Location and Weyl formula for the eigenvalues of non self-adjoint operators

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1. Introduction

Let $P(x, D_x)$ be a second order differential operator with $C^\infty(\Omega)$ real-valued coefficients in a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$. Consider a boundary problem

$$\begin{cases} P(x, D_x)u = f \text{ in } \Omega, \\ B(x, D_x)u = g \text{ on } \partial\Omega. \end{cases} \tag{1}$$

where $B(x, D_x)$ is a differential operator with order less or equal to 1 and the principal symbol $P(x, \xi)$ of $P(x, D_x)$ satisfies $p(x, \xi) \geq c_0|\xi|^2$, $c_0 > 0$. Assume that there exists $0 < \varphi < \pi$ such that the problem

$$\begin{cases} (P(x, D_x) - z)u = f \text{ in } \Omega, \\ B(x, D_x)u = g \text{ on } \partial\Omega. \end{cases} \tag{2}$$

is parameter elliptic for every $z \in \Gamma_\psi = \{z : \arg z = \psi\}$, $0 < |\psi| \leq \varphi$. Then following a classical result of Agranovich-Vishik (1964) we can find a closed operator A with domain $D(A) \subset H^2(\Omega)$ related to the problem (1). Moreover, for every closed angle $Q = \{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta\} \subset \{z \in \mathbb{C} : |\arg z| < \varphi\}$ which does not contain $\mathbb{R}^+$ there exists $a_Q > 0$ such that the resolvent $(A - z)^{-1}$ exists for $z \in Q, |z| \geq a_Q$. The operator A has a discrete spectrum in $\mathbb{C}$ with eigenvalues with finite multiplicities.

Let λ_j be the eigenvalues of A ordered as

$$0 \leq |\lambda_1| \leq \ldots \leq |\lambda_m| \leq \ldots.$$

In general A is not a self-adjoint operator and the analysis of the asymptotics of the counting function $N(r) = \#\{|\lambda_j| \leq r\}$ as $r \to +\infty$ is a difficult problem. In particular, it is quite complicated to obtain a Weyl formula for $N(r)$ with a remainder and many authors obtained results which yield only the leading term of the asymptotics. Even for elliptic boundary problem the result of Agranovich-Vishik says that in the domain $0 < \psi < |\arg z| < \varphi$ we can have only finite number eigenvalues but we could have a bigger eigenvalues-free domains. As we will discuss in the talk , to have a better remainder we must obtain a eigenvalues-free regions outside some parabolic neighborhoods of the real axis and this is crucial for Weyl formula. To do this a fine semi-classical analysis is applied.

In mathematical physics there are problems which are not related to parameter -elliptic boundary problems. Therefore, the results of Agranovich-Vishik canot be applied and the analysis of the eigenvalues-free regions must be studied by another approach.

In this direction we have the following problems:

I. Prove the discreteness of the spectrum of A in some subset $U \subset \mathbb{C}$.

II. Find eigenvalues-free domains having the form

$$|\operatorname{Im} z| \geq C_{\pm\delta}(|\operatorname{Re} z| + 1)^{\delta_\pm}, \pm \operatorname{Re} z \geq 0, 0 < \delta_\pm < 1.$$

III. Establish a Weyl asymptotic for the counting function

$$N(r) = cr^d + \mathcal{O}(r^{d-\kappa}),\ 0 < \kappa < 1.$$

In this talk we treat the problems (II) and (III). The problem (I) is easer to deal with and the analysis of (II) in many cases implies that $A - z$ is a Fredholm operator for z in a suitable regions.

2. Two spectral problems related to scattering theory.

I. Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of Ω is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R} \times \Omega, \\ \partial_\nu u - \gamma(x) u_t = 0 \text{ on } \mathbb{R} \times \Gamma, \\ u(0,x) = f_0,\ u_t(0,x) = f_1 \end{cases} \tag{3}$$

with initial data $f = (f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathcal{H}$. Here ν is the unit outward normal to Γ pointing into Ω and $\gamma(x) \geq 0$ is a C^∞ function on Γ. The solution of (3) is given by $(u, u_t) = V(t)f = e^{tG}f$, $t \geq 0$, where $V(t)$ is a contraction semi-group in $\mathcal{H}$ The spectrum of G in $\mathrm{Re}\, z < 0$ is formed by isolated eigenvalues with finite multiplicity, while the continuous spectrum of G coincides with $\mathbf{i}\mathbb{R}$. Next if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda \gamma f_1 = 0 \text{ on } \Gamma. \end{cases} \tag{4}$$

Notice that if $Gf = \lambda f$ with $\operatorname{Re}\lambda < 0$, $f \neq 0$, then $(u(t,x), u_t(t,x)) = V(t)f = e^{\lambda t} f(x)$ is a solution of (1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they perturb the inverse scattering problems. Recently it was proved that if we have a least one eigenvalue λ of G with $\operatorname{Re}\lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\mathrm{R}an\, W_- \neq \mathrm{R}an\, W_+$. Hence we canot define the scattering operator S related to (3) by $S = W_+^{-1} W_-$. We may define S by another evolution operator.

For dissipative boundary problems the $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. In this case for odd dimensions Lax and Phillips (1973) proved that $\mathbf{i}z_0$ is an eigenvalue of G. Consequently, the analysis of the location of the eigenvalues of G is important for the inverse scattering problems.

Now we will discuss another important spectral problem for scattering leading to non self-adjoint operator. The inhomogeneous medium in K is characterized by a smooth function $n(x) > 0$ in $\bar{K}$, called contrast. The scattering problem is related to an incident wave u_i which satisfies $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^d$ and the total wave $u = u_i + u_s$ which satisfies the transmission problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus \bar{K}, \\ \Delta u + k^2 n(x) u = \text{ in } K, \\ u^+ = u^- \text{ on } \Gamma, \\ \left(\frac{\partial u}{\partial \nu}\right)^+ = \left(\frac{\partial u}{\partial \nu}\right)^- \text{ on } \Gamma. \end{cases} , \tag{5}$$

where $f^{\pm} = \lim_{\epsilon \to 0} f(x \pm \epsilon \nu)$ for $x \in \Gamma$. Here $k > 0$ and the outgoing scattering wave u_s satisfies the outgoing Sommerfeld radiation condition

$$\lim_{r \to +\infty} r^{(1-d)/2} \left(\frac{\partial u_s}{\partial r} - \mathbf{i} k u_s \right) = 0$$

uniformly with respect to $\theta = x/r \in \mathbb{S}^{d-1}$, $r = |x|$.

