(w, M) \right\}$$

Theorem [S. Bögli & PS, 2014]

Let

- $\mathcal{H}$ and $\mathcal{H}_n$, $n \in \mathbb{N}$, subspaces of a Hilbert space $\mathcal{H}_0$
- $A \in \mathcal{C}(\mathcal{H})$, $A_n \in \mathcal{C}(\mathcal{H}_n)$ densely defined
- $K \subset \mathbb{C}$ compact and $\varepsilon > 0$

If

(a) $\exists z_0 \in \cap_{n\in\mathbb{N}} \rho(A_n) \cap \rho(A)$:

$$\|(A_n - z_0)^{-1} P_{\mathcal{H}_n} - (A - z_0)^{-1} P_{\mathcal{H}}\| \to 0$$

(b) $z \mapsto \|(A - z)^{-1}\|$ is non-constant on any open subset of $\rho(A)$

(c) $\overline{\sigma_\varepsilon(A) \cap K} = \overline{\sigma_\varepsilon(A)} \cap K \neq \emptyset$

then

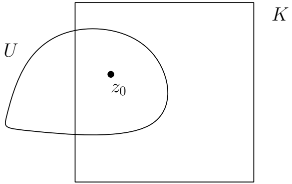
$$d_{\mathrm{H}}\left(\overline{\sigma_\varepsilon(A_n)} \cap K, \overline{\sigma_\varepsilon(A)} \cap K\right) \to 0, \quad n \to \infty.$$

Remarks

- previous results by Hansen (PhD thesis, 2008), problems on ∂K
- assumption (c) can be avoided by using a different distance (suitable for unbounded sets)
- assumption (b) cannot be omitted

Example

- A such that $\|(A-z)^{-1}\| = M$ for $z \in U$, U open
- $A_n = \left(1 - \frac{1}{n}\right)A$, $n \in \mathbb{N}$; take $z_0 \in U$



$$\|(A_n - z_0)^{-1}\| = \frac{n}{n-1}\left\|\left(A_n - \frac{n}{n-1}z_0\right)^{-1}\right\| = \frac{n}{n-1}M > M, \qquad \text{for all } n > n_0$$

- so $z_0 \in \sigma_{\frac{1}{M}}(A_n)$ and $U \cap \sigma_{\frac{1}{M}}(A) = \emptyset$

No convergence for $\sigma_{\frac{1}{M}}$

$$d_{\mathrm{H}}\left(\overline{\sigma_{\frac{1}{M}}(A_n)} \cap K, \overline{\sigma_{\frac{1}{M}}(A)} \cap K\right) \geq \operatorname{dist}(z_0, K \setminus U) > 0$$

- Banach space $\mathcal{X}$, $A \in \mathcal{C}(\mathcal{X})$, $M > 0$

Can $\{z \in \rho(A) : \|(A-z)^{-1}\| = M\}$ have an open subset in $\mathbb{C}$?

Pseudospectrum (two definitions)

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ z \in \mathbb{C} \ : \|(A-z)^{-1}\| > \frac{1}{\varepsilon} \right\}$$

$$\Sigma_\varepsilon(A) := \sigma(A) \cup \left\{ z \in \mathbb{C} \ : \|(A-z)^{-1}\| \geq \frac{1}{\varepsilon} \right\}$$

- does $\Sigma_\varepsilon(A) = \overline{\sigma_\varepsilon(A)}$ hold?

Resolvent as a holomorphic function

- $(A-z)^{-1}$ is a holomorphic function on $\rho(A)$
- maximum modulus principle?

Holomorphic matrix-valued function[7]

$$A(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

- $\|A(z)\| = 1$ for $|z| \leq 1$
- but $(A-z)^{-1}$ is a very special function

[7] E. Shargorodsky. *Bull. Lond. Math. Soc.* 40 (2008), pp. 493–504.

A remark on the geometry of Banach spaces

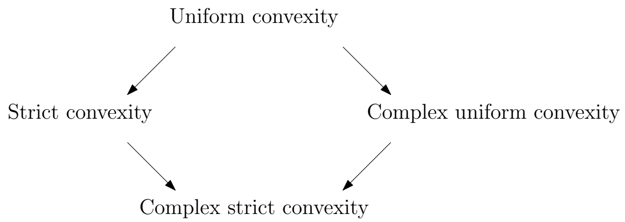
Uniformly convex Banach space

A Banach space $\mathcal{X}$ is uniformly convex, if for every $\varepsilon > 0$ exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$ with $\|x\| = \|y\| = 1$:

$$\|x - y\| \geq \varepsilon \quad \Longrightarrow \quad \left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$$

- geometrically: the unit ball is "uniformly round"
- Hilbert spaces are uniformly convex, L^p spaces, $1 < p < \infty$ are uniformly convex[8]

Various other convexities



[8] J. A. Clarkson. *Trans. Amer. Math. Soc.* 40 (1936), pp. 396–414.

