NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS IN COMPLEX GEOMETRY

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Outline of the Talk

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- PDE formulation: fix a Hermitian metric $g_{ij}(x)$ on X. Then the condition of constant scalar curvature for the unknown metric $g'_{ij}(x) = e^{u(x)} g_{ij}(x)$ is given by the equation

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- Starting from the works of Yau, Donaldson-Uhlenbeck-Yau and others in the last 40 years, the current belief is that the same relation between

algebraic geometry, *differential geometry*, *PDE*

  will hold quite generally, albeit in a more sophisticated form: a "canonical metric" characterizing the underlying algebraic structure should still exist, possibly with singularities. The singularities reflect global constraints, and will not occur iff if the structure is "stable" in a suitable algebraic geometric sense.

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- "Canonical metrics" often arise as the minima of an energy functional, whose more general critical points are then referred to as "extremal metrics".

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- When $\dim X = 4$, the Hodge $\star$ operator is an isomorphism of the space Λ^2 of 2-forms on X, with eigenspaces $\Lambda_{\pm}$ of eigenvalues ± 1 respectively. A connection A is said to be self-dual or anti-self-dual if it satisfies the corresponding condition below

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- If we decompose $F_A = F_+ + F_-$, with $\star F_\pm = \pm F_\pm$, it is readily seen that $I(A) = \|F_+\|^2 + \|F_-\|^2$, and $c_2(E) = \|F_+\|^2 - \|F_-\|^2$, where $c_2(E)$ is the second Chern class of E. It is then clear that the minimum of $I(A)$ is achieved at F either self-dual or anti-self-dual, depending on the sign of $c_2(E)$.

- If (X, ω) is a 2-dimensional complex Kähler manifold, the condition of self-duality turns out to have a remarkable interpretation. In that case

$$\Lambda_+ = \Lambda^{2,0} \oplus [\omega] \oplus \Lambda^{0,2}$$

Thus the condition that the curvature F be self-dual is equivalent to the vanishing of the $(2,0)$ and $(0,2)$ components of F, as well as of the projection of the $(1,1)$ component along $[\omega]$,

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- The vanishing of the $(2,0)$ and $(0,2)$ components of F is equivalent to the connection A being integrable, i.e., that it defines a holomorphic structure on the bundle E. Thus a self-dual connection is equivalent to a connection on a holomorphic vector bundle satisfying the above Hermitian-Einstein equation.

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- To find such a Hermitian-Einstein connection, we can also start from a holomorphic vector bundle E, and find a metric $H_{\bar{\alpha}\beta}$ so that $g^{j\bar{k}} F_{\bar{k}j}{}^{\alpha}{}_{\beta} = 0$, where F is the curvature of the Chern unitary connection, $\nabla_{\bar{j}} = \partial_{\bar{j}}$, $\Gamma^{\alpha}_{j\beta} = H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta}$.

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- The two points of view are equivalent, up to gauge transformations. This equivalence between Hermitian-Einstein equations and minima for Yang-Mills also generalizes to higher dimensions.

Kähler metrics of constant scalar curvature

Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X, and assume that L is positive, in the sense that the cohomology class $c_1(L)$ contains a representative which is a strictly positive $(1,1)$-form.

- A central question in Kähler geometry is when $c_1(L)$ contains a metric ω of constant scalar curvature R. Such metrics are minima of the Calabi functional

$$c_1(L) \ni \omega \;\to\; I(\omega) = \int_X R(\omega)^2\, \omega^n$$

Extremal metrics are critical points of this functional. Explicitly, their equation is

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- A general conjecture, due to Yau, is the existence of a metric of constant scalar curvature in $c_1(L)$ should be equivalent to the stability of $L \to X$ in the sense of geometric invariant theory. A precise notion of stability, called K-stability, has been proposed by G. Tian and S. Donaldson.
- The conjecture has been proved recently for the case $L = K_X^{-1}$ by X.X. Chen, S. Donaldson, and S. Sun. But it remains open in the general case.