If the incident wave has the form $u_i = e^{\mathbf{i}k\langle x,\omega\rangle}$, $\omega \in \mathbb{S}^{d-1}$, then

$$u_s(r\theta, k) = e^{\mathbf{i}kr} r^{-(d-1)/2}\Big(a(k,\theta,\omega) + \mathcal{O}(\frac{1}{r})\Big),\ r \to +\infty.$$

The function $a(k,\theta,\omega)$ is called scattering amplitude and the far-field operator $F(k) : L^2(\mathbb{S}^{d-1}) \longrightarrow L^2(\mathbb{S}^{d-1})$ has the form

$$(F(k)f)(\theta) = \int_{\mathbb{S}^{d-1}} a(k,\theta,\omega)f(\omega)d\omega.$$

The inverse scattering problem of the reconstruction of K based on the linear sampling method of Colton and Kress breaks down for frequencies k such that $F(k)$ has a non trivial kernel or co-kernel. If Ker $F(k) \neq \{0\}$ for $k \in \mathbb{R}$, then $\lambda = k^2$ is such that the problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } K, \\ \Delta v + k^2 n(x) v = 0 \text{ in } K, \\ u = v,\ \partial_\nu u = \partial_\nu v \text{ on } \Gamma \end{cases} \tag{6}$$

has a non-trivial solution $(u, v) \neq 0$.

We consider a more general setting. A complex number $\lambda \in \mathbb{C}, \lambda \neq 0$, is called interior transmission eigenvalue (ITE) if the following problem has a non-trivial solution $(u, v) \neq 0$:

$$\begin{cases} (\nabla c_1(x) \nabla + \lambda n_1(x))\, u_1 = 0 \text{ in } K, \\ (\nabla c_2(x) \nabla + \lambda n_2(x))\, u_2 = 0 \text{ in } K, \\ u_1 = u_2, \;\; c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 \text{ on } \Gamma, \end{cases} \tag{7}$$

where ν denotes the exterior unit normal to Γ, $c_j(x), n_j(x) \in C^\infty(\overline{K})$, $j = 1, 2$ are strictly positive real-valued functions. For the analysis of (ITE) one imposes the condition

$$d(x) = c_1(x) n_1(x) - c_2(x) n_2(x) \neq 0, \quad \forall x \in \Gamma. \tag{8}$$

Partial cases: 1) isotropic case: $c_1(x) = c_2(x)$, $\forall x \in \Gamma$, $n_1(x) \neq n_2(x)\, \forall x \in \Gamma$. 2) anisotropic case: $c_1(x) \neq c_2(x)$, $\forall x \in \Gamma$.

3. Eigenvalues-free regions

For the eigenvalues of G we examine two cases:
$(A): \ -1 < \gamma(x) - 1 < 0, \ \forall x \in \Gamma, \ (B): \ \gamma(x) - 1 > 0, \ \forall x \in \Gamma.$

Theorem 1

In the case (A) for every ϵ, $0 < \epsilon \ll 1$, the eigenvalues of G lie in the region

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \ |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1), \ \operatorname{Re} z < 0\}.$$

In the case (B) for every ϵ, $0 < \epsilon \ll 1$, and every $N \in \mathbb{N}$ the eigenvalues of G lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where

$$\mathcal{R}_N = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \ \operatorname{Re} z < 0\}.$$

In 1975 A. Majda proved that the eigenvalues of G lie in the region

$$case(A) : E_1 = \{z \in \mathbb{C} : \ |\operatorname{Re} z| \leq C_1(|\operatorname{Im} z|^{3/4} + 1), \ \operatorname{Re} z < 0\},$$

case (B): $\sigma_p(G) \subset E_1 \cup E_2$, where

$$E_2 = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq C_2(|\operatorname{Re} z|^{1/2} + 1), \ \operatorname{Re} z < 0\}.$$

The case $\gamma(x) = 1, \ \forall x \in \Gamma$ is special because for the ball $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ it was shown by Majda that there are no eigenvalues of G.

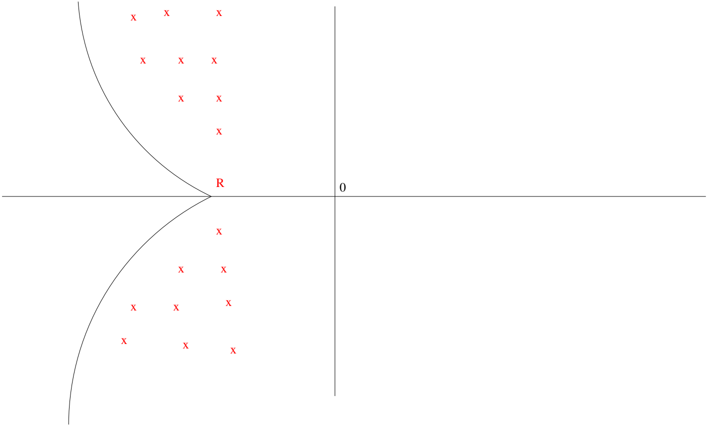

Figure: Eigenvalues for $0 < \gamma(x) < 1$

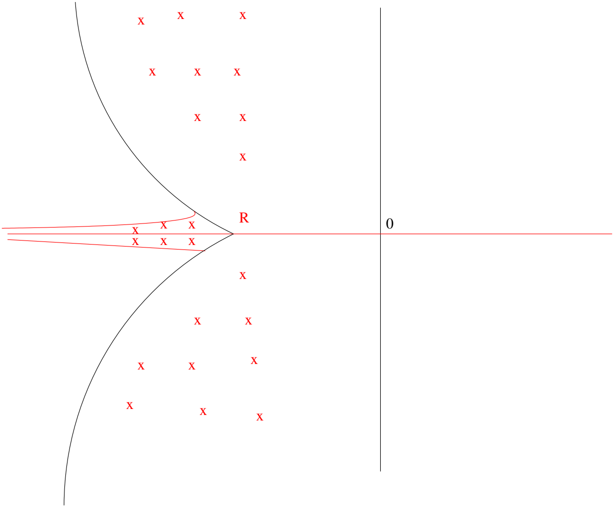

Figure: Eigenvalues for $\gamma(x) > 1$

Theorem 2 (Vodev)