Known results

- $\mathcal{X}$ a Banach space and $A \in \mathcal{C}(\mathcal{X})$
- $z \mapsto \|(A-z)^{-1}\|$ cannot be constant on an open subset $U \subset \rho(A)$ if
  i) (1976) Globevnik[9]: $A \in \mathscr{B}(\mathcal{X})$ and U belongs to unbounded component of $\rho(A)$
  ii) $A \in \mathscr{B}(\mathcal{X})$
    - (1976) Globevnik[8] if $\mathcal{X}$ is complex uniformly convex (*e.g.* Hilbert space, L^p-space with $1 \leq p < \infty$)
    - (1994) Daniluk for Hilbert spaces
    - (1997) Böttcher-Grudsky-Silbermann[10] for L^p-spaces with $1 < p < \infty$
    - (1998) Harrabi[11] if $\mathcal{X}$ finite-dimensional
    - (2008)Shargorodsky[12] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex uniformly convex (covers also $p = \infty$)
  iii) A generates a C_0 semigroup
    - (2010) Shargorodsky[13] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex uniformly convex
  iv) A has compact resolvent
    - (2015) Davies-Shargorodsky[14] if $\mathcal{X}$ or $\mathcal{X}^*$ is complex strictly convex

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[9] J. Globevnik. *Illinois J. Math.* 20 (1976), pp. 503–506.
[10] A. Böttcher, S. M. Grudsky, and B. Silbermann. *New York J. Math.* 3 (1997), pp. 1–31.
[11] A. Harrabi. *RAIRO Modél. Math. Anal. Numér.* 32 (1998), pp. 671–680.
[12] E. Shargorodsky. *Bull. Lond. Math. Soc.* 40 (2008), pp. 493–504.
[13] E. Shargorodsky. *Bull. Lond. Math. Soc.* 42 (2010), pp. 1031–1034.
[14] E. B. Davies and E. Shargorodsky. *Mathematika* online first (2015).

Example with constant resolvent norm[15]

- $\alpha_k := k+1$ and $\beta_k := 1 + 1/\alpha_k$, $k \in \mathbb{N}$
- 2×2 blocks
$$B_k := \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}, \quad k \in \mathbb{N},$$
- operator in $\ell^2(\mathbb{N})$: $A := \operatorname{diag}(B_1, B_2, B_3, \dots)$
- $\sigma(A) = \cup_{k\in\mathbb{N}} \sigma(B_k) = \{\pm\sqrt{k+2} : k \in \mathbb{N}\}$
- inverse of the block
$$(B_k - z)^{-1} = \frac{1}{\alpha_k \beta_k - z^2} \begin{pmatrix} z & \alpha_k \\ \beta_k & z \end{pmatrix}$$
- for $|z| < 1$:
$$\lim_{k\to\infty} \|(B_k - z)^{-1}\| = \left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\| = 1$$
- for $|z| < 1/2$:
$$\|(B_k - z)^{-1}\| \le \frac{1}{\alpha_k\beta_k - |z|^2} \left(\left\| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix} \right\| \right)$$
$$= \frac{|z| + \alpha_k}{\alpha_k\beta_k - |z|^2} \le \frac{1/2 + \alpha_k}{\alpha_k\beta_k - 1/4} = \frac{1/2 + \alpha_k}{3/4 + \alpha_k} < 1$$

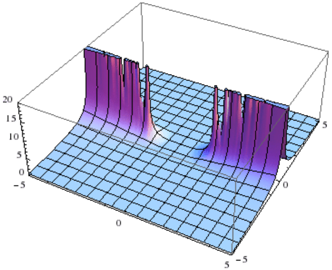
[15] E. Shargorodsky. *Bull. Lond. Math. Soc.* 40 (2008), pp. 493–504.