Elliptic vs Parabolic Methods

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Elliptic equations and the method of continuity

The equations that we just surveyed are all elliptic non-linear partial differential equations. One prevalent method is the method of continuity. Say the equation is of the form

$$F(u_{j\bar{k}}, u_j, u, z) = 0.$$

Schematically, the method of continuity would consist in introducing a one-parameter family of equations

$$F_t(u_{j\bar{k}}, u_j, u, z) = 0, \qquad 0 \leq t \leq 1$$

with $F_t = F$ when $t = 1$, and F_0 manifestly admitting a solution u_0. The goal is to show that the set of parameters t for which $F_t = 0$ admits a solution $u(t)$ is both open and closed. Openness usually follows readily from the implicit function theorem. The diffculty is usually closedness, which would follow from a priori estimates of the form

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For example, consider the Hermitian-Einstein equation

$$g^{j\bar{k}} F_{\bar{k}j}{}^{\alpha}{}_{\beta} - \mu \delta^{\alpha}{}_{\beta} = 0.$$

The family of equations proposed by K. Uhlenbeck and S.T. Yau is

$$\Lambda F - \mu I = -t \log h, \qquad h \equiv H_0^{-1} H.$$

Geometric heat flows

In the parabolic approach, one considers a flow where whose stationary points are given precisely by the elliptic equations to be solved. The problem becomes the one of establish the long-time existence, and convergence of the flow.

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- For example, for the equation $\Delta u - R(x) + ce^u = 0$ of the two-dimensional uniformization theorem, one may consider

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- This other flow may be technically more difficult than the previous one, because it no no longer uniformly parabolic. But it retains the geometric feature of the problem, because it is equivalent to the Ricci flow for the metric $g_{ij}(t) \equiv e^{u(x,t)} g_{ij}(x)$,

$$\dot{g}_{ij}(t) = -(R_{ij}(g(t)) - cg_{ij}(t)), \qquad g_{ij}(0) = g_{ij}$$

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- Because of their geometric origin, geometric flows may provide information even when no stationary points exist: for example, they may converge with a jump in the topology ("bubbling") or complex structure of the underlying geometry ("stability"), and/or detect a cyclic behavior ("soliton").

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Let $E \to X$ be a holomorphic Hermitian vector bundle over a compact Kähler manifold X. Consider the Yang-Mills flow of unitary connections $A^{\alpha}_{j\beta}$

$$\dot{A}^{\alpha}_{j\beta} = \nabla^{\bar{k}} F_{\bar{k}j}{}^{\alpha}{}_{\beta}$$

or equivalently (up to gauge transformations), the Donaldson flow of metrics $H_{\bar{\gamma}\beta}$,

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Theorem 1 (Donaldson, Uhlenbeck and Yau) The holomorphic, irreducible vector bundle $E \to X$ admits a Hermitian-Einstein metric (equivalently, a minimum of the Yang-Mills functional) if and only if $E \to X$ is stable in the sense of Mumford-Takemoto (i.e., for any sub sheaf $E' \subset E$, $E' \neq E$, we have $\mu(E') < \mu(E)$, where $\mu(E) =< \mathrm{Tr}\, F \wedge \omega^{n-1} > /\mathrm{rank}\, E$.)

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- In particular, when E is stable, the flows converge.

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The stable case

Theorem 1 (Donaldson, Uhlenbeck and Yau) The holomorphic, irreducible vector bundle $E \to X$ admits a Hermitian-Einstein metric (equivalently, a minimum of the Yang-Mills functional) if and only if $E \to X$ is stable in the sense of Mumford-Takemoto (i.e., for any sub sheaf $E' \subset E$, $E' \neq E$, we have $\mu(E') < \mu(E)$, where $\mu(E) =< \mathrm{Tr}\, F \wedge \omega^{n-1} > /\mathrm{rank}\, E$.)