Assume (8) *fulfilled together with the condition*
$c_1(x) = c_2(x)$, $\partial_\nu c_1(x) = \partial_\nu c_2(x)$, $\forall x \in \Gamma$. *Then for every* $0 < \epsilon \ll 1$ *the (ITE) lie the region*

$$\Lambda_+ := \{z \in \mathbb{C} : \operatorname{Re}\lambda \geq 0, \, |\operatorname{Im}\lambda| \leq C_\epsilon(\operatorname{Re}\lambda + 1)^{3/4+\epsilon}\}$$

and there are only a finite number (ITE) with $\operatorname{Re}\lambda < 0$. *If* $c_1(x) \neq c_2(x), \forall x \in \Gamma$, *the (ITE) lie in*

$$\Lambda'_+ := \{z \in \mathbb{C} : \operatorname{Re}\lambda \geq 0, \, |\operatorname{Im}\lambda| \leq C_\epsilon(\operatorname{Re}\lambda + 1)^{4/5+\epsilon}\}.$$

If $(c_1(x) - c_2(x))d(x) > 0$, $\forall x \in \Gamma$ *we have only a finite number (ITE) with* $\operatorname{Re}\lambda < 0$. *Moreover, if we assume that* $(c_1(x) - c_2(x))d(x) < 0$, $\forall x \in \Gamma$, *then for* $\operatorname{Re}\lambda \geq 0$ *the* (*ITE*) *are in* Λ_+, *while for* $\operatorname{Re}\lambda < 0$ *and every* $N \geq 1$ *there exists* $C_N > 0$ *such that* (*ITE*) *lie in*

$$\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq C_N(|\operatorname{Re}\lambda| + 1)^{-N}, \, \operatorname{Re}\lambda \leq 0\}.$$

A weaker result in a partial case $(n_1(x) = 1, n_2(x) > 1$ in $K)$ with eigenvalues-free region

$$\{z \in \mathbb{C} : \operatorname{Re}\lambda \geq 0, \, |\operatorname{Im}\lambda| \geq C(\operatorname{Re}\lambda + 1)^{24/25}\}$$

has been obtained by Hitrik and al.

Strictly convex obstacles

Theorem 3

Assume K strictly convex. In the case $\gamma(x) - 1 > 0$ for every $N \in \mathbb{N}$ outside the region $\mathcal{R}_N$ we have only finite number eigenvalues of the generator G. Moreover, there exists $\epsilon_0 > 0$ such that all eigenvalues of G satisfy $\mathrm{Re}\,\lambda_j \leq -\epsilon_0$.

Theorem 4 (Vodev)

Assume K strictly convex and $c_1(x) = c_2(x), \partial_\nu c_1(x) = \partial_\nu c_2(x)$, $x \in \Gamma$. Then for every $\epsilon > 0$ the (ITE) lie in the region

$$\Lambda_+ := \{z \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0, \, |\mathrm{Im}\,\lambda| \leq C_\epsilon(\mathrm{Re}\,\lambda + 1)^{1/2+\epsilon}\}$$

and there are only a finite number (ITE) with $\mathrm{Re}\,\lambda < 0$.

The result of Theorem 4 is almost optimal since for the interval $K = \{|x| \leq 1\}$ the eigenvalues lie in

$$\Lambda_+ := \{z \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0, \, |\mathrm{Im}\,\lambda| \leq C_\epsilon(\mathrm{Re}\,\lambda + 1)^{1/2}\}$$

and this domain cannot be improved (Silvester, Ha and Stefanov). Theorem 3 is also almost optimal, since in $\mathcal{R}_N$ we have infinite number eigenvalues.

4. Weyl formula for the eigenvalues

To obtain a Weyl formula for the (ITE) introduce the coefficients

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_K \Big(\frac{n_j(x)}{c_j(x)} \Big)^{d/2} dx,\ j = 1, 2,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

In the anisotropic case $c_1(x) = 1, n_1(x) = 1, c_2(x) \neq 1, c_2(x)n_2(x) \neq 1, \forall x \in \bar{K}$, the asymptotics

$$N(r) \sim (\tau_1 + \tau_2) r^d,\ r \to +\infty. \tag{9}$$

has been obtained by Lakshatanov and Vainberg (2012) under some additional assumptions which guarantee that the boundary problem is parameter-elliptic. By the results of Agranovich and Vishik outside every angle $D_\alpha = \{z \in \mathbb{C} :\ |\arg z| \leq \alpha\}$ we have only a finite number of (ITE) and the following estimate holds

$$\|(z - \mathcal{A})^{-1}\| \leq C_\alpha |z|^{-1},\ z \notin D_\alpha,\ |z| \gg 1.$$

The authors applied directly a result of Boimanov-Kostjuchenko (1990) leading to (9).