Example with constant resolvent norm

- operator in $\ell^2(\mathbb{N})$: $A = \operatorname{diag}(B_1, B_2, B_3, \dots)$
- for $|z| < 1/2$:

$$\|A\| = \sup_{k\in\mathbb{N}} \|(B_k - z)^{-1}\| = 1$$

Numerics



- it seems that

$$\forall z \in \rho(A), \quad \|(A - z)^{-1}\| \geq 1$$

Theorem [S. Bögli & PS, 2014]

Let $\mathcal{X}$ be a complex uniformly convex Banach space, $A \in \mathcal{C}(\mathcal{X})$. If there exist an open subset $U \subset \rho(A)$ and a constant $M > 0$ such that

$$\|(A-z)^{-1}\| = M, \quad z \in U,$$

then

$$\forall z \in \rho(A), \quad \|(A-z)^{-1}\| \geq M.$$

Sketch of the proof

- $F(z) := (A-z)^{-1}$ is analytic function with $\|F(\cdot)\| \equiv M$ on U
- take $z_0 \in U$ and $\{e_k\}_k \subset \mathcal{H}$ with $\|e_k\| = 1$ and $\|(A-z_0)^{-1} e_k\| \to M$.
- Globevnik & Vidav[16]: $\|F'(z_0) e_k\| \to 0$
- the 1st resolvent identity twice:
  $(A-z)^{-1} e_k = (A-z_0)^{-1} e_k + (z-z_0)\Big(I + (z-z_0)(A-z)^{-1}\Big) \underbrace{(A-z_0)^{-2} e_k}_{=F'(z_0)e_k \to 0}$
- hence

$$\|(A-z)^{-1}\| \geq \lim_{k\to\infty} \|(A-z)^{-1} e_k\| = \lim_{k\to\infty} \|(A-z_0)^{-1} e_k\| = M \qquad \square$$

[16] J. Globevnik and I. Vidav. *J. Funct. Anal.* 15 (1974), pp. 394–403.

Corollaries

i) If there exists a path $\gamma : [0,\infty) \to \rho(A)$ such that

$$\lim_{s\to\infty} |\gamma(s)| = \infty, \qquad \lim_{s\to\infty} \|(A-\gamma(s))^{-1}\| = 0,$$

then resolvent norm cannot be constant on any open subset of $\rho(A)$.

ii) This applies if $A \in \mathscr{B}(\mathcal{X})$ since

$$\|(A-z)^{-1}\| \le (|z| - \|A\|)^{-1}, \quad |z| > \|A\|.$$

iii) This applies if A generates a C_0 semigroup since, by Hille-Yosida Theorem,

$$\exists\, C > 0,\, \omega \in \mathbb{R} : \quad \|(A-z)^{-1}\| \le C(z-\omega)^{-1}, \quad z \in (\omega, +\infty).$$

Operator matrix[17]

$$T = \begin{pmatrix} 0 & f(A) \\ A & 0 \end{pmatrix} \quad \text{in} \quad \mathcal{H} \oplus \mathcal{H}$$

- $A = A^* > 0$ in $\mathcal{H}$ (with discrete spectrum), $f : \mathbb{R} \to \mathbb{R}$ continuous

a) $\lim_{x\to+\infty} f(x) = 0 \implies \rho(T) = \emptyset$

b) $\lim_{x\to+\infty} f(x) = C > 0$ and $f(x) \geq C \implies$ constant $\|(T-z)^{-1}\|$ on $\Omega \subset \rho(T)$

  - Shargorodsky's example: $A = \operatorname{diag}(2, 3, 4, \dots)$ and $f(x) = 1 + 1/x$

c) $f(x) = |x|^\beta$, $\beta \in (0,1) \implies \|(T - re^{i\phi})^{-1}\| = \mathcal{O}(r^{-2\beta/(\beta+1)})$ if $\phi \notin \{0, \pi\}$.

  - decay $\implies \|(T-z)^{-1}\|$ is not constant on any open set
  - decay not sufficient to generate a C_0 semigroup

[17] A. V. Balakrishnan and R. Triggiani. *Appl. Math. Lett.* 6 (1993), pp. 33–37.