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The Yang-Mills Flow on holomorphic vector bundles

Let $E \to X$ be a holomorphic Hermitian vector bundle over a compact Kähler manifold X. Consider the Yang-Mills flow of unitary connections $A^{\alpha}_{j\beta}$

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The general case

Theorem 2 (S. Bando, Y.T. Siu, G. Daskalopoulos, R. Wentworth, A. Jacob, B. Sibley)
Let $E \to X$ be a holomorphic vector bundle over a compact Kähler manifold X. Then
(a) The connections $A(t)$ converge, after suitable gauge transformations, on $X \setminus Z_{an}$ to a connection A_∞ on a bundle E_∞ over $X \setminus Z_{an}$ ($\dim Z_{an} \leq 4$).
(b) The bundle E_∞ extends as a reflexive sheaf, still denoted E_∞, over X.
(c) The sheaf E_∞ over X can be identified with the double dual $(Gr^{hns}(E))^{**}$ of the Harder-Narasimhan-Seshadri filtration over X.
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- The analytic set Z_{an} arises as the set long which the curvature of the connections blow up along the flow. It has been shown to coincide with the algebraic set Z_{alg} of singularities of $Gr^{hns}(E)$ by Daskalopoulos and Wentworth in dimension 2, and by Sibley and Wentworth in general.

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- When $c_1(X) < 0$ or $c_1(X) = 0$, it is not difficult, as shown by H.D. Cao using estimates of Yau and Aubin, to show that the flow converges to a Kähler-Einstein metric. We shall concentrate henceforth on the case $c_1(X) > 0$.

Partial results when $c_1(X) > 0$

- Perelman obtained some very powerful estimates for the Kähler-Ricci flow. They include in particular a uniform upper bound (independent of time) for $||R||_{C^0(X)}$, for the diameter of X with respect to the metric $g_{\bar{k}j}(t)$, and a uniform lower bound for the volume $B_r(z)/r^{2n}$ ("non-collapse").

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Major open questions

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- An even more obscure issue is to determine the behavior of the flow in general, even when the manifold X is unstable.
- As nothing is known in general, we shall examine the simplest case of the flow when $c_1(X) > 0$, and when the complex structure can be stable, semistable, or unstable. This is the case of S^2 with conic singularities, which we describe next.

Riemann surfaces with conic singularities

- Let M be a Riemann surface. We say that a metric $g(z)$ on M has a conic singularity at a point p if $g(z)$ is a smooth metric away from p, and satisfies near p

$$g(z) = e^{u(z)}|z-p|^{-2\beta}|dz|^2$$

where $u(z)$ is a bounded function near p, and $0 < \beta < 1$ is a fixed constant.

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- Let $p_1, \cdots, p_k$ be k given points, with corresponding weights $\beta_1, \cdots, \beta_k$, $0 < \beta_j < 1$. We consider metrics $g(z)$ with conic singularities at p_j with weight β_j whose Ricci current is in $c_1(M)$. This is equivalent to

$$\int_{M\setminus\beta} Ric(g) = \chi(M) - \sum_{j=1}^{k} \beta_j \equiv \chi(M, \beta).$$

We shall also normalize metrics so that $\int_M g(z) = 2$.

- A metric $g(z)$ with conic singularities on (M, β) is said to be Kähler-Einstein if

$$Ric(g) = \frac{1}{2}\chi(M,\beta)g(z) \qquad \text{on } M \setminus \beta$$

- It is said to be a soliton if there exists a holomorphic vector field V vanishing at $p_1, \cdots, p_k$ so that

$$Ric(g) = \frac{1}{2}\chi(M,\beta)g(z) + L_V g(z) \qquad \text{on } M \setminus \beta$$

where L_V denotes the Lie derivative with respect to V.

Classification of conic singularities on a Riemann surface

We consider a compact Riemann surface (M, β) with conic singularities $\beta = \sum_{j=1}^{k} \beta_j [p_j]$, $0 < \beta_j < 1$. Recall that its Euler characteristic is defined by $\chi(M, \beta) = \chi(M) - \sum_{j=1}^{k} \beta_j$.

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- When $\chi(M, \beta) \leq 0$, there exists a unique Kähler-Einstein metric with conic singularities at β.
- Thus the difficult case is when $\chi(M, \beta) \geq 0$, or equivalently,

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The situation in this last case has been described as follows by Troyanov (see also Luo and Tian):

- $k = 1$: there exists a unique soliton ("tear drop")
- $k = 2,\ \beta_1 = \beta_2$: there exists a unique Kähler-Einstein metric ("football")
- $k = 2,\ \beta_1 \neq \beta_2$: there exists a unique soliton
- $k \geq 3$: there exists a unique Kähler-Einstein metric if and only if

$$2 \max \beta_j < \sum_{j=1}^{k} \beta_j.$$

This last condition is now known as Troyanov's condition.