The isotropic case $c_1(x) = c_2(x) = 1, \ \forall x \in \bar{K}, \ n_1(x) = 1, \ n_2(x) \neq 1, \ \forall x \in \Gamma$, is more difficult since the corresponding operator A has domain

$$D(A) = \{(u, v) \in L^2(K) \times L^2(K) : \ \Delta u \in L^2(K), \ \Delta v \in L^2(K),$$

$$u - v = 0, \partial_\nu (u - v) = 0 \text{ on } \Gamma\}$$

Thus $D(A)$ is not included in $H^2(K)$, and the problem is not parameter-elliptic. In this case Robbiano (2013) obtained (9) by establishing the asymptotics

$$\sum_j \frac{1}{|\lambda_j|^p + t} = \alpha t^{-1+\frac{d}{2p}} + o(t^{-1+\frac{d}{2p}}), \ t \to +\infty.$$

where $p \in \mathbb{N}$ is sufficiently large. An application of the Tauberian theorem of Hardy-Littlewood yields the result. A similar result with leading term $3\sqrt{3}(\tau_1 + \tau_2)$ has been obtained previously by Dimassi and Petkov (2013). By this argument one obtains a very week estimate for the remainder which can be estimated by the principal term divided by a logarithmic factor. To get better results, it is important to take into account parabolic eigenvalues-free regions and to apply different techniques which are not based on Tauberian theorems.

Theorem 5 (-P., Vodev)

Under the condition (8), assume that there are no (ITE) in the region

$$\{\lambda \in \mathbb{C} : \, |\operatorname{Im} \lambda| \geq C(|\operatorname{Re} \lambda| + 1)^{1-\frac{\kappa}{2}}\}, \; C > 0, 0 < \kappa \leq 1. \tag{10}$$

Then for every $0 < \epsilon \ll 1$ *we have the asymptotics*

$$N(r) = (\tau_1 + \tau_2) r^d + \mathcal{O}_\epsilon(r^{d-\kappa+\epsilon}), \; r \to +\infty. \tag{11}$$

• For arbitrary obstacles and $c_1(x) = c_2(x), \; \partial_\nu c_1(x) = \partial_\nu c_2(x), \forall x \in \Gamma$, we can take $\kappa = \frac{1}{2} - \epsilon$ and we obtain a remainder $\mathcal{O}_\epsilon(r^{d-1/2+\epsilon})$.

• For strictly convex obstacles we may take $\kappa = 1 - \epsilon, \; \forall \epsilon$. Consequently, we have in this case a remainder $O_\epsilon(r^{d-1+\epsilon})$.

• The optimal result should be to have a eigenvalues-free region with $\kappa = 1$ but it is an open problem. This optimal result is known only for the interval $K = \{x \in \mathbb{R} : |x| \leq 1\}$. Even for the ball $|x| \leq 1$ in $\mathbb{R}^d, d \geq 2$ this is an open problem.

5. Location of eigenvalues

The proofs of Theorems 1-4 are based on a fine semi-classical analysis. Introduce three regions in $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$:

$$Z_1 = \{z \in \mathbb{C} :\ \mathrm{Re}\, z = 1, h^{1/2-\epsilon} \leq \mathrm{Im}\, z \leq 1,\ 0 < \epsilon \ll 1\},$$

$Z_2 = \{z \in \mathbb{C} :\ \mathrm{Re}\, z = -1, |\mathrm{Im}\, z| \leq 1\},\ Z_3 = \{z \in \mathbb{C} :\ |\mathrm{Re}\, z| \leq 1, \mathrm{Im}\, z = 1\}$. Consider the semi-classical problem

$$\begin{cases} (P(h) - z)u = 0 \text{ in } \mathbb{R}^d \setminus K,\ u \in H^2(\mathbb{R}^d \setminus K), \\ u = f \text{ on } \Gamma, \end{cases} \tag{12}$$

where $P(h) = -\frac{h^2}{n(x)} \nabla c(x) \nabla$. Let $D_\nu = -i\partial_\nu$, and let γ_0 denote the trace on Γ. The problem is to construct a semi-classical parametrix for the problem (12) in $Z_1 \cup Z_2 \cup Z_3$ and to find an approximation for the semi-classical Dirichlet-to-Neumann map (DN)

$$\mathcal{N}(z, h) : H_h^s(\Gamma) \ni f \longrightarrow \gamma_0 h D_\nu u \in H_h^{s-1}(\Gamma)$$

for domains with arbitrary geometry. Here $H_h^s(\Gamma)$ is the semi-classical Sobolev space with norm $\|\langle hD\rangle^s u\|_{L^2(\Gamma)}$.

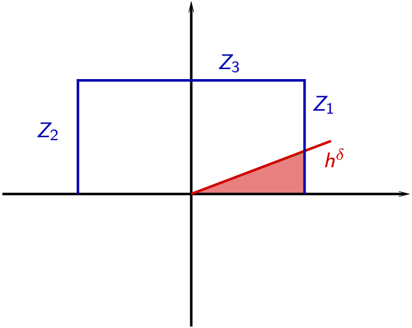

Figure: Contours $Z_1, Z_2, Z_3, \delta = 1/2 - \epsilon$

Strategy:

Step 1. Let $u = (u_1, u_2) \neq 0$ be an eigenfunction of G and let $f = u_1|_\Gamma$. Then $(-\Delta + \lambda^2)u_1 = 0$ and $\partial_\nu u_1 - \lambda\gamma u_1 = 0$ on Γ. Set

$$\lambda = \frac{\mathbf{i}\sqrt{z}}{h},\ 0 < h \ll 1, z \in Z_1 \cup Z_2 \cup Z_3.$$

We obtain the problem

$$(-h^2\Delta - z)u_1 = 0, \text{in } \mathbb{R}^d \setminus K,\ N(z,h)f - \sqrt{z}\gamma f = 0 \text{ on } \Gamma.$$

Step 2. One search an approximation $\|N(z,h) - Op_h(\rho)\|_{L^2 \to L^2} = \mathcal{O}(h^\alpha)$ by a h-pseudo-differential operator $Op_h(\rho)$ and we get

$$C(z,h)f := Op_h(\rho)f - \sqrt{z}\gamma f = \mathcal{O}(h^\alpha)f,\ \alpha > 0.$$

Here α depend on the zone Z_j, $j = 1.2, 3$.