Operator

$$A = -\Delta + Q \quad \text{in} \quad L^2(\mathbb{R}^d)$$

- Q is complex valued and such that A has compact resolvent

Approximations

$$A_n = -\Delta + Q \quad \text{in} \quad L^2(\Omega_n)$$

- $\{\Omega_n\}_n$ are expanding bounded suff. regular domains that exhaust $\mathbb{R}^d$; *e.g.*

$$\Omega_n = B_n(0), \quad n \in \mathbb{N}$$

- Dirichlet, Neumann or Robin BC are imposed on $\partial\Omega_n$
- if Robin BC: $\sup \|\gamma_n\|_\infty < \infty$, where $\partial_\nu f + \gamma_n f = 0$ at $\partial\Omega_n$

Questions

- Does $\sigma_\varepsilon(A_n)$ converge to $\sigma_\varepsilon(A)$?
- Does $\sigma(A_n)$ converge to $\sigma(A)$? In what sense?

m-sectorial case

- 1D example: $Q(x) = (1+\mathrm{i})x^2 + \mathrm{i}\delta(x)$
- decomposition: $Q = Q_0 + W$
  1. sectoriality: $L^1_{\mathrm{loc}}(\mathbb{R}^d) \ni Q_0$ has values in a sector with semi-angle $< \pi/2$
  2. growth at ∞: $|Q_0(x)| \to \infty$ as $|x| \to \infty$
  3. W: possibly singular, but $-\Delta$-form bounded with bound < 1
- the operator A introduced via closed sectorial forms

non-m-sectorial case

- 1D example: $Q(x) = \mathrm{i}x^3 - x^2 + \mathrm{i}x^{-1/4}$
- decomposition: $Q = Q_0 - U + W$, $\operatorname{Re} Q_0 \geq 0$, $U \geq 0$, $U \operatorname{Re} Q_0 = 0$
  1. regularity: $Q_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$|\nabla Q_0|^2 \leq a + b|Q_0|^2, \qquad U^2 \leq a_U + b_U |\operatorname{Im} Q_0|^2 \quad \text{with} \quad b_U < 1$$

  2. growth at ∞: $|Q_0(x)| \to \infty$ as $|x| \to \infty$
  3. W: possibly singular, but $-\Delta$-bounded with bound < 1
- operator A introduced via Kato's Thm. (m-accretive Schrödinger operators[18])

[18] D. E. Edmunds and W. D. Evans. Spectral Theory and Differential Operators. Oxford University Press, 1987.

Theorem [S. Bögli, PS, C. Tretter]

Under assumptions on potential Q, boundary conditions and domains Ω_n above,

$$\left\|(A_n - z)^{-1}\chi_{\Omega_n} - (A-z)^{-1}\right\| \to 0, \quad z \in \rho(A).$$

Steps in the proof

- detailed analysis of form-domains or domains of A, A_n
- strong resolvent convergence (form & operator approach)
- collective compactness[19]: for every $I \subset \mathbb{N}$ infinite, any sequence of $\phi_n \in \operatorname{Dom}(A_n)$, $n \in I$, such that $\{\|A_n\phi_n\| + \|\phi_n\|\}_{n\in I}$ is bounded, has a convergent subsequence in $L^2(\mathbb{R}^d)$.

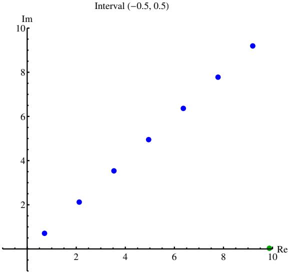
Corollary: pseudospectral convergence

$$d_{\mathrm{H}}\left(\overline{\sigma_\varepsilon(A_n)} \cap K, \overline{\sigma_\varepsilon(A)} \cap K\right) \to 0, \quad n \to \infty.$$

- compact resolvent $\Rightarrow$ resolvent is not constant on any open set

[19] P. M. Anselone and T. W. Palmer. *Pacific J. Math.* 25 (1968), pp. 417–422.

$$A := -\partial_x^2 + \mathrm{i}x^2 \text{ in } L^2(\mathbb{R}), \quad \Omega_n = (-n, n) \ + \ \text{Dirichlet BC at } \pm n$$



N-s-a operators in general

norm resolvent convergence $\not\Longrightarrow$ convergence of spectra

Corollary: spectral exactness

1. Every eigenvalue λ of A is approximated:
there is $\{\lambda_n\}_n$, $\lambda_n \in \sigma(A_n)$, such that $\lambda_n \to \lambda$ as $n \to \infty$.
2. No pollution: every accumulation point of $\{\lambda_n\}_n$ is an eigenvalue of A :
If $\{\lambda_n\}_n$, $\lambda_n \in \sigma(A_n)$, having an accumulation point λ, then $\lambda \in \sigma(A)$.

Theorem [S .Bögli, PS, C. Tretter]

Let assumptions on potential Q, boundary conditions and domains Ω_n hold.