Theorem 3 (P., J. Song, J. Sturm, X. Wang) Let (S^2, β) be the sphere, and consider initial metrics of the form

$$g_0(z) = e^{u_0(z)} \prod_{j=1}^{k} (\frac{1+|z|^2}{|z-p_j|^2})^{\beta_j} g_{FS}(z)$$

where $g_{FS}(z)$ is the Fubini-Study metric on S^2, and $u_0(z)$ is a smooth function on S^2. We also assume that $g_0(z)$ has been normalized to have area 2.
Then the Kähler-Ricci flow

$$\dot{g}(z,t) = -Ric(g(z,t)) + \frac{1}{2}\chi(S^2,\beta) g(z,t) \text{ on } S^2 \setminus \beta, g(z,0) = g_0(z)$$

admits a unique solution for all time $t \in [0,\infty)$, satisfying
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- The long-time existence of the Kähler-Ricci flow with conic singularities had been established before by H. Yin and Mazzeo, Rubinstein, and Sesum, for different classes of initial data and functions.
- Yin's results are for classes defined by the special Schauder spaces that he introduced for conic singularities. Mazzeo, Rubinstein, and Sesum's results are for classes of polyhomogeneous metrics.

Definition Let (S^2, β) be the sphere with conic singularities $\beta = \sum_{j=1}^{k} \beta_j [p_j]$. Then we say that

(a) (S^2, β) is stable if $\sum_{j=1}^{k} \beta_j \geq 2$ or $2\max_j \beta_j < \sum_{j=1}^{k} \beta_j$.

(b) (S^2, β) is semistable if $\sum_{j=1}^{k} \beta_j < 2$ or $2\max_j \beta_j = \sum_{j=1}^{k} \beta_j$.

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Theorem 4 (P., J. Song, J. Sturm, and X. Wang) Consider the Kähler-Ricci flow on (S^2,β) with initial data $g_0(z)$ as described in Theorem 3. To be specific, assume that β_k is the largest weight.
(a) If (S^2,β) is stable, then the Kähler-Ricci flow converges in $C^\infty(S^2\setminus\beta)$ to the unique Kähler-Einstein metric on (S^2,β).
(b) If (S^2,β) is semistable, then the Kähler-Ricci flow converges in the Gromov-Hausdorff sense to the Kähler-Einstein metric on (S^2,β_∞), where

$$\beta_\infty=\beta_k[p_\infty]+\beta_k[q_\infty].$$

(c) If (S^2,β) is unstable, then the Kähler-Ricci flow converges in the Gromov-Hausdorff sense to the soliton on (S^2,β_∞), where

$$\beta_\infty=\beta_k[p_\infty]+\Big(\sum_{j=1}^{k-1}\beta_j\Big)[q_\infty].$$

In both cases (b) and (c), the points $p_1,\cdots,p_{k-1}$ converge in Gromov-Hausdorff distance to the point q_∞, while the point p_k converges to the point p_∞.

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- We note that the sphere S^2 with marked points $p_1, \cdots, p_k$ has a complex structure that varies with the points $p_1, \cdots, p_k$. In particular, in the semistable and unstable cases, where the limiting space is the sphere with only 2 marked points p_∞ and q_∞, we have the phenomenon of the complex structure jumping in the limit.

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- In this, the situation turns out then to be very similar to the Yang-Mills flow, with reparametrizations playing the role of gauge transformations, and the complex structure of the bundle also jumping in the limit.