Step 3. We wish to invert the operator $C(z,h)$ and deduce $f = \mathcal{O}(h^\beta)f, \beta > 0$. This implies $f = 0$ and hence $u = 0$. It is not sufficient to prove that C is invertible and we must examine the norm $\|C^{-1}C\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

We need to introduce some $h-$pseudo-differential operators.
We say that $a(x, \xi; h) \in S_\delta^{k,m}(\Gamma)$ if the following conditions are satisfied:

(i) for $|\xi| \geq L \gg 1$ we have

$$|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha,\gamma,L}(1 + |\xi|)^{m-|\gamma|}, \ \forall \alpha, \forall \gamma.$$

(ii) for $|\xi| \leq L$ we have

$$|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha,\gamma,L} h^{-k-\delta(|\alpha|+|\gamma|)}, \ \forall \alpha, \forall \gamma.$$

Then for $a \in S_\delta^{k,m}(\Gamma)$, we consider the operator

$$\Big(Op_h(a)f\Big)(x) = (2\pi h)^{-d+1} \int\int e^{i\langle x-y,\xi\rangle/h} a(x,\xi) f(y) dy d\xi.$$

We have a calculus for the $h-$ pseudodifferential operators with symbols in $S_\delta^{k,m}$ if $0 < \delta < 1/2$. In particular, if $a \in S_\delta^{0,1}$, $b \in S_\delta^{0,-1}$, one gets

$$\|Op_h(a)Op_h(b) - Op_h(ab)\|_{L^2} \leq Ch^{1-2\delta}.$$

Close to the boundary introduce geodesic normal coordinates (x', x_d) in a neighborhood of a point $x_0 \in \Gamma$ with $x_d = 0$ on Γ(we take $x_d = \mathrm{dist}(x, \Gamma)$). For simplicity, we treat the case $c_1 = c_2 = 1$. Then the symbol of $-h^2\Delta$ becomes $\xi_d^2 + r(x, \xi') + hq(x, \xi')$ and $r(x', 0, \xi') = r_0(x', \xi')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta|_\Gamma$ on Γ. For $z \in Z_1 \cup Z_2 \cup Z_3$, let $\rho \in C^\infty(T^*(\Gamma)$ be the root of the equation

$$\rho^2 + r_0(x', \xi') - n(x', 0)z = 0$$

with $\mathrm{Im}\, \rho > 0$. Then we have the following

Proposition 1 (Vodev, (2014))

Given $0 < \epsilon \ll 1$, *there exists* $0 < h_0(\epsilon) \ll 1$ *such that for* $z \in Z_1$ *and* $0 < h \leq h_0$ *we have*

$$\|\gamma_0 hD_\nu - Op_h(\rho + hb)f\|_{H_s^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\,\mathrm{Im}\, z|}}\|f\|_{L^2(\Gamma)}, \tag{13}$$

where $b \in S_0^{0,0}$ *does not depend on* z, h *and the function* $n(x)$. *Moreover, for* $z \in Z_2 \cup Z_3$ *the above estimate holds with* $|\,\mathrm{Im}\, z|$ *replaced by* 1.

Vodev established Prop. 1 for bounded domains K, but with some modification of the proof the same result holds for $\mathbb{R}^d \setminus K$.

In the case (A) we have $0 < \epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0,\ \epsilon_0 > 0,\ \forall x \in \Gamma$. If $u \neq 0$. Set $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$. The boundary condition implies

$$N(z,h)f - \gamma\sqrt{z}f = 0.$$

According to Prop. 1, for $1 \geq \operatorname{Im} z \geq h^\delta,\ \delta = 1/2 - \epsilon$, we have

$$\|Op_h(\rho)f - \gamma\sqrt{z}f\|_{L^2(\Gamma)} \leq C\frac{h}{\sqrt{|\operatorname{Im} z|}}\|f\|_{L^2(\Gamma)}, \tag{14}$$

where for $z \in Z_2 \cup Z_3$ the estimate holds with $|\operatorname{Im} z|$ replaced by 1. Consider the symbol

$$c(x',\xi',z) = \rho(x',\xi',z) - \gamma\sqrt{z}$$

$$= \frac{(1-\gamma^2)z - r_0(x',\xi')}{\rho(x',\xi',z) + \gamma\sqrt{z}}.$$

We show that $c(x',\xi',z) \in S^{0,1}_\delta$ is elliptic, while $|\operatorname{Im} z| c^{-1} \in S^{0,-1}_\delta$.

Thus

$$\|Op_h(c^{-1})g\|_{L^2(\Gamma)} \leq C|\operatorname{Im} z|^{-1}\|g\|_{L^2(\Gamma)}$$

and we deduce

$$\|Op_h(c^{-1})Op_h(c)f\|_{L^2(\Gamma)} \leq C_1 \frac{h}{|\operatorname{Im} z|^{3/2}}\|f\|_{L^2(\Gamma)}.$$

A more fine analysis shows that

$$\|Op_h(c^{-1})Op_h(c)f - f\|_{L^2(\Gamma)} \leq C_2 \frac{h}{|\operatorname{Im} z|^{2}}\|f\|_{L^2(\Gamma)}.$$

Consequently, one concludes that

$$\|f\|_{L^2(\Gamma)} \leq C_3\left(h^{1-2\delta} + h^{1-\frac{3}{2}\delta}\right)\|f\|_{L^2(\Gamma)}. \tag{15}$$

Since $\delta = 1/2 - \epsilon,\ 0 < \epsilon \ll 1$, from (15) we obtain $f = 0$ for $0 < h \leq h_0(\epsilon)$ small enough. The case $\gamma(x) - 1 > 0$ is more complicated and the argument, exploited in the case $\gamma(x) - 1 < 0$, breaks down since for $\operatorname{Re} z = -1$, $\operatorname{Im} z = 0$ the symbol

$$[(1 + r_0(x', \xi') - \gamma(x')]$$

is not elliptic and it may vanish for some (x_0', ξ_0').