- $\lambda \in \sigma(A)$ an eigenvalue of algebraic multiplicity m
- $\mathcal{L}_\lambda$ the corresponding algebraic eigenspace
- $\{\lambda_{1;n}, \ldots, \lambda_{m;n}\} \subset \sigma(A_n)$ be the eigenvalues of A_n converging to λ as $n \to \infty$

Then there is $C \geq 0$, independent of n, such that

$$\left|\lambda - \frac{1}{m}\sum_{j=1}^{m} \lambda_{j;n}\right| \leq C \max_{\substack{\phi \in \mathcal{L}_\lambda \\ \|\phi\|=1}} \left\|\phi \restriction \mathbb{R}^d \setminus \Omega_n\right\|$$

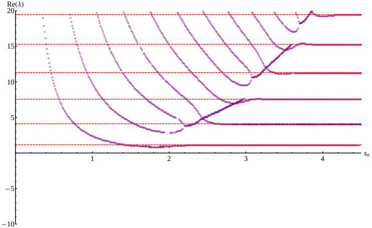
Remarks

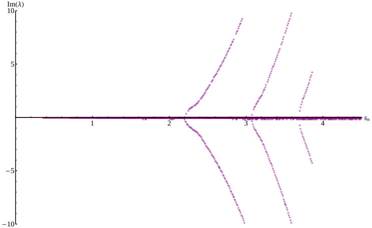
- analogous for individual eigenvalues (no average), but with an additional power (if Jordan blocks)
- proof based on the norm resolvent convergence and paper of Osborn[20]

[20] J. E. Osborn. *Math. Comput.* 29 (1975), pp. 712–725.

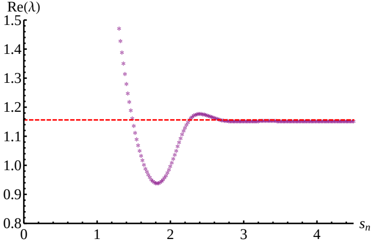
$$A = -\partial_x^2 + \mathrm{i}x^3, \quad \mathrm{Dom}\,(A) = W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}\,(x^3)$$

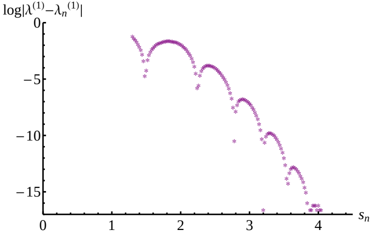
- $\sigma(A) \subset \mathbb{R}$





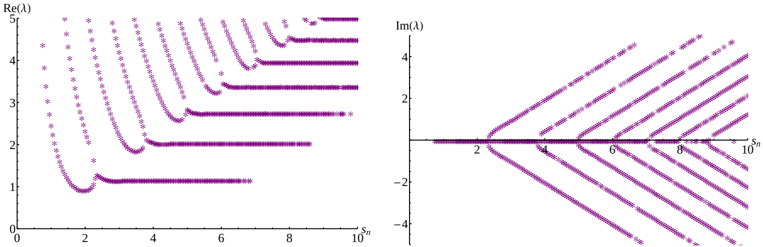
- the first eigenvalue and the rate (Dirichlet BC)





$$A = -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}\,(A) = W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}\,(x)$$

- $\sigma(A) = \emptyset$
- all eigenvalues escape to infinity



- "approximation of the lowest" eigenvalue[21]

$$\lim_{n\to\infty} (\inf \mathrm{Re}\,\sigma(A_n)) = \frac{|\mu_1|}{2}, \quad \mu_1 \approx -2.338$$

[21] K. Beauchard et al. *ESAIM Control Optim. Calc. Var.* 21 (2015), pp. 487–512.

Main results

- convergence of pseudospectrum in Hausdorff distance
  - norm resolvent convergence
  - resolvent norm not constant on any open set
- global mininum of the resolvent norm
  - complex uniformly convex space

$$\|(A-z)^{-1}\| = M \quad \text{on open } U \subset \mathbb{C} \Rightarrow \forall z \in \rho(A), \quad \|(A-z)^{-1}\| \geq M$$

- spectral and pseudospectral convergence for domain truncation of $-\Delta + Q$
  - various sectoriality and regularity assumptions on Q
  - norm resolvent convergence
  - pseudospectral convergence, spectral exactness, convergence rates

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Mathematical aspects of the physics with non-self-adjoint operators

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Marseille, colonie grecque by Pierre Puvis de Chavannes (1869)
Musée des beaux-arts de Marseille

Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

A quotation from the preface to the 2007 book *Linear operators and their spectra* by E. B. Davies

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