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- We note that the sphere S^2 with marked points $p_1, \cdots, p_k$ has a complex structure that varies with the points $p_1, \cdots, p_k$. In particular, in the semistable and unstable cases, where the limiting space is the sphere with only 2 marked points p_∞ and q_∞, we have the phenomenon of the complex structure jumping in the limit.
- At any finite time, the conic singularities are the same as at the original time. The reason that the jumping can occur in the limit is because Gromov-Hausdorff convergence allows a reparametrization at each finite time, and the reparametrizations may not converge.
- In this, the situation turns out then to be very similar to the Yang-Mills flow, with reparametrizations playing the role of gauge transformations, and the complex structure of the bundle also jumping in the limit.
- The proof of Theorems 3 and 4 requires several ingredients. The long-time existence is proved by approximating the initial data and the flow by smooth metrics with curvature uniformly bounded from below. The convergence requires an extension to this setting of Perelman's functionals, the compactness results of Cheeger and Colding for manifolds with Ricci curvature bounded from below, and the partial C^0 estimate of Donaldson and Sun. A key step is to show that the limiting manifold is the sphere with a finite number of conic singularities.

Non-Kähler geometry and Strominger systems

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Motivation and history

- Superstring theory is at the present time the only candidate for a unified theory of all physical interactions, including quantum gravity. But it requires 10 space-time dimensions. To reconcile this with our experience, it is suggested that this 10-dimensional space time is of the form of $M^{3,1} \times X$, where $M^{3,1}$ is Minkowski 4-dimensional space, and X is a small 6-dimensional space.

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- The effective theory on $M^{3,1}$ depends on what X is. The consistency constraints of superstrings, together with the requirement that the effective theory be $N = 1$ supersymmetric, have led Candelas, Horowitz, Strominger, and Witten to propose that X should be a Calabi-Yau manifold, that is, a compact 3-dimensional complex manifold which is Ricci-flat,

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- It was noted by Strominger that the ansatz introduced by Candelas, Horowitz, Strominger, and Witten can be relaxed to metrics with torsion. The Strominger system is a system of equations obtained by Strominger by allowing metrics with torsion for compactifications of the heterotic string still admitting supersymmetry. As such, they generalize the Ricci-flat condition above, and also incorporate a Hermitian-Einstein condition for an associated holomorphic vector bundle.

The Strominger system

The Strominger system is a system of equations for a compact, 3-dimensional complex manifold X, equipped with a no-where vanishing holomorphic 3-form Ω, and a holomorphic vector bundle $E \to X$. The equations are for a Hermitian metric ω on X, and a Hermitian metric $H_{\bar\alpha\beta}$ on E, satisfying the following system

$$
\begin{aligned}
&(1)\ F^{2,0} = F^{0,2} = 0, \qquad g^{j\bar k}F_{\bar k j} = 0\\
&(2)\ i\partial\bar\partial\omega + \alpha\,(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F)) = 0\\
&(3)\ d^\dagger\omega = i(\bar\partial - \partial)\ln ||\Omega||_\omega
\end{aligned}
$$

- Here F is the curvature of the bundle E with respect to the metric $H_{\bar\alpha\beta}$, $F = -\bar\partial(H^{-1}\partial H)$. F is viewed as a $(1,1)$-form, valued in $End(E)$. Similarly, the Riemann curvature tensor Rm is viewed as a $(1,1)$-form, valued in $End(T^{1,0}(X))$.
- The expression $||\Omega||_\omega$ is defined by $||\Omega||_\omega^2 = |\Omega|^2\omega^{-3}$.
- $\alpha > 0$ is a parameter, called the string tension.
- It was pointed out by Jun Li and Yau that the third equation can be replaced by (3')

$$d(||\Omega||_\omega\omega^2) = 0$$

  which is a variation of the condition that ω be "balanced".
- The condition (2) shows clearly that, in general, ω is not Kähler.

A geometric construction of Goldstein and Prokushkin

Goldstein and Prokushkin have shown how to obtain a certain class of Calabi-Yau 3-folds which are toxic vibrations over a $K3$ surface. More precisely, let (S, ω_S) be a $K3$ surface with Ricci flat Kähler metric ω_S. Then to any pair $\kappa_1, \kappa_2 \in H^2(S, \mathbf{Z})$ of anti-self-dual $(1,1)$-forms on S, Goldstein and Prokushkin can associate a toric vibration $\pi : X \to S$, with nowhere vanishing holomorphic 3-form given by $\Omega = \theta \wedge \pi^*(\Omega_S)$, for a $(1,0)$-form θ. Then the form

$$\omega_0 = \pi^*(\omega_S) + i\theta \wedge \bar{\theta}$$

is a balanced Hermitian metric on X, i.e. $d\omega_0^2 = 0$.