6. Idea of the proof of Theorem 5

Step 1. We pass to a semi-classical setting. Set $Z = \{z \in \mathbb{C};\ \frac{1}{2} \leq |\mathrm{Re}\, z| \leq 3,\ |\mathrm{Im}\, z| \leq 1\}$ and consider for $z \in Z$ and $0 < h \ll 1$ the operator

$$hT(z/h^2) := c_1\mathcal{N}_1(z,h) - c_2\mathcal{N}_2(z,h),$$

where the DN-maps $N_j(z,h)$ were defined in the previous section. Let $G_D^{(j)}$, $j = 1, 2$, be the Dirichlet self-adjoint realization of the operator $L_j := -n_j^{-1}\nabla c_j \nabla$ in the space $H_j = L^2(K, n_j(x)dx)$. Set $\mathcal{H} = H_1 \oplus H_2$. Let $R(\lambda)$ be the resolvent of the transmission boundary problem. We omit in the notation $j = 1, 2$ and consider the operators

$$F(z,h) = \mathcal{N}(z,h) - \tilde{\mathcal{N}}(z,h) = \mathcal{N}(z,h)Op_h(\chi) - \gamma_0 D_\nu (h^2 G_D - z)^{-1}\frac{c}{n}Op_h(p),$$

where $\chi(x', \xi') = \Phi(\delta_0 r_0(x', \xi'))$ with $\Phi(\sigma) = 1$ for $|\sigma| \leq 1$ and $\Phi(\sigma) = 0$ for $|\sigma| \geq 2$, while $0 < \delta_0 \ll 1$ is small enough. Here $\tilde{\mathcal{N}}(z,h)$ is the parametrix of the DN operator $\mathcal{N}(z,h)Op_h(1-\chi)$ in the domain where $r_0(x', \xi') > \frac{1}{\delta_0}$ and p is some symbol with behavior $\mathcal{O}(h^N)$. The number N will be taken large enough and it depends only on the parametrix construction.

The operator $F(z, h)$ is meromorphic with values in trace class operators and we denote by $\mu_j(F(z, h))$ its characteristic eigenvalues.

Lemma 1

If z/h^2 does not belong to spec G_D, then for every integer $0 \le m \le N/4$ we have

$$\mu_j(F(z, h)) \le \frac{C}{\delta(z, h)} \Big(hj^{1/(d-1)}\Big)^{-2m}, \ \forall j,$$

where $\delta(z, h) := \min\{1, \mathrm{dist}\,\{z, \mathrm{spec}\, h^2 G_D\}\} > 0$ and $C > 0$ depends on m and N but is independent of z, h, j.

Let

$$T(\lambda) := c_1\gamma_0 D_\nu K_1(\lambda) - c_2\gamma_0 D_\nu K_2(\lambda),$$

where $K_j(\lambda)f = u$, and u is the solution of the problem

$$\begin{cases} \Big(L_j - \lambda\Big)u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma. \end{cases} .$$

Step 2.

Theorem 6

Assume that $T(\lambda)^{-1}$ is a meromorphic function with residue of finite rank. Let $\delta \subset \mathbb{C}$ be a simple closed positively oriented curve which avoids the eigenvalues of $G_D^{(j)}$, $j = 1, 2$, as well as the poles of $T(\lambda)^{-1}$. Then we have the identity

$$\mathrm{tr}_{\mathcal{H}}\, (2\pi i)^{-1} \int_\delta R(\lambda) d\lambda = \sum_{j=1}^{2} \mathrm{tr}_{H_j}\, (2\pi i)^{-1} \int_\delta (G_D^{(j)} - \lambda)^{-1} d\lambda$$
$$-\mathrm{tr}_{L^2(\Gamma)}\, (2\pi i)^{-1} \int_\delta T(\lambda)^{-1} \frac{dT(\lambda)}{d\lambda} d\lambda. \quad (16)$$

Let us mention that if $R(\lambda)$ is an operator-valued meromorphic function with residue of finite rank, the multiplicity of a pole $\lambda_k \in \mathbb{C}$ of $R(\lambda)$ is defined by

$$\mathrm{mult}\, (\lambda_k) = \mathrm{rank}\, (2\pi i)^{-1} \int_{|\lambda - \lambda_k| = \epsilon} R(\lambda) d\lambda,\ 0 < \epsilon \ll 1.$$

On the other hand, the rank above is equal to the trace and on the left hand side of (16) we have the sum of the mutiplicities of the (ITE) lying in the domain $\omega_\delta \subset \mathbb{C}$ bounded by δ. Clearly, the terms with $(G_D^{(j)} - \lambda)^{-1}$ yield the sum of eigenvalues of $G_D^{(j)}$ in ω_δ counted with their multiplicities.

Step3. It is possible to construct invertible, bounded operator $E(z,h) : H_h^s(\Gamma) \to H_h^{s+1}(\Gamma)$ with bounded inverse $E(z,h)^{-1} : H_h^s(\Gamma) \to H_h^{s-1}(\Gamma)$, $\forall s \in \mathbb{R}$ so that

$$hT(z/h^2) = E^{-1}(z,h)(I + \mathcal{K}(z,h)),$$

$$(hT(z/h^2))^{-1} = (I + \mathcal{K}(z,h))^{-1} E(z,h)$$

with a trace class operator

$$\mathcal{K}(z,h) = E(z,h)(c_1 F_1(z,h) - c_2 F_2(z,h)) + \mathcal{L}(z,h).$$

Moreover, the operators $E(z,h), E^{-1}(z,h)$ are holomorphic with respect to z in Z, while $\mathcal{K}(z,h)$ is memoromorphic operator-valued function in this region. Then

$$\mathrm{tr} \int_\delta T^{-1}(z/h^2) \frac{d}{dz} T(z/h^2) dz = \mathrm{tr} \int_\delta (I + \mathcal{K}(z,h))^{-1} \frac{d}{dz} \mathcal{K}(z,h) dz.$$

Set $g_h(z) := \det(I + \mathcal{K}(z,h))$ and denote by $M_\delta(h)$ the number of the poles $\{\lambda_k\}$ of $R(\lambda)$ such that $h^2\lambda_k$ are in ω_δ. Similarly, we denote by $M_\delta^{(j)}(h)$ the number of the eigenvalues ν_k of $G_D^{(j)}$ such that $h^2\nu_k \in \omega_\delta$. Then using the well-known formula

$$\mathrm{tr}\,(I + \mathcal{K}(x,h))^{-1}\frac{\partial \mathcal{K}(z,h)}{\partial z} = \frac{\partial}{\partial z}\log\det(I + \mathcal{K}(z,h)),$$

we get from (16) the following

Lemma 2

Let $\delta \subset Z$ be closed positively oriented curve which avoid the eigenvalues of $h^2 G_D^{(j)}, j = 1,2$ as well as the poles of $T(z/h^2)^{-1}$. Then we have

$$M_\delta(h) = M_\delta^{(1)}(h) + M_\delta^{(2)}(h) + \frac{1}{2\pi i}\int_\delta \frac{d}{dz}\log g_h(z)dz. \tag{17}$$

Observe that $z_0 \in Z \setminus \mathrm{spec}\,(h^2 G_D^{(1)}) \cup \mathrm{spec}\,(h^2 G_D^{(2)})$ is a zero of $g_h(z)$ if and only if z_0 is a pole of $R(z/h^2)$ and hence z_0/h^2 is an (ITE).

Step 4.

Lemma 3

Let $0 < \kappa \leq 1$ be as in Theorem 5. Then, given any $0 < \epsilon \ll 1$, the operator $I + \mathcal{K}(z, h)$ is invertible on $L^2(\Gamma)$ for $z \in Z$, $|\operatorname{Im} z| \geq h^{\kappa - \epsilon}$ and the inverse operator satisfies in this region the estimate

$$\left\| (I + \mathcal{K}(z, h))^{-1} \right\|_{L^2 \to L^2} \leq C_\epsilon h^{-\ell}$$

with constants $C > 0$, $\ell > 0$. For these values of z we have

$$\log \frac{1}{|g_h(z)|} \leq C_\epsilon h^{1-d-\epsilon}, \; 0 < \epsilon \ll 1. \tag{18}$$

Moreover, for these z the function $g_h(z)$ is holomorphic and we have

$$\left| \frac{d}{dz} \log g_h(z) \right| \leq \frac{C_\epsilon h^{1-d-2\epsilon}}{|\operatorname{Im} z|} \tag{19}$$

for $z \in W := \{z \in \mathbb{C} : \; 2/3 \leq |\operatorname{Re} z| \leq 5/2, \; 2h^{\kappa-\epsilon} \leq |\operatorname{Im} z| \leq 1/2\}$.

Proposition 2

For every $0 < \epsilon \ll 1$ and $A > 0$, independent of h, we have the asymptotics

$$I(h) := \sharp\Big\{z_k,\ z_k/h^2 \text{ is (ITE)}:\ 1 - Ah^{\kappa-\epsilon} \leq |\mathrm{Re}\, z_k| \leq 2 + Ah^{\kappa-\epsilon},\ |\mathrm{Im}\, z_k| \leq h^{\kappa-\epsilon}\Big\}$$

$$= (2^{d/2} - 1)(\tau_1 + \tau_2)h^{-d} + \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-3\epsilon}). \tag{20}$$

We will discuss only the case of (ITE) with $\mathrm{Re}\, z_k > 0$, since the case $\mathrm{Re}\, z_k < 0$ is similar (and even simpler since the function $g_h(z)$ does not have poles in $\mathrm{Re}\, z < 0$). Consider the points

$$w_1^{\pm} = 1 - Ah^{\kappa-\epsilon} \pm \frac{\mathbf{i}}{3},\ w_2^{\pm} = 2 + Ah^{\kappa-\epsilon} \pm \frac{\mathbf{i}}{3},$$

$$\widetilde{w}_1^{\pm} = 1 - Ah^{\kappa-\epsilon} \pm \mathbf{i}3h^{\kappa-\epsilon},\ \widetilde{w}_2^{\pm} = 2 + Ah^{\kappa-\epsilon} \pm \mathbf{i}3h^{\kappa-\epsilon}$$

and set

$$\Theta_1 = \left\{z \in \mathbb{C} : 1 - 2(A+1)h^{\kappa-\epsilon} \leq \mathrm{Re}\, z \leq 1 + h^{\kappa-\epsilon},\ |\mathrm{Im}\, z| \leq 4h^{\kappa-\epsilon}\right\},$$

$$\Theta_2 = \left\{z \in \mathbb{C} : 2 - h^{\kappa-\epsilon} \leq \mathrm{Re}\, z \leq 2 + 2(A+1)h^{\kappa-\epsilon},\ |\mathrm{Im}\, z| \leq 4h^{\kappa-\epsilon}\right\}.$$

Lemma 4

There exist positively oriented piecewise smooth curves $\widetilde{\gamma}_1 \subset \Theta_1$ and $\widetilde{\gamma}_2 \subset \Theta_2$, where $\widetilde{\gamma}_1$ connects the point $\widetilde{w}_1^-$ with $\widetilde{w}_1^+$, while $\widetilde{\gamma}_2$ connects the point $\widetilde{w}_2^+$ with $\widetilde{w}_2^-$, such that

$$\left| \mathrm{Im} \int_{\widetilde{\gamma}_j} \frac{d}{dz} \log g_h(z) dz \right| \leq C_\epsilon h^{-d+\kappa-2\epsilon}, \quad j = 1, 2. \tag{21}$$