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A special solution of Fu and Yau

Fu and Yau found a solution of the Strominger system with X given by the Goldstein-Prokushkin construction, and the following ansatz for the metric on X,

$$\omega_u = \pi^*(e^u \omega_S) + i\theta \wedge \bar{\theta}$$

where u is a function on S. This reduces the Strominger system to a two-dimensional Monge-Ampère equation with gradient terms,

$$i\partial\bar{\partial}(e^u - fe^{-u}) \wedge \omega + \alpha i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \mu = 0$$

where $f \geq 0$ is a known function, and μ is a $(2,2)$-form with average 0.

Explicitly, the above equation can be rewritten as

$$\frac{\det g'_{\bar{k}j}}{\det g_{\bar{k}j}} = (e^u + fe^{-u})^2 - 2\alpha(e^u - \alpha fe^{-u})|\nabla u|^2 - 4\alpha^2 e^{-u} < \nabla u, \nabla f > +2\alpha^2 e^{-u}\Delta f - 2\alpha\mu$$

where the Hermitian form $g'_{\bar{k}j}$ is defined by

$$g'_{\bar{k}j} = (e^u + fe^{-u})g_{\bar{k}j} + 4\alpha\, u_{\bar{k}j} > 0$$

It is then natural to impose the ellipticity condition $g'_{\bar{k}j} > 0$.

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Theorem 5 (Fu and Yau) Consider the above complex Monge-Ampère equation in dimension $\dim S = 2$ with the above ellipticity condition. Then there exists a solution $u \in C^\infty(S)$ satisfying the above ellipticity condition. In particular, we obtain in this way a solution of the Strominger system with X a toric fibration over the $K3$ surface S.

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- The C^0 estimate is by Moser iteration, for both lower and upper bounds.
- A new, particularly difficult step is to show that, for $||e^{-u}||_{L^4} = A$ sufficiently small, the quantity $\det g'_{\bar{k}j}/\det g_{\bar{k}j}$ is bounded from below, away from 0. Thus the equation remains non-degenerate.

A generalization to n-dimensions of the Fu-Yau equation

As an n-dimensional analogue of the equation which they solved in dimension 2, Fu and Yau proposed the study of the following complex Hessian equation

$$i\partial\bar{\partial}(e^u - fe^{-u}) \wedge \omega^{n-1} + \alpha i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u \wedge \omega^{n-2} + \mu = 0$$

under the following ellipticity condition

$$\omega' \equiv (e^u + fe^{-u})\omega + 2n\alpha i\partial\bar{\partial}u > 0$$

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Theorem 6 (P., S. Picard, X. Zhang) Assume that $u \in C^4(X)$ is a solution of the above equation satisfying the above ellipticity condition. Then we have the following a priori estimates:
(a) Let $||e^{-u}||_{L^{4(n-1)}} = A$. Then if A is sufficiently small, we have $\inf e^{-u} \leq C_0 A$ and $\sup e^u \leq C_1 A^{-1}$, for constants C_0, C_1 depending only on ω, f, and μ.
(b) We have $||\nabla u||_{C^0} \leq C$.
(c) Assume that for any $0 < \kappa < 1$, there is an A so that $\sigma_2(\omega') \geq \kappa$. Then there exists a constant $C > 0$ so that $C^{-1}\omega < \omega' < C\omega$.
(d) Assume that $||u||_{C^2} \leq C$. Then there exists a constant C' so that $||u||_{C^{2,\alpha}} \leq C'$.

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- The estimate in (c) makes use of some new techniques introduced for real Hessian equations by P. Guan, Ren, and Wang.
- The estimate in (d) follows from results of Tosatti, Yu Wang, B. Weinkove, and X. Yang, which is itself built on earlier work of Yu Wang.
- Unlike in dimension 2, it is not known if the hypothesis in (c) holds.