We apply Lemma 2 with a contour $\delta = \gamma_1 \cup \gamma_3 \cup \gamma_2 \cup \gamma_4$, where $\gamma_3 \subset W$ is the segment $[w_1^+, w_2^+]$ on the line passing through the points w_1^+ and w_2^+, and $\gamma_4 \subset W$ is the segment $[w_2^-, w_1^-]$ on the line passing through the points w_2^- and w_1^-. Next, $\gamma_1 = [w_1^-, \widetilde{w}_1^-] \cup \widetilde{\gamma}_1 \cup [\widetilde{w}_1^+, w_1^+]$, $\gamma_2 = [w_2^+, \widetilde{w}_2^+] \cup \widetilde{\gamma}_2 \cup [\widetilde{w}_2^-, w_2^-]$ (see Figure). Since $\gamma_j \subset W$, $|\gamma_j| = \mathcal{O}(1)$, $j = 3, 4$, by (19) we have

$$\left| \int_{\gamma_j} \frac{d}{dz} \log g_h(z) dz \right| \leq \int_{\gamma_j} \left| \frac{d}{dz} \log g_h(z) \right| |dz|$$

$$\leq C_\epsilon h^{-d+1-2\epsilon} \int_{\gamma_j} |dz| \leq C_\epsilon h^{-d+1-2\epsilon}, \quad j = 3, 4$$

Applying (19) once more, we have

$$\left| \int_{[w_j^\pm, \widetilde{w}_j^\pm]} \frac{d}{dz} \log g_h(z) dz \right| \leq C_\epsilon h^{-d+1-2\epsilon} \int_{3h^{\kappa-\epsilon}}^{1/2} \frac{d\sigma}{\sigma} \leq C_\epsilon h^{-d+1-3\epsilon}, \quad j = 1, 2.$$

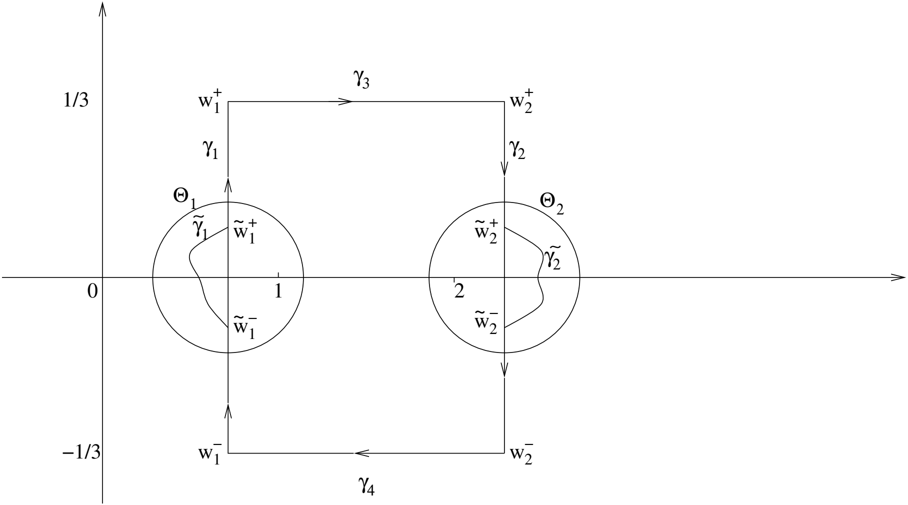

Figure: contour δ

Choose $h = \frac{\sqrt{2}}{r}$, $r \gg 1$. The asymptotics (20) avec eigenvalues $\lambda_k = \frac{z_k r^2}{2}$ yields

$$\{\lambda \in \mathbb{C} : \lambda \text{ is } (ITE),\ \frac{r^2}{2} - Ar^{2-\kappa+\epsilon} \leq |\mathrm{Re}\,\lambda| \leq r^2 + Ar^{2-\kappa+\epsilon},\ |\mathrm{Im}\,\lambda| \leq r^{2-\kappa+\epsilon}\}$$

$$= (1 - 2^{-d/2})(\tau_1 + \tau_2)r^d + \mathcal{O}_{\epsilon,A}(r^{d-\kappa+3\epsilon}),$$

where κ is described in Theorem 5. By applying this asymptotics with different A, we conclude that the same asymptotics holds for the (ITE) in the region

$$\{\lambda \in \mathbb{C} : \frac{r^2}{2} \leq |\lambda| \leq r^2,\ |\mathrm{Im}\,\lambda| \leq r^{2(1-\frac{\kappa}{2}+\epsilon)}\}, 0 < \epsilon \ll 1.$$

According to our assumption, there are no (ITE) in the region $\{\lambda \in \mathbb{C} : \frac{r^2}{2} \leq |\lambda| \leq r^2,\ |\mathrm{Im}\,\lambda| \geq r^{2(1-\frac{\kappa}{2}+\epsilon)}\}$ for every $0 < \epsilon \ll 1$, provided $r \geq r_0(\epsilon) \gg 1$. Thus we get the asymptotics

$$N(r) - N(r/\sqrt{2}) = (1 - 2^{-d/2})(\tau_1 + \tau_2)r^d + \mathcal{O}_\varepsilon(r^{d-\kappa+\varepsilon}), \quad r \geq r_0(\epsilon),$$

for every $0 < \varepsilon \ll 1$. This implies easily the result of Theorem 5.

Concluding remarks.

1. The idea of Theorem 6 to use an integral of the resolvent $R(\lambda)$ comes back from the works of Sjöstrand and Vodev and Popov and Vodev for the counting function for the resonances in a neighbourhood of $\mathbb{R}$ for the elasticity system and the transmission problem, respectively.

2. To our best knowledge Theorem 5 is the first result where a precise relation between the eigenvalues-free region and the order of the remainder in the Weyl formula was established.

3. The argument of the proof of Theorem 5 works for more general boundary problems, assuming that we have established an eigenvalues-free region. We compare the resolvent of the boundary problem with that of the Dirichlet one and eigenvalues appear as the points where some operator $T(\lambda)$ is not invertible. In particular, we proved an analogue of Theorem 6 for the eigenvalues of the generator G lying in $\omega \subset\subset \{\mathrm{Re}\, z < 0\}$.

4. The approach for the analysis of eigenvalues-free regions can be applied also for more general problems including Maxwell system etc. The construction of a parametrix for $-h^2\Delta - z$ works replacing $-\Delta$ by another second order elliptic operator $P(x, hD_x)$.

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Thank you !