C^2 estimates for Hessian equations with gradients

Let (X, ω) be a compact Kähler manifold. Let $\chi(u, z)$ be a $(1, 1)$-form on X, and consider the following complex Hessian equation, which is a more general equation than the Fu-Yau equation,

$$(\chi(z, u) + i\partial\bar{\partial}u)^k \wedge \omega^{n-k} = \psi(z, u, Du)\omega^n$$

where $\psi(z, u, p)$ is a given positive function, and the solution u is required to be χ-plurisubharmonic, i.e. $\omega' = \chi(z, u) + i\partial\bar{\partial}u > 0$.

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Theorem 7 (P., S. Picard, X. Zhang) Assume that u is a C^4 solution of the above equation which is χ-plurisubharmonic. Then there exists a constant C depending only on $||u||_{C^0}$, $||Du||_{C^0}$, and the upper and lower bounds of ψ in the range limited by the bounds on u and Du so that

$$||D\bar{D}u||_{C^0} \leq C.$$

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- This estimate also requires an extension of a technique of Guan, Ren, and Wang.
- Actually, it holds under a weaker assumption that χ-plurisubharmonicity, namely that ω' be in the $\Gamma_{k+1} = \{M; \sigma_1(M) > 0, \sigma_2(M) > 0, \cdots, \sigma_{k+1}(M) > 0\}$. But we expect that the optimal condition should be that ω' be in the Γ_k cone, which is an open problem.

A possible general approach: the Anomaly Flow

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If we already know the holomorphic vector bundle $E \to X$ in the Strominger system, then the Donaldson-Uhlenbeck-Yau theorem determines the corresponding Hermitian-Einstein metric $H_{\bar{\alpha}\beta}$. The system reduces then to the problem of solving for ω satisfying simultaneously the equation (2) (whose origin is a cancellation of anomaly requirement), and the condition (3) of being balanced.

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One possible strategy is to adopt a particular ansatz for ω that guarantees that it is balanced, and then try to find a solution of (2) with that ansatz. For example, if ω_0 is balanced, then any metric ω of the form

$$\omega^2 = \omega_0^2 + i\partial\bar{\partial}(u\tilde{\omega})$$

with $d\omega_0^2 = 0$ and $\tilde{\omega}$ any $(1,1)$-form, is automatically balanced, and we can then solve for u. The drawback is that no particular ansatz seems more compelling than the others, and the resulting equations all seem very complicated and unnatural.

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Because of this, it has been recently proposed by P., S. Picard, and X. Zhang to look for solutions of the Strominger system as stationary points of the following flow of positive $(2,2)$-forms, called the "Anomaly flow",

$$\partial_t(||\Omega||_\omega \omega^2) = i\partial\bar{\partial}\omega + \alpha(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F))$$

with $\omega = \omega_0$ when $t = 0$, where ω_0 is a balanced metric.

Theorem 8 (P., S. Picard, X. Zhang) The above flow of positive $(2,2)$-forms defines a vector field on the space of positive $(1,1)$-forms.
(a) The corresponding flow preserves the balanced property of the metric $\omega(t)$.
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- It had been shown a while ago by Michelson that, given a positive $(n-1,n-1)$-form Ψ, there is a unique positive $(1,1)$-form ω so that $\omega^{n-1} = \Psi$. It turns out that ω can be expressed algebraically in Ψ. In fact, $\star\omega = \Psi$.

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- An indication that the anomaly flow is natural is provided by the fact that it preserves the ansatz of Fu Yau for toric fibrations over $K3$ surfaces. In fact, it reduces to the following "parabolic Fu-Yau equation",

$$\dot{u} = e^{-u}(\Delta(e^u - fe^{-u}) + \alpha \det u_{\bar{k}j} + \mu)$$

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- More generally, the general theory of the anomaly flow, such as criteria for long-time existence and convergence, is yet to be developed. It is also known that the Fu-Yau ansatz for the Strominger equation is not always solvable, so algebraic-geometric conditions for the existence of solutions would be very important, if at all available.